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DIAGONAL AND VON NEUMANN REGULAR MATRICES OVER A DEDEKIND DOMAIN

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Abstract: Some well known results for matrices over a principal ideal domain are generalized to matrices over a Dedekind domain:

– necessary and sufficient conditions are obtained for a matrix to be diagonalizable and an algorithm is given to execute this diagonalization;

– the class of von Neumann regular matrices is characterized.

It is shown how to diagonalize a matrix of which it was known in literature that is was diagonalizable, but for which no constructive way was available to achieve the diagonalization. Also, an answer is given to question formulated by Den Boer.

Relations to results of L. Gerstein and M. Newman and to the calculation of the Moore–Penrose inverse of a matrix are mentioned.

1 – Introduction

Let $A \in \mathcal{M}_{m\times n}(S)$, S a (commutative) principal ideal domain (PID). An algorithm is known (see [5]) to obtain invertible matrices $P \in \mathcal{M}_m(S)$ and $Q \in \mathcal{M}_n(S)$ such that $A = P\,\mathrm{diag}[d_1, ..., d_r, 0, ..., 0]Q$, $d_1, ..., d_r \in S$ and $d_i \mid d_{i+1}$, $i \in \{1, ..., r-1\}$. It has also been shown (see [10]) that A will be von Neumann regular (i.e. $X \in \mathcal{M}_{n\times m}(S)$ exists such that $A \times A = A$) if and only if a diagonal reduction exists in which the diagonal elements $d_1, ..., d_r$ can be taken equal to 1.

Let $A \in \mathcal{M}_{m\times n}(R)$, R a (commutative) Dedekind domain. If A is of determinantal rank r, the ideal generated by the $i \times i$ determinants ($i \leq r$) formed out of A will be denoted by δ_i and the i-th invariant factor ideal by ε_i (so $\delta_{i-1}\,\varepsilon_i = \delta_i$, $\forall\, i \in \{1, .., r\}$, $\delta_0 = R$). By the class of A we understand the class of the ideal generated by an arbitrary non-zero row of $r \times r$ minors.

It is known (see [6]) that an arbitrary matrix over R may not necessarily be diagonalizable (in the above sense), even if it would be von Neumann regular.

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An almost diagonal form was obtained by Krull (see [7]). A will be called Krull-equivalent (notation $A \sim_{\mathrm{K}} B$, instead of $A \sim B$ for the usual equivalence) to a matrix B (which does not have to be of the same size) if and only if invertible matrices P and Q and elements $u, v \in \mathbb{N}$ exist such that

$$\begin{bmatrix} P & 0 \\ 0 & 1_u \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1_v \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} .$$

Any matrix A will then be Krull-equivalent to its "Krull normal form" $K(A)$:

$$K(A) = \begin{bmatrix} a_1 & 0 & \ldots & & & 0 \\ \gamma_1 a_1 & 0 & & & & \\ 0 & a_2 & & & & \\ 0 & \gamma_2 a_2 & & & & \\ \vdots & & & & & \vdots \\ & & & a_{r-1} & & \\ & & & \gamma_{r-1} a_{r-1} & 0 & 0 \\ & & & 0 & a_{r1} & a_{r2} \\ 0 & & \ldots & 0 & \gamma_r a_{r1} & \gamma_r a_{r2} \end{bmatrix} ,$$

where:

$\gamma_i \in \mathbb{Q}$, the field of fractions of R;

$(a_i, \gamma_i a_i) = \varepsilon_i$, $\forall\, i \in \{1, ..., r-1\}$;

(a_{r1}, a_{r2}) and $(\gamma_r a_{r1}, \gamma_r a_{r2})$, $\gamma_r \in \mathbb{Q}$, have ε_r as greatest common divisor ideal and their class is the class of A.

2 – Diagonalization of matrices over a Dedekind domain

Proposition 1. *The following are equivalent:*

i) $A \sim \mathrm{diag}[d_1, ..., d_r, 0, ..., 0]$, *a diagonal matrix with* $d_i | d_{i+1}$, $\forall\, i \in \{1, ..., r-1\}$*;*

ii) *All* ε_i *are principal ideals and the class of* A *is the class of* R*.*

Proof:

i)⇒ii) If $A \sim \mathrm{diag}[d_1, ..., d_r, 0, ..., 0]$, both have the same invariant factor ideals and class. Thus $\varepsilon_i = d_i R_i$.

ii)⇒i) Let $A = [a_{ij}]$, $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$, be of rank $r \geq 2$. (If A is of rank 1, one applies on A the operations that will be used below on the matrix B).

Elements $z_1, ..., z_m$ can be found such that $\sum_{i=1}^{m} z_i a_{ij} = a_j$, $j \in \{1, ..., n\}$ and $a_1 R + ... + a_j R + ... + a_m R = \varepsilon_1 = d_1 R$ (see [7]). It follows (see [2]), that an invertible matrix X can be constructed such that $[z_1\ z_2\ \ldots\ z_m]X = [1\ 0\ \ldots\ 0]$. Writing $X^{-1} = [T'^{\mathrm{T}}\ T^{\mathrm{T}}]^{\mathrm{T}}$, where $T' \in \mathcal{M}_{1\times m}(R)$, $T \in \mathcal{M}_{(m-1)\times m}(R)$, it is obtained that

$$\begin{bmatrix} z_1\ z_2\ \ldots\ z_m \\ T \end{bmatrix} = \begin{bmatrix} z_1\ z_2\ \ldots\ z_m \\ T \end{bmatrix} X \begin{bmatrix} T' \\ T \end{bmatrix} = 1_m \begin{bmatrix} T' \\ T \end{bmatrix} = X^{-1} .$$

With these notations:

$$\begin{bmatrix} z_1\ z_2\ \ldots\ z_m \\ T \end{bmatrix} A = \begin{bmatrix} a_1\ a_2\ \ldots\ a_m \\ A_1 \end{bmatrix} .$$

This last matrix can be reduced to $\begin{bmatrix} d_1\ 0\ \ldots\ 0 \\ A_2 \end{bmatrix}$ and then to $\begin{bmatrix} d_1 & 0\ \ldots\ 0 \\ 0 & \\ \vdots & \\ 0 & A_3 \end{bmatrix}$. This procedure can be repeated until a matrix B of rank 1 is obtained such that

$$A = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{r-1} & \\ & & & & B \end{bmatrix} , \qquad d_i \mid d_{i+1}, \quad i \in \{1, \ldots, r-2\} .$$

Since the class of B is the class of R, the ideal generated by an arbitrary row of B will be $\varepsilon_r = d_r R$. Thus, an invertible matrix U can be constructed such that

$$B.U = \begin{bmatrix} b_1 & 0\ \ldots\ 0 \\ & V \\ b_{m-r+1} & \end{bmatrix} .$$

But $\operatorname{rank} B = 1$, so $V = 0$. Finally, the remaining column $[b_1, \ldots, b_{m+r-1}]^{\mathrm{T}}$ is reduced to a column of which all the elements are 0, except the first. ∎

Corollary. *Any matrix over a PID can be diagonalized.* ∎

Application. R. Puystjens and J. Van Geel have considered (see [11]) the following matrix over $\mathbf{Z}[\sqrt{-5}]$:

$$A = \begin{bmatrix} 3 & 1+2\sqrt{-5} \\ 1-2\sqrt{-5} & 3 \end{bmatrix} .$$

Using techniques about matrices over graded rings, they could show that it must be equivalent to $\operatorname{diag}[1, -12]$; K. Coppieters could, with a computer, obtain invertible matrices P and Q such that $A = P \operatorname{diag}[1, -12] Q$.

By the algorithm given in the proof of the theorem, it is possible to obtain these matrices P and Q in an explicit way.

We use elements $z_1 = 2 - 2\sqrt{-5}$ and $z_2 = 3 - 2\sqrt{-5}$ for which

$$a_1 \mathbf{Z}[\sqrt{-5}] + a_2 \mathbf{Z}[\sqrt{-5}] = \mathbf{Z}[\sqrt{-5}],$$
$$a_1 = 3z_1 + (1 - 2\sqrt{-5})\, z_2\,, \qquad a_2 = (1 + 2\sqrt{-5})\, z_1 + 3z_2\ .$$

Then

$$P'A = \begin{bmatrix} z_1 & z_2 \\ 151 - 8\sqrt{-5} & 167 + 4\sqrt{-5} \end{bmatrix} A = \begin{bmatrix} -11 - 14\sqrt{-5} & 31 - 4\sqrt{-5} \\ 6(110 - 59\sqrt{-5}) & 6(122 + 51\sqrt{-5}) \end{bmatrix} .$$

Now the first row can be reduced to

$$[1\ 0]^{\mathrm{T}}:\ A \sim \begin{bmatrix} 1 & 0 \\ 6(2 - 3\sqrt{-5}) & -12 \end{bmatrix} .$$

If all these calculations are put together, invertible matrices P and Q are obtained such that $PAQ = \operatorname{diag}[1, -12]$.

3 – Von Neumann regular matrices over a Dedekind domain

Proposition 2. *The following are equivalent:*

i) *A is von Neumann regular;*

ii) *The invariant factor ideals of A are the unit ideal;*

iii) *There exist invertible matrices $P \in \mathcal{M}_m(R)$ and $Q \in \mathcal{M}_n(R)$ such that*

$$A = P \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix} Q\ ,$$

where $E = E^2 \in \mathcal{M}_2(R)$ and $\operatorname{class} E = \operatorname{class} A$.

Proof:

i)$\Leftrightarrow$ii) If A is von Neumann regular, $\delta_r = (1)$. So $\delta_{r-1}\varepsilon_r = (1)$, and since both ideals are integral ideals, δ_{r-1} and ε_r are the unit ideal. In this way it is obtained that all invariant factor ideals ε_i will be generated by the unit element 1. On the other hand, if all ideals $\varepsilon_i = (1)$ it follows that $\delta_r = (1)$.

ii)$\Rightarrow$iii) If in the Krull normal form K_A, $\varepsilon_i = (a_i, \gamma_i a_i) = (1)$, for all $i < r$, then integral elements x_i and y_i exist such that $x_i a_i + y_i(\gamma_i a_i) = 1$. The 2×2

matrices $\begin{bmatrix} x_i & y_i \\ -\gamma_i a_i & a_i \end{bmatrix}$ are integral invertible matrices. Then:

$$\begin{bmatrix} x_1 & y_1 & & & & & & \\ -\gamma_1 a_1 & a_1 & & & & & & \\ & & x_2 & y_2 & & & & \\ & & -\gamma_2 a_2 & a_2 & & & & \\ & & & & \ddots & & & \\ & & & & & x_{r-1} & y_{r-1} & \\ & & & & & -\gamma_{r-1} a_{r-1} & a_{r-1} & \\ & & & & & & & 1 \end{bmatrix} K_A =$$

$$= \begin{bmatrix} 1 & 0 & & & & & & & \\ 0 & 0 & & & & & & & \\ & 1 & 0 & & & & & & \\ & 0 & 0 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & 0 & & & \\ & & & & 0 & 0 & & & \\ & & & & & & a_{r1} & a_{r2} \\ & & & & & & \gamma_r a_{r1} & \gamma_r a_{r2} \end{bmatrix}.$$

So $A \sim_{\mathrm{K}} \begin{bmatrix} 1 & & & \\ & a_{r1} & a_{r2} & \\ & \gamma_r a_{r1} & \gamma_r a_{r2} & \\ & & & 0 \end{bmatrix}$ and hence $A \sim_{\mathrm{K}} \begin{bmatrix} 1 & & \\ & a_{r1} & a_{r2} \\ & \gamma_r a_{r1} & \gamma_r a_{r2} \end{bmatrix}$.

The following cases can be distinguished:

a) $m = r$ or $n = r$.

In these cases the class of the matrix is the class of ε_r, that is the principal class. So A is right invertible.

Instead of $\begin{bmatrix} a_{r1} & a_{r2} \\ \gamma_r a_{r1} & \gamma_r a_{r2} \end{bmatrix}$, we can take in these cases the 2×2 matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Consequently,

$$A \sim_{\mathrm{K}} \begin{bmatrix} 1_{r-1} & & \\ & 1 & 0 \\ & 0 & 0 \end{bmatrix} \quad \text{or} \quad A \sim_{\mathrm{K}} 1_r \ .$$

L.S. Levy (see [7]) showed that for matrices, over a Dedekind domain, that are of the same size, the Krull equivalence implies the usual equivalence of matrices. It follows that $A \sim [1_r \ 0]_{r,n}$ or $A^{\mathrm{T}} \sim [1_r \ 0]_{r,m}$.

b) $m > r$ and $n > r$.

The 2×2 matrix $\begin{bmatrix} a_{r1} & a_{r2} \\ \gamma_r a_{r1} & \gamma_r a_{r2} \end{bmatrix}$ is of the same class as A, of rank 1 and has invariant factor ideal (1). So there exist integral elements x, y, z and t such that $xa_{r1} + ya_{r2} + z\gamma_r a_{r1} + t\gamma_r a_{r2} = 1$ or else $xa_{r1} + ya_{r2} + \gamma_r(za_{r1} + ta_{r2}) = 1$. Call $c_{r1} = xa_{r1} + ya_{r2}$ and $c_{r2} = za_{r1} + ta_{r2}$. Then $c_{r1}R + c_{r2}R = a_{r1}R + a_{r2}R$ and so

$$\begin{bmatrix} c_{r1} & c_{r2} \\ \gamma_r c_{r1} & \gamma_r c_{r2} \end{bmatrix} \sim_{\mathrm{K}} \begin{bmatrix} a_{r1} & a_{r2} \\ \gamma_r a_{r1} & \gamma_r a_{r2} \end{bmatrix} .$$

But

$$\begin{bmatrix} c_{r1} & c_{r2} \\ \gamma_r c_{r1} & \gamma_r c_{r2} \end{bmatrix}^2 = \begin{bmatrix} c_{r1} & c_{r2} \\ \gamma_r c_{r1} & \gamma_r c_{r2} \end{bmatrix}$$

since $c_{r1} + \gamma_r c_{r1} = 1$. So A is Krull-equivalent to $\begin{bmatrix} 1_{r-1} & 0 \\ 0 & E \end{bmatrix}$, and $E = \begin{bmatrix} c_{r1} & c_{r2} \\ \gamma_r c_{r1} & \gamma_r c_{r2} \end{bmatrix}$ is idempotent. Now $\begin{bmatrix} 1_{r-1} & 0 \\ 0 & E \end{bmatrix} \in \mathcal{M}_{(r+1)\times(r+1)}(R)$, and since $r + 1 \leq m$, $r + 1 \leq n$, one can eventually add zero columns and (or) rows to obtain that $\operatorname{diag}[1_{r-1}, E, 0] \in \mathcal{M}_{m\times n}(R)$. Since A is also Krull-equivalent to $\operatorname{diag}[1_{r-1}, E, 0]$, these matrices of the same size will be equivalent too. ∎

Corollary. *Any von Neumann regular matrix over a PID is equivalent to a* $\operatorname{diag}[1_r, 0]$*-matrix.* ∎

Application. Den Boer (see [1]) stated the following question (it was a part of a larger conjecture): "If a ring is neither a Bezout domain nor a local domain, then one can find irreducible matrices $A_1, ..., A_r, B_1, ..., B_s$ over R such that

$$\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_s \end{bmatrix}$$

are equivalent but where $r \neq s$ and the zero blocks are of different size". Using the proposition, such an example can be provided. Consider

$$\left.\begin{aligned} A_1 &= \begin{bmatrix} -6 & 1 - 2\sqrt{-5} \\ -2(1 + 2\sqrt{-5}) & 7 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 7 & -1 + 2\sqrt{-5} \\ 2(1 + 2\sqrt{-5}) & -6 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}\right\} \quad \text{over } \mathbf{Z}[\sqrt{-5}] .$$

Then A_1, A_2 and B_1 are irreducible (i.e. they cannot be diagonalized); yet, the matrices $\operatorname{diag}[A_1, A_2]$ and $\operatorname{diag}[B_1, 0]$ are equivalent.

4 – Relations to other results

a) Notation

After a permutation of columns in the matrix $[K(A)\ 0_{2r,r-1}]$ a matrix

$$K'(A) = \left[\begin{array}{cc:cc:c:cc:cc}
a_1 & 0 & & & \cdots & & & & 0 \\
\gamma_1 a_1 & 0 & & & & & & & \\
\hdashline
 & & a_2 & 0 & & & & & \\
 & & \gamma_2 a_2 & 0 & & & & & \\
\hdashline
 & & & & \ddots & & & & \vdots \\
 & & & & & a_{r-1} & 0 & & \\
 & & & & & \gamma_{r-1} a_{r-1} & 0 & 0 & 0 \\
\hdashline
 & & & & & & & a_{r1} & a_{r2} \\
0 & & & & \cdots & & & \gamma_r a_{r1} & \gamma_r a_{r2}
\end{array}\right]$$

is obtained, which can be denoted shortly as

$$K'(A) = \operatorname{diag}\left[\begin{bmatrix} a_1 & 0 \\ \gamma_1 a_1 & 0 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ \gamma_2 a_2 & 0 \end{bmatrix}, ..., \begin{bmatrix} a_{r-1} & 0 \\ \gamma_{r-1} a_{r-1} & 0 \end{bmatrix}, \begin{bmatrix} a_{r1} & a_{r2} \\ \gamma_r a_{r1} & \gamma_r a_{r2} \end{bmatrix}\right].$$

All matrices with the same invariant factor ideals and of the same class, are Krull-equivalent to $K'(A)$. $K'(A)$ is unique up to left and (or) right multiplication by matrices of the form $\operatorname{diag}[U_1, ..., U_r]$, $U_i \in \mathcal{M}_{2\times 2}(R)$, invertible $(i \in \{1, ..., r\})$.

Let B be another matrix over R, of the same rank r and with

$$K'(B) = \operatorname{diag}\left[\begin{bmatrix} b_1 & 0 \\ \rho_1 b_1 & 0 \end{bmatrix}, \begin{bmatrix} b_2 & 0 \\ \rho_2 b_2 & 0 \end{bmatrix}, ..., \begin{bmatrix} b_{r-1} & 0 \\ \rho_{r-1} b_{r-1} & 0 \end{bmatrix}, \begin{bmatrix} b_{r1} & b_{r2} \\ \rho_r b_{r1} & \rho_r b_{r2} \end{bmatrix}\right],$$

with $y_i, \rho_i \in Q$, $\forall\, i$. Let $\sigma_i \in Q$, $(c_i, \sigma_i c_i) = (a_i, \gamma_i a_i).(b_i, \rho_i b_i)$, $\forall\, i \in \{1, ..., r-1\}$, and (c_{r1}, c_{r2}) and $(\sigma_r c_{r1}, \sigma_r c_{r2})$ have the product of the g.c.d. ideals of (a_{r1}, a_{r2}) and $(\sigma_r a_{r1}, \sigma_r a_{r2})$ and of (b_{r1}, b_{r2}) and $(\rho_r b_{r1}, \rho_r b_{r2})$ as g.c.d. ideal. Suppose

$$K'(A) \circ K'(B) = \operatorname{diag}\left[\begin{bmatrix} c_1 & 0 \\ \sigma_1 c_1 & 0 \end{bmatrix}, ..., \begin{bmatrix} c_{r-1} & 0 \\ \sigma_{r-1} c_{r-1} & 0 \end{bmatrix}, \begin{bmatrix} c_{r1} & c_{r2} \\ \sigma_r c_{r1} & \sigma_r c_{r2} \end{bmatrix}\right]$$

has the same class as the product of the classes of A and B. Then $K'(A) \circ K'(B)$ is unique, up to multiplication by matrices of the form $\operatorname{diag}[U_1, U_2, ..., U_r]$, $U_i \in \mathcal{M}_{2\times 2}(R)$ invertible.

b) Relation to results of L. Gerstein and M. Newman

A multiplicative property for the invariant factors of module homomorphisms over a Dedekind domain has been proved in [3]. With the given notations it becomes:

Proposition 3. *If $A, B \in \mathcal{M}_n(R)$, R a Dedekind domain, and* $\det A$ *and* $\det B$ *are relatively prime non-zero ideals, then $K'(AB)$ corresponds to $K'(A) \circ K'(B)$, up to multiplication by matrices with invertible* 2 *by* 2 *matrices on the diagonal.* ∎

Let $S(A)$ denote the Smith canonical form of a matrix over a PID; it is seen that the previous proposition generalizes a well-known property for matrices over a PID:

Corollary. *If $A, B \in \mathcal{M}_n(S)$, S a PID, and* $\det A$ *and* $\det B$ *are relatively prime, then $S(AB)$ corresponds to $S(A)\,S(B)$, up to multiplication by matrices with invertible elements on the diagonal.* ∎

M. Newman proved the following result in the more particular case of matrices over a PID (see [9]):

Proposition 4. *Let $A \in \mathcal{M}_r(R)$ and $B \in \mathcal{M}_s(R)$, R a Dedekind domain, have Krull normal forms $K'(A)$ and $K'(B)$ as above (for $K'(B)$: change r into s).*

If $\det A$ *and* $\det B$ *are relatively prime non-zero ideals, and $r \leq s$, then*

$$K'\left(\begin{bmatrix} A & \\ & B \end{bmatrix}\right) = \operatorname{diag}\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} b_1 & 0 \\ \rho_1 b_1 & 0 \end{bmatrix}, \ldots, \right.$$
$$\left. \begin{bmatrix} b_{s-r} & 0 \\ \rho_{s-r} b_{s-r} & 0 \end{bmatrix}, \begin{bmatrix} c_1 & 0 \\ \sigma_1 c_1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} c_r & 0 \\ \sigma_r c_r & 0 \end{bmatrix}\right] ,$$

where

$$\begin{cases} (c_1, \sigma_1 c_1) = (a_1, \gamma_1 a_1).(b_{s-r+1}, \rho_{s-r+1} b_{s-r+1}), \\ \ldots, \\ (c_r, \sigma_r c_r) = (a_r, \gamma_r a_r).(b_r, \rho_r b_r), \end{cases} \qquad \sigma_1, \ldots, \sigma_r \in \mathbb{Q} .$$

Proof: Note that the classes of all the Krull forms that are used, are the one of R, and that since $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 1_s \end{bmatrix} \begin{bmatrix} 1_r & 0 \\ 0 & B \end{bmatrix}$ the result of Gerstein can be used. ∎

c) Moore–Penrose inverse of a matrix over a Dedekind domain

The notations and definitions about the Moore–Penrose inverse A^{ψ} of a matrix A, that are used in the following proposition, can be found in [4].

Proposition 5. *Let $A \in \mathcal{M}_{m\times n}(R)$, R a Dedekind domain, and suppose $*$ is an involution on the matrices over R. Then A will have a Moore–Penrose inverse with respect to the involution $*$*

if and only if

i) *There exist invertible matrices $P \in \mathcal{M}_m(R)$ and $Q \in \mathcal{M}_n(R)$ such that*

$$A = P \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix} Q, \quad \text{with } E^2 = E \in \mathcal{M}_{2\times 2}(R)\ ;$$

ii) *There exist matrices $N, N' \in \mathcal{M}_s(R)$ ($s = \inf\{m,n\}$) such that*

$$\left[\begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,s} N\right]^* = \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,s} N$$

and

$$\begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,s} N N' = \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,s}\ ;$$

iii) *and for which*

$$\alpha = N' N^* \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{m,s}^* P^* P \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{m,s} N + 1_s - N' \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,s} N\ ,$$

$$\beta = \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,n} Q\, Q^* \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,n}^* N' + 1_s - \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,s}$$

are invertible. In that case

$$A^{\psi} = Q^* \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,n}^* N' \beta^{-1} \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{s,s} N\, \alpha^{-1} N' N^* \begin{bmatrix} 1_{r-1} & & \\ & E & \\ & & 0 \end{bmatrix}_{m,s}^* P^*\ .$$

Proof: Cf. [4]. ∎

If $m = n$, one can find invertible matrices N and $N' = N^{-1}$ in the previous proposition.

It is possible to obtain a constructive method to reduce the matrix $\begin{bmatrix} A & & \\ & 0 & \\ & & 0 \end{bmatrix}$ to the form given above. For a large class of matrices and involutions this contains enough information in order to obtain A^{ψ}.

d) Example

Consider the matrices over the ring $R = \mathbf{Z}[\sqrt{-5}]$, with the symplectic involution: if $Z = (z_{ij})_{1\le i\le m,\,1\le j\le n}$, $z_{ij} \in R$, then $Z^* = ((-1)^{i+j}\,c(z_{ji}))_{1\le j\le n,\,1\le i\le m}$, where $c(z_{ji})$ denotes the conjugate of z_{ji}. Take

$$A = \begin{bmatrix} -18-6\sqrt{-5} & -9+9\sqrt{-5} & -11 \\ -2+4\sqrt{-5} & 9 & 2+2\sqrt{-5} \\ -6-2\sqrt{-5} & -3+3\sqrt{-5} & -4 \end{bmatrix} \in \mathcal{M}_{3\times 3}(R)\ .$$

Since the invariant factors are the unit ideal, A is von Neumann regular and can be made equivalent to an idempotent matrix. Invertible matrices can be constructed such that:

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & (1+\sqrt{-5})\,(3+\sqrt{-5}) \\ 0 & 3(-1+\sqrt{-5}) & -2(3+\sqrt{-5}) \end{bmatrix}\ .$$

Take

$$P = \begin{bmatrix} -11 & 0 & 3 & & \\ 2(1+\sqrt{-5}) & 1 & 0 & & \\ -4 & 0 & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & 0 & -(3-3\sqrt{-5}) & -3(1-\sqrt{-5})\,(3-\sqrt{-5}) & -27 \\ 0 & -3 & 0 & 28 & 3(1+\sqrt{-5})\,(3+\sqrt{-5}) \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3(1-\sqrt{-5}) & 2(3+\sqrt{-5}) \\ 0 & 1 & 0 & -9 & -(1+\sqrt{-5})\,(3+\sqrt{-5}) \end{bmatrix},$$

then:

$$\begin{bmatrix} A & & \\ & 0 & \\ & & 0 \end{bmatrix} = P \begin{bmatrix} 1 & & & & \\ & -27 & -14(1+\sqrt{-5}) & & \\ & 9(1-\sqrt{-5}) & 28 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} Q$$

$$= P \begin{bmatrix} 1 & & & \\ & E & & \\ & & 0 & \\ & & & 0 \end{bmatrix} Q \quad \text{with } \; E^2 = E \; \text{ 2 by 2.}$$

If we want to check if A has a Moore–Penrose inverse with respect to the symplectic involution, then we could start by looking for an invertible matrix N such that $\begin{bmatrix} 1 & & & \\ & E & & \\ & & 0 & \\ & & & 2 \end{bmatrix} N$ is symmetric. A symmetric 2×2 matrix for the symplectic involution is of the form $\begin{bmatrix} z_1 & t \\ -c(t) & z_2 \end{bmatrix}$ with $z_1, z_2 \in \mathbf{Z}$, $t \in R$.

If such a matrix is of the same class as E, then $-c(t) = \gamma_r z_1$, $\gamma_r t = z_2$ so $-c(\gamma_r)\gamma_r z_1 = z_2$. Since $\gamma_r = -\frac{1-\sqrt{-5}}{3}$, $z_1 = 3z$, $z_2 = -2z$, $t = -\frac{3}{1-\sqrt{-5}}(-2)z = (1+\sqrt{-5})z$ for some $z \in \mathbf{Z}$. Comparing invariant factor ideals, we take $z = 1$. So, we try find N such that

$$\begin{bmatrix} 1 & & \\ & E & \\ & & 0 \end{bmatrix} N = \begin{bmatrix} 1 & & & & \\ & 3 & 1+\sqrt{-5} & & \\ & -1+\sqrt{-5} & -2 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}.$$

Using Krull's methods, it is found that

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1+\sqrt{-5} & 14(1+\sqrt{-5}) & 28 \\ 0 & -(1-\sqrt{-5}) & -2 & -27 & -9(1-\sqrt{-5}) \\ 0 & 0 & 1 & 14 & 0 \\ 0 & 1 & 0 & 0 & 9 \end{bmatrix}.$$

With the same notations as in the proposition, we consider

$$\alpha = \begin{bmatrix} 113 & -279(1-\sqrt{-5}) & -868 & 0 & 0 \\ -31(1+\sqrt{-5}) & 487 & 252(1+\sqrt{-5}) & 0 & 0 \\ 62 & -162(1-\sqrt{-5}) & -503 & 0 & 0 \\ 0 & \dots & & 1 & 0 \\ 0 & \dots & & 0 & 1 \end{bmatrix},$$

$$\beta = \begin{bmatrix} 1 & & & & \\ & 55 & 28(1+\sqrt{-5}) & & \\ & -18(1-\sqrt{-5}) & -5 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

The matrices α and β are invertible, and thus $\begin{bmatrix} A & \\ & 0_{2,2} \end{bmatrix}^{\psi}$ can be calculated by

$$\begin{bmatrix} A & \\ & 0_{2,2} \end{bmatrix}^{\psi} = Q^* \begin{bmatrix} 1 & & \\ & E & \\ & & 0_{2,2} \end{bmatrix}^* N^{-1}\,\beta^{-1}\,\alpha^{-1}\,N^* \begin{bmatrix} 1 & & \\ & E & \\ & & 0_{2,2} \end{bmatrix}^* P^* \,.$$

But in this case $\begin{bmatrix} A & \\ & 0_{2,2} \end{bmatrix}^{\psi} = \begin{bmatrix} A^{\psi} & \\ & 0_{2,2} \end{bmatrix}$ whence

$$A^{\psi} = \begin{bmatrix} -4(3-\sqrt{-5}) & -11(1-\sqrt{-5})\,(3-\sqrt{-5}) & -22(3-\sqrt{-5}) \\ 6(1+\sqrt{-5}) & 99 & 33(1+\sqrt{-5}) \\ 1 & 3(1-\sqrt{-5}) & 6 \end{bmatrix}.$$

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