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ON ATOMISTIC LATTICES

Andrzej Walendziak

Abstract: Any atomistic lattice is strong whereas the converse does not hold, in general. In the present paper we prove that, if in a strong atomic J-lattice, each atom has a complement, then the lattice is atomistic. This result generalizes the theorem of [2].

1 – Basic notions

Let L be a lattice. If L contains a least or a greatest element, these elements will be denoted by 0 or 1, respectively. An element $u \in L$ is called join-irreducible iff, for all $a, b \in L$, $u = a \vee b$ implies $u = a$ or $u = b$. Denote by $J(L)$ the set of all join-irreducible elements of L. We write $a \prec b$ ($a, b \in L$) if $a < b$ and if $a \le c < b$ implies $c = a$ for all c. An element $p \in L$ is called an atom if $0 \prec p$. By $[a, b]$ ($a \le b$, $a, b \in L$) we denote an interval, that is the set of all $c \in L$ for which $a \le c \le b$.

A lattice L is called atomic, if L has a least element and the interval $[0, a]$ contains an atom for each $a > 0$. If for any $a, b \in L$ with $a < b$ there is an element $p \in [a, b]$ such that $a \prec p$, then we say that L is strongly atomic. A lattice L is called atomistic if every element of L is a join of atoms. If each element of a given lattice L is a join of elements of $J(L)$, then we shall call L a J-lattice. Note that any atomistic lattice is a J-lattice.

A complete lattice L is called upper continuous iff, for every $a \in L$ and for every chain $C \subseteq L$,

$$a \wedge \bigvee C = \bigvee (a \wedge c\colon\ c \in C)\ .$$

The dual of an upper continuous lattice is called lower continuous.

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For our investigations we also need the concept of a strong lattice. For lattices of finite length the definition of strongness is given by Stern [2] by a property

(S) $\qquad u \in J(L) - \{0\},\ \ a \in L \quad \text{and} \quad u \le a \vee u' \quad \text{imply} \quad u \le a\ ,$

where u' denotes the unique lower cover of u.

We extend the notion of strongness from lattices of finite length to arbitrary lattices. Namely, we say that a lattice L is strong if the following condition is satisfied:

$(S)'$ $\qquad u \in J(L) - \{0\},\ \ a, b \in L \quad \text{and} \quad b < u \le a \vee b \quad \text{imply} \quad u \le a\ .$

It is easy to see that in lattices of finite length properties (S) and (S') are equivalent.

We remark that any atomistic lattice is strong. (Indeed, each nonzero join-irreducible element of an atomistic lattice is an atom.)

2 – Results

The first major result is

Theorem 1. *Let L be an atomic J-lattice. If each atom of L has a complement, then L is strong iff L is atomistic.*

Proof: Assume that the lattice L is strong. We show that L is atomistic. Let a be a nonzero element of L. Since L is a J-lattice, $a = \bigvee(u\colon\ u \in U \le J(L))$. Suppose that a join-irreducible element $u \in U$ is not an atom. Since L is atomic, there exists an atom $p \in L$ such that $p < u$. Let $\overline{p}$ be a complement of p. This means that $1 = p \vee \overline{p}$ and $0 = p \wedge \overline{p}$. Then $p < u \le \overline{p}$ and strongness implies $u \le \overline{p}$. Thus we have $p < \overline{p}$, which contradicts the fact that $p \wedge \overline{p} = 0$. It follows that L is atomistic. The converse is clear. ∎

Now we prove the following

Theorem 2. *A lower continuous strongly atomic lattice in which each atom has a complement is atomistic iff it is strong.*

Proof: Observe that if a lattice L is lower continuous and strongly atomic, then L is a J-lattice. Indeed, let $a \in L$ and $b := \bigvee(u \in J(L)\colon\ u \le a)$. Assume that $b < a$. Since L is strongly atomic there exists an element $p \in L$ such that $b \prec p \le a$. Consider the set $T := \{t \in L\colon\ b \vee t = p\}$. T is nonempty, since $p \in T$. Let C be a chain in T. The lower continuity yields

$$b \vee \bigwedge C = \bigwedge(b \vee c\colon\ c \in C) = p\ .$$

Thus $\bigwedge C \in T$ and by the dual of Zorn's lemma T contains a minimal element v. Clearly, $v \in J(L)$ and $v \leq a$. Consequently, $v \leq b$, and hence $p = b \vee v = b$, a contradiction. Thus $a = \bigvee(u \in J(L)\colon\ u \leq a)$, which shows that L is a J-lattice. Now the assertion follows from Theorem 1. ∎

We recall that a lattice L satisfies the descending chain condition (DCC) if each nonempty subset of L contains a minimal element. It is obvious that any lattice satisfying the DCC is lower continous and strongly atomic. Therefore, we obtain the following

Corollary 1. *Let a lattice L satisfy the DCC and let each atom of L have a complement. Then L is atomistic iff it is strong.*

Remark 1. Since every lattice of finite length satisfies the DCC, this corollary implies the theorem of [2].

We know (see [1], Theorem 4.1) that every upper continuous, semimodular, atomistic lattice is relatively complemented. This together with Corollary 1 yields

Corollary 2. *Let L be a semimodular, upper continuous, strong lattice with DCC. If each atom of L has a complement, then L is relatively complemented.*

Remark 2. Corollary 2 generalizes the corollary of [2].

REFERENCES

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Andrzej Walendziak,
Department of Mathematics, Agricultural and Pedagogical University,
PL-08110 Siedlce – POLAND