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A PROBLEM OF DIOPHANTOS–FERMAT AND CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

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Abstract: One shows that, if $(U_n)_{n\geq 0}$ is the sequence of Chebyshev polynomials of the second kind, then the product of any two distinct elements of the set

$$\Big\{U_m, U_{m+2r}, U_{m+4r}; \, 4\cdot U_{m+r}\cdot U_{m+3r}\Big\}\,, \qquad m,r\in\mathbb{N}\,,$$

increased by $U_a^2\cdot U_b^2$, for suitable positive integers a and b, is a perfect square.

This generalizes a result obtained by José Morgado in [4].

1 – Introduction

Chebyshev polynomials $(U_n)_{n\geq 0}$ are defined by the recurrence relation

$$U_{n+1}(x) = 2\cdot x\cdot U_n(x) - U_{n-1}(x)\,, \qquad (\forall)\, x\in\mathbb{C}, \quad (\forall)\, n\in\mathbb{N}^*\,, \tag{1.1}$$

where $U_0(x)=1$ and $U_1(x)=2x$.

An important property of these polynomials is given by the formula

$$U_{k-1}(\cos\varphi) = \frac{\sin k\varphi}{\sin\varphi}\,, \qquad (\forall)\,\varphi\in\mathbb{C}, \quad \sin\varphi\neq 0, \quad (\forall)\, k\in\mathbb{N}^*\,. \tag{1.2}$$

Also one has the relations

$$U_k\left(\frac{i}{2}\right) = i^k\cdot F_{k+1}\,, \qquad (\forall)\, k\in\mathbb{N}\,, \tag{1.3}$$

where $i^2=-1$ and $(F_n)_{n\geq 0}$ is the sequence of so-called Fibonacci numbers:

$$F_{n+1} = F_n + F_{n-1}\,, \qquad (\forall) n\in\mathbb{N}^*, \quad F_0=0, \quad F_1=1\,. \tag{1.4}$$

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We are going to prove the following

Theorem. *If $(U_n)_{n\geq 0}$ is the sequence of Chebyshev polynomials of the second kind, then the product of any two distinct elements of the set*

$$\Big\{U_m, U_{m+2r}, U_{m+4r};\ 4\cdot U_{m+r}\cdot U_{m+2r}\cdot U_{m+3r}\Big\}\,, \qquad m,r\in\mathbb{N}\ ,$$

increased by $U_a^2\cdot U_b^2$ for suitable positive integers a and b, is a perfect square, $(\forall)\, m,r\in\mathbb{N}$, $(\forall)\, x\in\mathbb{C}$.

Proof: One has the identity

$$U_m\cdot U_{m+r+s}+U_{r-1}\cdot U_{s-1}=U_{m+r}\cdot U_{m+s}\,, \quad (\forall)\, x\in\mathbb{C},\ \ (\forall) m,r,s\in\mathbb{N}^*\ . \tag{2.1}$$

Indeed, let x be an element of $\mathbb{C}$; then $(\exists)\,\varphi\in\mathbb{C}$ such that $x=\cos\varphi$. One has

$$\begin{aligned}
&U_m(x)\cdot U_{m+r+s}(x)+U_{r-1}(x)\cdot U_{s-1}(x)=\\
&=U_m(\cos\varphi)\cdot U_{m+r+s}(\cos\varphi)+U_{r-1}(\cos\varphi)\cdot U_{s-1}(\cos\varphi)\\
&=\frac{\sin(m+1)\,\varphi}{\sin\varphi}\cdot\frac{\sin(m+r+s+1)}{\sin\varphi}+\frac{\sin r\varphi}{\sin\varphi}\cdot\frac{\sin s\varphi}{\sin\varphi}\\
&=\frac{\Big[\cos(r+s)\,\varphi-\cos(2m+r+s+2)\,\varphi\Big]+\Big[\cos(r-s)\,\varphi-\cos(r+s)\,\varphi\Big]}{2\cdot\sin^2\varphi}\\
&=\frac{\cos(r-s)\,\varphi-\cos(2m+r+s+2)\,\varphi}{2\cdot\sin^2\varphi}=\frac{\sin(m+r+1)\,\varphi}{\sin\varphi}\cdot\frac{\sin(m+s+1)\,\varphi}{\sin\varphi}\\
&=U_{m+r}(\cos\varphi)\cdot U_{m+s}(\cos\varphi)=U_{m+r}(x)\cdot U_{m+s}(x)\ ,\ \text{q.e.d.}
\end{aligned}$$

By setting $s=r$ in (2.1), one obtains

$$U_m\cdot U_{m+2r}+U_{r-1}^2=U_{m+r}^2\,, \qquad m,r\in\mathbb{N}^*\ , \tag{2.2}$$

which proves a part of the Theorem above, with $a=r-1$, $b=0$.

Now, let us replace, in (2.2), m by $m+2r$; then

$$U_{m+2r}\cdot U_{m+4r}+U_{r-1}^2=U_{m+3r}^2\ , \tag{2.3}$$

which proves also a part of the Theorem above, with $a=r-1$, $b=0$.

By replacing, in (2.2), r by $2r$, it results

$$U_m \cdot U_{m+4r} + U_{2r-1}^2 = U_{m+2r}^2 \quad (a = 2r-1, \ \ b = 0) \ . \tag{2.4}$$

From the identity (2.1), it follows

$$U_{r-1}^2 \cdot U_{s-1}^2 = (U_{m+r} \cdot U_{m+s} - U_m \cdot U_{m+r+s})^2$$

and so

$$\begin{aligned} 4 \cdot U_m \cdot U_{m+r} \cdot U_{m+s} \cdot U_{m+r+s} + U_{r-1}^2 \cdot U_{s-1}^2 &= \\ = (U_{m+r} \cdot U_{m+s} + U_m \cdot U_{m+r+s})^2 \ , &\quad m, r, s \in \mathbb{N}^* \ . \end{aligned} \tag{2.5}$$

If, in (2.5), one sets $s = 2r$, one obtains

$$\begin{aligned} 4 \cdot U_m \cdot U_{m+r} \cdot U_{m+2r} \cdot U_{m+3r} + U_{r-1}^2 \cdot U_{2r-1}^2 &= \\ = (U_{m+r} \cdot U_{m+2r} + U_m \cdot U_{m+3r})^2 &\quad (a = r-1, \ \ b = 2r-1) \ . \end{aligned} \tag{2.6}$$

By replacing m by $m + r$, in (2.6), it follows

$$\begin{aligned} 4 \cdot U_{m+r} \cdot U_{m+2r} \cdot U_{m+3r} \cdot U_{m+4r} + U_{r-1}^2 \cdot U_{2r-1}^2 &= \\ = (U_{m+2r} \cdot U_{m+3r} + U_{m+r} \cdot U_{m+4r})^2 \ . \end{aligned} \tag{2.7}$$

Finally, from (2.5), it results, for $s = r$ and replacing m by $m + r$,

$$4 \cdot U_{m+r} \cdot U_{m+2r}^2 \cdot U_{m+3r} + U_{r-1}^4 = (U_{m+2r}^2 + U_{m+r} \cdot U_{m+3r})^2 \quad (a = b = r-1) \ . \tag{2.8}$$

The relations (2.2)–(2.8) show that the Theorem holds. ∎

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According to (1.3), from the relations (2.2)–(2.8) one obtains the following identities for Fibonacci numbers:

$$F_m \cdot F_{m+2r} + (-1)^m \cdot F_r^2 = F_{m+r}^2 \ , \tag{3.1}$$

$$F_{m+2r} \cdot F_{m+4r} + (-1)^m \cdot F_r^2 = F_{m+3r}^2 \ , \tag{3.2}$$

$$F_m \cdot F_{m+4r} + (-1)^m \cdot F_{2r}^2 = F_{m+2r}^2 \ , \tag{3.3}$$

$$\begin{aligned} 4 \cdot F_m \cdot F_{m+r} \cdot F_{m+2r} \cdot F_{m+3r} + F_r^2 \cdot F_{2r}^2 &= \\ = (F_{m+r} \cdot F_{m+2r} + F_m \cdot F_{m+3r})^2 \ , \end{aligned} \tag{3.4}$$

$$(3.5)\qquad 4\cdot F_{m+r}\cdot F_{m+2r}\cdot F_{m+3r}\cdot F_{m+4r}+F_r^2\cdot F_{2r}^2 = \\ = (F_{m+2r}\cdot F_{m+3r}+F_{m+r}\cdot F_{m+4r})^2\ ,$$

$$(3.6)\qquad 4\cdot F_{m+r}\cdot F_{m+2r}^2\cdot F_{m+3r}+F_r^4 = \\ = (F_{m+2r}^2+F_{m+r}\cdot F_{m+3r})^2\ , \qquad m,r\in\mathbb{N}\ .$$

These identities show that the product of any two distinct elements of the set

$$\Big\{F_m, F_{m+2r}, F_{m+4r};\ 4\cdot F_{m+r}\cdot F_{m+2r}\cdot F_{m+3r}\Big\}\ , \qquad m,r\in\mathbb{N}\ ,$$

increased by $F_a^2\cdot F_b^2$ or $-F_a^2\cdot F_b^2$, where F_a and F_b are suitable elements of the Fibonacci sequence $(F_n)_{n\geq 0}$, is a perfect square.

This remark is in fact the José Morgado's result given in [4].

Moreover, the José Morgado's result is also a generalization of some results of V.E. Hoggatt and E.G. Bergum, given in [1] and [2], about a problem of Diophantos–Fermat.

This problem claims to find four natural numbers such that the product of any two, added by unity, is a square.

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