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DIOPHANTINE QUADRUPLES FOR SQUARES OF FIBONACCI AND LUCAS NUMBERS

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Abstract: Let n be an integer. A set of positive integers is said to have the property $D(n)$ if the product of its any two distinct elements increased by n is a perfect square. In this paper, the sets of four numbers represented in terms of Fibonacci numbers with the property $D(F_n^2)$ and $D(L_n^2)$, where (F_n) is the Fibonacci sequence and (L_n) is the Lucas sequence, are constructed. Among other things, it is proved that the set

$$\Big\{2F_{n-1},\, 2F_{n+1},\, 2F_n^3F_{n+1}F_{n+2},\, 2F_{n+1}F_{n+2}F_{n+3}(2F_{n+1}^2-F_n^2)\Big\}$$

has the property $D(F_n^2)$ and that the sets

$$\Big\{2F_{n-2},\, 2F_{n+2},\, 2F_nL_{n-1}L_n^2L_{n+1},\, 10F_nL_{n-1}L_{n+1}[L_{n-1}L_{n+1}-(-1)^n]\Big\}\ ,$$

$$\Big\{F_{n-3}F_{n-2}F_{n+1},\, F_{n-1}F_{n+2}F_{n+3},\, F_nL_n^2,\, 4F_{n-1}^2F_nF_{n+1}^2(2F_{n-1}F_{n+1}-F_n^2)\Big\}$$

have the property $D(L_n^2)$.

1 – Introduction

Let (F_n) be the Fibonacci sequence. In [10] Morgado has showed that the product of any two distinct elements of the set

$$\Big\{F_n,\, F_{n+2r},\, F_{n+4r},\, 4F_{n+r}F_{n+2r}F_{n+3r}\Big\}$$

increased by $F_a^2F_b^2$ or $-F_a^2F_b^2$, for suitable positive integers a and b, is a perfect square.

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Let n be a positive integer. The aim of this paper is to find out four distinct positive integers, represented in terms of Fibonacci numbers, having the property that the product of any two of them increased by F_n^2 is a perfect square. It is natural to suppose that at least one of them is not divisible by F_n. Our starting point is the identity

$$4F_{n-1} \cdot F_{n+1} + F_n^2 = L_n^2 \ ,$$

where $L_n = F_{n-1} + F_{n+1}$ is n^{th} Lucas number.

Generally, we say that the set of positive integers $\{a_1, ..., a_m\}$ has the *property of Diophantus of order* n, in brief $D(n)$, if, forall $i, j = 1, ..., m$, $i \neq j$, the following holds: $a_i a_j + n = b_{ij}^2$, where b_{ij} is an integer. The set $\{a_1, ..., a_m\}$ is called Diophantine m-tuple. In [4] it is showed that a set $\{a, b\}$ with the property $D(e^2)$, $e \in \mathbf{Z}$, can be extended to the set $\{a, b, c, d\}$ with the same property, if ab is not a perfect square.

Let it be $ab + e^2 = k^2$. The manner of constructing is as follows: let s and t be a positive integer solution of the Pellian equation $S^2 - ab\,T^2 = 1$ (since ab is not a perfect square, s and t exist). Let us define two double sequences $y_{n,m}$ and $z_{n,m}$, $n, m \in \mathbf{Z}$, as follows:

$$y_{0,0} = e, \quad z_{0,0} = e, \quad y_{1,0} = k + a, \quad z_{1,0} = k + b \ ,$$

$$y_{-1,0} = k - a, \quad z_{-1,0} = k - b \ ,$$

$$y_{n+1,0} = \frac{2k}{e}\, y_{n,0} - y_{n-1,0}, \quad z_{n+1,0} = \frac{2k}{e}\, z_{n,0} - z_{n-1,0} \ , \qquad n \in \mathbf{Z} \ ,$$

$$y_{n,1} = s\, y_{n,0} + a\, t\, z_{n,0}, \quad z_{n,1} = b\, t\, y_{n,0} + s\, z_{n,0} \ , \qquad n \in \mathbf{Z} \ ,$$

$$y_{n,m+1} = 2\, s\, y_{n,m} - y_{n,m-1}, \quad z_{n,m+1} = 2\, s\, z_{n,m} - z_{n,m-1} \ , \qquad n, m \in \mathbf{Z} \ .$$

Let us set $x_{n,m} = (y_{n,m}^2 - e^2)/a$. According to [4, Theorem 2], if $x_{n,m}$ and $x_{n+1,m}$ are positive integers, then the set $\{a, b, x_{n,m}, x_{n+1,m}\}$ has the property $D(e^2)$. It is also proved that the sets $\{a, b, x_{0,m}, x_{1,m}\}$, $m \in \mathbf{Z} \backslash \{-1, 0\}$, and $\{a, b, x_{-1,m}, x_{0,m}\}$, $m \in \mathbf{Z} \backslash \{0, 1\}$, have the property $D(e^2)$. It is enough to find out one positive integer solution of the Pellian equation $S^2 - abT^2 = 1$ to extend a set $\{a, b\}$ with the property $D(e^2)$ to a set $\{a, b, c, d\}$ with the same property.

2 – Quadruples with the property $D(F_n^2)$

For any positive integer n, it holds

$$4\, F_{n-1}\, F_{n+1} + F_n^2 = L_n^2 \ . \tag{1}$$

Indeed, $L_n^2 - F_n^2 = (F_{n-1} + F_{n+1} - F_n)\,(F_{n-1} + F_{n+1} + F_n) = 2F_{n-1} \cdot 2F_{n+1} = 4F_{n-1}F_{n+1}$. Therefore, the sets $\{2F_{n-1}, 2F_{n+1}\}$, $\{F_{n-1}, 4F_{n+1}\}$, $\{4F_{n-1}, F_{n+1}\}$ have the property $D(F_n^2)$. In order to extend these sets to quadruples with the property $D(F_n^2)$ by applying the construction described in the introduction, we have to find a solution of Pellian equation $S^2 - 4F_{n-1}F_{n+1}T^2 = 1$. One solution can be found from the identity

$$4\,F_{n-1}\,F_n^2\,F_{n+1} + 1 = (F_n^2 + F_{n-1}\,F_{n+1})^2\ . \tag{2}$$

(see [10]). Hence, it can be put: $s = F_n^2 + F_{n-1}F_{n+1}$, $t = F_n$. In this way, we can get an infinite number of sets with the property $D(F_n^2)$. Particularly, the following theorem holds:

Theorem 1. *For all integers $n \geq 2$, the sets*

$$\Big\{2F_{n-1},\, 2F_{n+1},\, 2F_n^3F_{n+1}F_{n+2},\, 2F_{n+1}F_{n+2}F_{n+3}(2F_{n+1}^2 - F_n^2)\Big\}\ ,$$

$$\Big\{F_{n-1},\, 4F_{n+1},\, F_n^3F_{n+2}F_{n+3},\, F_{n+1}F_{n+2}F_{n+4}[F_{n+2}^2 + 2(-1)^n]\Big\}$$

and

$$\Big\{4F_{n-1},\, F_{n+1},\, F_n^3L_nL_{n+1},\, F_{n+1}F_{2n+4}[F_{2n+2} + 2(-1)^n]\Big\}$$

have the property $D(F_n^2)$.

For all integers $n \geq 3$, the sets

$$\Big\{2F_{n-1},\, 2F_{n+1},\, 2F_{n-2}F_{n-1}F_n^3,\, 2F_n^3F_{n+1}F_{n+2}\Big\}\ ,$$

$$\Big\{F_{n-1},\, 4F_{n+1},\, F_{n-2}F_{n-1}F_{n+1}(2F_n^2 - F_{n-1}^2),\, F_n^3F_{n+2}F_{n+3}\Big\}$$

and

$$\Big\{4F_{n-1},\, F_{n+1},\, F_{n-2}F_{2n-2}F_{2n-1},\, F_n^3L_nL_{n+1}\Big\}$$

have the property $D(F_n^2)$.

Proof: We will apply the construction described in the introduction. We are going to show that all the sets from the Theorem 1 are of the form $\{a, b, x_{0,1}, x_{1,1}\}$ or $\{a, b, x_{-1,1}, x_{0,1}\}$.

Looking at the equations (1) and (2), we see that $e = F_n$, $k = L_n$, $s = F_n^2 + F_{n-1}F_{n+1}$, $t = F_n$. In order to simplify, let us put: $F_n = v$ and $F_{n+1} = u$. Then, $u^2 - uv - v^2 = (-1)^n$, so that $(u^2 - uv - v^2)^2 = 1$ (see [12, p. 34]).

1) $a = 2F_{n-1}$, $b = 2F_{n+1}$.

Respectively, it holds: $y_{0,0} = z_{0,0} = F_n$, $y_{1,0} = 3F_{n-1} + F_{n+1}$, $z_{1,0} = F_{n-1} + 3F_{n+1}$, $y_{-1,0} = F_n$, $z_{-1,0} = -F_n$. Hence,

$$y_{0,1} = v(u^2 + vu - v^2) \ ,$$
$$y_{1,1} = 4u^3 + vu^2 - 3v^2u - v^3 \ ,$$
$$y_{-1,1} = v(u^2 - 3vu + 3v^2) \ ,$$

so that

$$x_{0,1} = \left[y_{0,1}^2 - v^2(u^2 - vu - v^2)^2\right]/2(u-v) = 2v^3u(v+u) = 2F_n^3F_{n+1}F_{n+2} \ ,$$
$$x_{1,1} = 2u(v+u)\,(v+2u)\,(2u^2 - v^2) = 2F_{n+1}F_{n+2}F_{n+3}(2F_{n+1}^2 - F_n^2) \ ,$$
$$x_{-1,1} = 2v^3(u-v)\,(2v-u) = 2F_{n-2}F_{n-1}F_n^3 \ .$$

2) $a = F_{n-1}$, $b = 4F_{n+1}$.

In this case, it holds: $y_{0,0} = z_{0,0} = F_n$, $y_{1,0} = 2F_{n-1} + F_{n+1}$, $z_{1,0} = F_{n-1} + 5F_{n+1}$, $y_{-1,0} = F_{n+1}$, $z_{-1,0} = -F_{n+3}$, so that

$$y_{0,1} = vu^2 \ ,$$
$$y_{1,1} = 3u^3 + vu^2 - 2v^2u - v^3 \ ,$$
$$y_{-1,1} = u^3 - 3vu^2 + 2v^2u + v^3$$

and

$$x_{0,1} = v^3(v+u)\,(v+2u) = F_n^3F_{n+2}F_{n+3} \ ,$$
$$x_{1,1} = u(v+u)\,(2v+3u)\,(3u^2 - v^2) = F_{n+1}F_{n+2}F_{n+4}[F_{n+2}^2 + 2(-1)^n] \ ,$$
$$x_{-1,1} = u(u-v)\,(2v-u)\,(v^2 + 2vu - u^2) = F_{n-2}F_{n-1}F_{n+1}(2F_n^2 - F_{n-1}^2) \ .$$

3) $a = 4F_{n-1}$, $b = F_{n+1}$.

Hence, $y_{0,0} = z_{0,0} = F_n$, $y_{1,0} = 5F_{n-1} + F_{n+1}$, $z_{1,0} = F_{n-1} + 2F_{n+1}$, $y_{-1,0} = -F_{n-3}$, $z_{-1,0} = F_{n-1}$,

$$y_{0,1} = v(u^2 + 3vu - 3v^2) \ ,$$
$$y_{1,1} = 6u^3 + vu^2 - 5v^2u - v^3 \ ,$$
$$y_{-1,1} = 7v^3 - 13v^2u + 9vu^2 - 2u^3 \ ,$$

and, finally,

$$x_{0,1} = v^3(2u-v)\,(2v+u) = F_n^3 L_n L_{n+1} \ ,$$
$$x_{1,1} = u(v+u)\,(v+3u)\,(3u^2-2v^2) = F_{n+1}F_{2n+4}[F_{2n+2}+2(-1)^n] \ ,$$
$$x_{-1,1} = (u-v)\,(2v-u)\,(3v-u)\,(2v^2-2vu+u^2) = F_{n-2}F_{2n-2}F_{2n-1} \ . \blacksquare$$

Theorem 1 may also be proved directly. For example, the following equations hold:

$$F_{n-1}\cdot 4F_{n+1} + F_n^2 = L_n^2 \ ,$$
$$F_{n-1}\cdot F_n^3 F_{n+2}F_{n+3} + F_n^2 = (F_n F_{n+1}^2)^2 \ ,$$
$$F_{n-1}\cdot F_{n+1}F_{n+2}F_{n+4}[F_{n+2}^2+2(-1)^n] + F_n^2 = [F_{n+1}F_{n+2}^2+(-1)^n F_{n+3}]^2 \ ,$$
$$4F_{n+1}\cdot F_n^3 F_{n+2}F_{n+3} + F_n^2 = \{F_n[2F_{n+1}F_{n+2}-(-1)^n]\}^2 \ ,$$
$$4F_{n+1}\cdot F_{n+1}F_{n+2}F_{n+4}[F_{n+2}^2+2(-1)^n] + F_n^2 = \{F_{n+3}[2F_{n+1}F_{n+2}+(-1)^n]\}^2 \ ,$$
$$F_n^3 F_{n+2}F_{n+3}\cdot F_{n+1}F_{n+2}F_{n+4}[F_{n+2}^2+2(-1)^n] + F_n^2 = \\ = \{F_n[F_{n+2}^4+(-1)^n F_{n+2}^2-1]\}^2 \ .$$

3 – Quadruples with the property $D(L_n^2)$

For any integer n, $n \geq 2$, the following holds

$$4F_{n-2}F_{n+2} + L_n^2 = 9F_n^2 \ .$$

Indeed, $9F_n^2 - L_n^2 = (3F_n - L_n)\,(3F_n + L_n) = 2(F_n - F_{n-1})\cdot 2(F_n + F_{n+1}) = 4F_{n-2}F_{n+2}$. By means of the identity

$$4F_{n-2}F_n^2F_{n+2} + 1 = (F_n^2 + F_{n-2}F_{n+2})^2$$

(see [10]) and the construction described in the introduction, the following theorem can be proved in the same way as it is done in the Theorem 1:

Theorem 2. *For any integer $n \geq 3$, the following sets have the property $D(L_n^2)$:*

$$\Big\{2F_{n-2},\ 2F_{n+2},\ 2F_nL_{n-1}L_n^2L_{n+1},\ 10F_nL_{n-1}L_{n+1}[L_{n-1}L_{n+1}-(-1)^n]\Big\} \ ,$$

$$\Big\{2F_{n-2},\, 2F_{n+2},\, 2F_{n-1}F_nF_{n+1}L_n^2,\, 2F_nL_{n-1}L_n^2L_{n+1}\Big\}\ ,$$

$$\Big\{F_{n-2},\, 4F_{n+2},\, F_nL_n^2(2F_n+F_{n+2})\,(F_n+2F_{n+2}),\\ L_{n-1}L_{n+1}(L_{n-1}+2L_{n+1})\,[L_n(F_{n-1}+2F_{n+1})-9(-1)^n]\Big\}\ ,$$

$$\Big\{F_{n-2},\, 4F_{n+2},\, F_{n-1}F_{n+1}(F_{n-1}+2F_{n+1})\,(F_{n+2}^2-3F_n^2),\\ F_nL_n^2(2F_n+F_{n+2})\,(F_n+2F_{n+2})\Big\}\ ,$$

$$\Big\{4F_{n-2},\, F_{n+2},\, F_nL_n^2(2F_{n-2}+F_n)\,(2F_n+F_{n+2}),\\ L_{n-1}L_{n+1}(2L_{n-1}+L_{n+1})\,[L_n(2F_{n-1}+F_{n+1})-9(-1)^n]\Big\}$$

and

$$\Big\{4F_{n-2},\, F_{n+2},\, F_{n-1}F_{n+1}(2F_{n-1}+F_{n+1})\,(3F_n^2-F_{n-2}^2),\\ F_nL_n^2(2F_{n-2}+F_n)\,(2F_n+F_{n+2})\Big\}\ .$$

There exists a direct way of proving the Theorem 2, too. For example:

$$2F_{n-2}\cdot 2F_{n+2}+L_n^2=(3F_n)^2\ ,$$
$$2F_{n-2}\cdot 2F_nL_{n-1}L_n^2L_{n+1}+L_n^2=\{L_n[2F_nL_{n-1}-(-1)^n]\}^2\ ,$$
$$2F_{n-2}\cdot 10F_nL_{n-1}L_{n+1}[L_{n-1}L_{n+1}-(-1)^n]+L_n^2=[2L_{n-1}^2L_{n+1}-5(-1)^nF_n]^2\ ,$$
$$2F_{n+2}\cdot 2F_nL_{n-1}L_n^2L_{n+1}+L_n^2=\{L_n[2F_{2n+1}-3(-1)^n]\}^2\ ,$$
$$2F_{n+2}\cdot 10F_nL_{n-1}L_{n+1}[L_{n-1}L_{n+1}-(-1)^n]+L_n^2=[2L_{n-1}L_{n+1}^2-5(-1)^nF_n]^2\ ,$$
$$2F_nL_{n-1}L_n^2L_{n+1}\cdot 10F_nL_{n-1}L_{n+1}[L_{n-1}L_{n+1}-(-1)^n]+L_n^2=\\ =[L_n(2L_{n-1}^2L_{n+1}^2-1)]^2\ .$$

4 – Morgado identity

Morgado has proved the following identity in [11]:

$$F_nF_{n+1}F_{n+2}F_{n+4}F_{n+5}F_{n+6}+L_{n+3}^2=\Big[F_{n+3}(2F_{n+2}F_{n+4}-F_{n+3}^2)\Big]^2\ .$$

Let us consider the problem of finding out the set $\{a,b,c,d\}$ with the property $D(L_{n+3}^2)$, such as $a\cdot b=F_nF_{n+1}F_{n+2}F_{n+4}F_{n+5}F_{n+6}$. We shall attempt to solve this problem by means of the construction described in the introduction. In this case, we are not going to use the Pellian equation $S^2 - abT^2 = 1$ but choose a and b so that we shall be able to get the solution of the problem only by considering the sequence $x_{n,0}$. As it is said in the introduction, if $x_{2,0}$ is a positive integer, then the set $\{a, b, x_{1,0}, x_{2,0}\}$ has the property $D(L_{n+3}^2)$. Hence, $y_{0,0} = e$, $y_{1,0} = k + a$, $y_{2,0} = \frac{2k}{e}(k+a) - e$ and

$$\begin{aligned} x_{2,0} &= \frac{y_{2,0}^2 - e^2}{a} = \frac{(y_{2,0} - e)\,(y_{2,0} + e)}{a} \\ &= \frac{\frac{2}{e}(k^2 + ak - e^2) \cdot \frac{2k}{e}(k+a)}{a} = \frac{4k(k+a)\,(k+b)}{e^2} \;. \end{aligned}$$

It will be shown that in our case numbers a and b can be chosen so that $x_{2,0} \in \mathbb{N}$ or $x_{-2,0} \in \mathbb{N}$. It holds:

Theorem 3. *Let n be a positive integer and $k_n = F_{n+3}(2F_{n+2}F_{n+4} - F_{n+3}^2)$. The the following sets have the property $D(L_{n+3}^2)$:*

$$\Big\{F_n F_{n+1} F_{n+2},\; F_{n+4} F_{n+5} F_{n+6},\; 4F_{n+3} L_{n+3}^2,\; 4k_n(4F_{n+3}\, k_n - 1)\Big\}\;,$$

$$\Big\{F_n F_{n+1} F_{n+4},\; F_{n+2} F_{n+5} F_{n+6},\; F_{n+3} L_{n+3}^2,\; 4k_n(F_{n+3}\, k_n + 1)\Big\}$$

and

$$\Big\{F_n F_{n+2} F_{n+5},\; F_{n+1} F_{n+4} F_{n+6},\; F_{n+3} L_{n+3}^2,\; 4k_n(F_{n+3}\, k_n - 1)\Big\}\;.$$

Proof: 1) $a = F_n F_{n+1} F_{n+2}$, $b = F_{n+4} F_{n+5} F_{n+6}$.

Then, $a + b = 6F_{n+3}(F_{n+2}^2 + F_{n+4}^2)$, so that

$$\begin{aligned} x_{1,0} &= (k^2 + 2ak + a^2 - e^2)/a = a + b + 2k \\ &= F_{n+3}\,(6F_{n+2}^2 + 6F_{n+4}^2 + 4F_{n+2}F_{n+4} - 2F_{n+2}^2 + 4F_{n+2}F_{n+4} - 2F_{n+4}^2) \\ &= 4F_{n+3} L_{n+3}^2\;, \end{aligned}$$

$$\begin{aligned} x_{2,0} &= \frac{4k}{L_{n+3}^2}\Big[k^2 + k(a+b) + ab\Big] \\ &= \frac{4k}{L_{n+3}^2}\,(k\, x_{1,0} - L_{n+3}^2) = 4k\,(4F_{n+3}\, k - 1)\;. \end{aligned}$$

2) $a = F_n F_{n+1} F_{n+4}$, $b = F_{n+2} F_{n+5} F_{n+6}$.

Hence, $a+b=F_{n+3}(10F_{n+2}F_{n+4}-F_{n+2}^2-F_{n+4}^2)$, so that

$$\begin{aligned} x_{-1,0} &= a+b-2k \\ &= F_{n+3}\,(10F_{n+2}F_{n+4}-F_{n+2}^2-F_{n+4}^2+2F_{n+2}^2-8F_{n+2}F_{n+4}+2F_{n+4}^2) \\ &= F_{n+3}L_{n+3}^2\ , \end{aligned}$$

$$\begin{aligned} x_{-2,0} &= -\frac{4k(a-k)\,(b-k)}{L_{n+3}^2} \\ &= \frac{4k}{L_{n+3}^2}\Big[k(a+b)-ab-k^2\Big] = \frac{4k}{L_{n+3}^2}\,(k\,x_{-1,0}+L_{n+3}^2) \\ &= 4k\,(F_{n+3}\,k+1)\ . \end{aligned}$$

3) $a=F_nF_{n+2}F_{n+5}$, $b=F_{n+1}F_{n+4}F_{n+6}$.
Now, $a+b=3F_{n+3}^3$, so that

$$\begin{aligned} x_{1,0} &= a+b+2k \\ &= F_{n+3}\,(3F_{n+2}^2-6F_{n+2}F_{n+4}+3F_{n+4}^2-2F_{n+2}^2+8F_{n+2}F_{n+4}-2F_{n+4}^2) \\ &= F_{n+3}L_{n+3}^2\ , \end{aligned}$$

$$x_{2,0} = \frac{4k}{L_{n+3}^2}\,(k\,x_{1,0}-L_{n+3}^2) = 4k\,(F_{n+3}k-1)\ . \ \blacksquare$$

Remark 1. It can be shown that by using notation as in the Theorem 3, the following holds:

$$\begin{aligned} 4F_{n+3}\,k_n-1 &= (5F_{n+3}F_{n+4}+F_{n+2}^2)\,(5F_{n+2}F_{n+3}-F_{n+4}^2)\ , \\ F_{n+3}\,k_n+1 &= F_{n+2}^2F_{n+4}^2\ , \\ F_{n+3}\,k_n-1 &= (F_{n+5}^2-2F_{n+4}^2)\,(F_{n+4}^2-2F_{n+3}^2)\ . \end{aligned}$$

5 – Fibonacci number triples

Some integer solutions of Pythagorean equation $x^2+y^2=z^2$ can be obtained using Fibonacci numbers (in that case, Pythagorean triple x,y,z is called Fibonacci number triple). Namely, the following relation (see [7]) is valid:

$$(F_nF_{n+3})^2+(2F_{n+1}F_{n+2})^2=(F_n^2+2F_{n+1}F_{n+2})^2\ . \tag{3}$$

On the basis of this relation, another Diophantine quadruple can be obtained. Let $a = F_n^2$, $b = F_{n+3}^2$. The aim is to find out the numbers x and y of the kind that the set $\{a, b, x, y\}$ may have the property $D(4F_{n+1}^2 F_{n+2}^2)$. In this case, ab is a perfect square and the construction described in the introduction cannot be applied. However, the following holds:

Theorem 4. *The set* $\{F_n^2, F_{n+3}^2, 4F_{n-1}F_{n+2}^3(F_{n+3}^2 - 2F_{n+2}^2), 4F_{n+1}^3 F_{n+4}(F_{n+3}^2 - 2F_{n+2}^2)\}$ *has the property* $D(4F_{n+1}^2 F_{n+2}^2)$, *for all* $n \in \mathbb{N}$.

Proof:

$$F_n^2 \cdot F_{n+3}^2 + 4F_{n+1}^2 F_{n+2}^2 = (F_{n+1}^2 + F_{n+2}^2)^2 = (F_n^2 + 2F_{n+1}F_{n+2})^2 \ .$$

We are now going to prove this theorem appealing to the Gelin–Cesáro identity: $F_{n-2}F_{n-1}F_{n+1}F_{n+2} + 1 = F_n^4$ and to the Morgado identity: $F_nF_{n+2}F_{n+3}F_{n+5} + 1 = (F_{n+4}^2 - 2F_{n+3}^2)^2$ (see [9]):

$$\begin{aligned}
&F_n^2 \cdot 4F_{n-1}F_{n+2}^3(F_{n+3}^2 - 2F_{n+2}^2) + 4F_{n+1}^2 F_{n+2}^2 = \\
&\qquad = 4F_{n+2}^2 \Big\{ F_{n-1}F_n^2 F_{n+2} F_{n+3}^2 - 2F_{n-1}F_n^2 F_{n+2}^3 \\
&\qquad\quad + F_{n+1}^2 \Big[F_{n+1}^4 - F_{n-1}F_nF_{n+2}(2F_{n+1} + F_n) \Big] \Big\} \\
&\qquad = 4F_{n+2}^2 \Big[F_{n+1}^6 - 2F_{n-1}F_nF_{n+1}^3 F_{n+2} \\
&\qquad\quad + F_{n-1}F_n^2 F_{n+2}(F_{n+2}^2 + 2F_{n+1}F_{n+2} + F_{n+1}^2 - 2F_{n+2}^2 - F_{n+1}^2) \Big] \\
&\qquad = \Big[2F_{n+2}(F_{n+1}^3 - F_{n-1}F_nF_{n+2}) \Big]^2 \ ,
\end{aligned}$$

$$\begin{aligned}
&F_n^2 \cdot 4F_{n+1}^3 F_{n+4}(F_{n+3}^2 - 2F_{n+2}^2) + 4F_{n+1}^2 F_{n+2}^2 = \\
&\qquad = 4F_{n+1}^2 \Big\{ F_n^2 F_{n+1} F_{n+3}^2 F_{n+4} - 2F_n^2 F_{n+1} F_{n+2}^2 F_{n+4} \\
&\qquad\quad + F_{n+2}^2 \Big[F_{n+2}^4 - F_nF_{n+1}(2F_{n+2} - F_n)F_{n+4} \Big] \Big\} \\
&\qquad = 4F_{n+1}^2 \Big[F_{n+2}^6 - 2F_nF_{n+1}F_{n+3}^3 F_{n+4} \\
&\qquad\quad + F_n^2 F_{n+1}F_{n+4}(F_{n+3} - F_{n+2})\,(F_{n+3} + F_{n+2}) \Big] \\
&\qquad = \Big[2F_{n+1}(F_{n+2}^3 - F_nF_{n+1}F_{n+4}) \Big]^2 \ ,
\end{aligned}$$

$$F_{n+3}^2 \cdot 4F_{n-1}F_{n+2}^3(F_{n+3}^2 - 2F_{n+2}^2) + 4F_{n+1}^2F_{n+2}^2 =$$
$$= 4F_{n+2}^2\Big\{F_{n-1}F_{n+2}F_{n+3}^4 - 2F_{n-1}F_{n+2}^3F_{n+3}^2$$
$$+ F_{n+1}^2\Big[F_{n+1}^4 - F_{n-1}(F_{n+3} - 2F_{n+1})F_{n+2}F_{n+3}\Big]\Big\}$$
$$= 4F_{n+2}^2\Big[F_{n+1}^6 + 2F_{n-1}F_{n+1}^3F_{n+2}F_{n+3}$$
$$+ F_{n-1}F_{n+2}F_{n+3}^2(F_{n+2}^2 + 2F_{n+1}F_{n+2} + F_{n+1}^2 - 2F_{n+2}^2 - F_{n+1}^2)\Big]$$
$$= \Big[2F_{n+2}(F_{n+1}^3 + F_{n-1}F_{n+2}F_{n+3})\Big]^2 ,$$

$$F_{n+3}^2 \cdot 4F_{n+1}^3F_{n+4}(F_{n+3}^2 - 2F_{n+2}^2) + 4F_{n+1}^2F_{n+2}^2 =$$
$$= 4F_{n+1}^2\Big\{F_{n+1}F_{n+3}^4F_{n+4} - 2F_{n+1}F_{n+2}^2F_{n+3}^2F_{n+4}$$
$$+ F_{n+2}^2\Big[F_{n+2}^4 - (2F_{n+2} - F_{n+3})F_{n+1}F_{n+3}F_{n+4}\Big]\Big\}$$
$$= 4F_{n+1}^2\Big[F_{n+2}^6 - 2F_{n+1}F_{n+2}^3F_{n+3}F_{n+4}$$
$$+ F_{n+1}F_{n+3}^2F_{n+4}(F_{n+3} - F_{n+2})\,(F_{n+3} + F_{n+2})\Big]$$
$$= \Big[2F_{n+1}(F_{n+1}F_{n+3}F_{n+4} - F_{n+2}^3)\Big]^2 ,$$

$$4F_{n-1}F_{n+2}^3(F_{n+3}^2 - 2F_{n+2}^2) \cdot 4F_{n+1}^3F_{n+4}(F_{n+3}^2 - 2F_{n+2}^2) + 4F_{n+1}^2F_{n+2}^2 =$$
$$= 4F_{n+1}^2F_{n+2}^2\Big[4F_{n-1}F_{n+1}F_{n+2}F_{n+4}(F_{n+3}^2 - 2F_{n+2}^2) + 1\Big]$$
$$= 4F_{n+1}^2F_{n+2}^2\Big[4F_{n-1}F_{n+1}F_{n+2}F_{n+4}(4F_{n-1}F_{n+1}F_{n+2}F_{n+4} + 1) + 1\Big]$$
$$= \Big[2F_{n+1}F_{n+2}(2F_{n-1}F_{n+1}F_{n+2}F_{n+4} + 1)\Big]^2 . \blacksquare$$

6 – Diophantine quadruples for the products of Fibonacci numbers

Up to now, we have considered the sets with the property $D(n)$ where n was a square of an integer. Let us now consider how to obtain the sets with the property $D(n)$ in which n is not a erfect square using Fibonacci numbers. In this connection, let us adduce the result of Arkin and Bergum [1]: the set

$$\left\{\frac{F_{12p} - F_{12r}}{4},\ 9F_{12p} - F_{12r},\ \frac{25F_{12p} - 9F_{12r}}{16},\ \frac{49F_{12p} - F_{12r}}{16}\right\}$$

has the property $D(F_{12p}F_{12r})$. This result is the direct consequence of the fact that the set $\{4m,\ 144m+8,\ 25m+1,\ 49m+3\}$ has the property $D(16m+1)$. We are going to show that the similar result can be obtained when there is a set of the form $\{a_im+b_i\colon\ i=1,2,3,4\}$ with the property $D(am+b)$, where $a,b,a_i,b_i\in\mathbf{Z}$. The sets of this form have been considered in [4] and can be obtained e.g. from more general formulas: the sets

$$(4)\quad \Big\{m,\ (3l+1)^2m+2l,\ (3l+2)^2m+2l+2,\ 9(2l+1)^2m+8l+4\Big\}\ ,$$

$$(5)\quad \Big\{4m,\ (3l-1)^2m+l-1,\ (3l+1)^2m+l+1,\ 36l^2m+4l\Big\}\ ,$$

$$(6)\quad \Big\{m,\ l^2m-2l-2,\ (l+1)^2m-2l,\ (2l+1)^2m-8l-4\Big\}\ ,$$

$$(7)\quad \Big\{4m,\ (l-1)^2m+l-3,\ (l+1)^2+l+3,\ 4l^2m+4l\Big\}\ ,$$

$$(8)\quad \Big\{9m+4(3l-1),\ (3l-1)^2m+2(l-1)\,(6l^2-4l+1),\ (3l+1)^2m+2l(6l^2+2l-1),\ (6l-1)^2m+4l(2l-1)\,(6l-1)\Big\}\ ,$$

$$(9)\quad \Big\{m,\ (3l+1)^2m+2l(3l+1),\ (3l+2)^2m+2(3l^2+3l+1),\ 9(l+1)^2m+2(l+1)\,(3l+2)\Big\}\ ,$$

have the properties $D(2(2l+1)m+1)$, $D(8lm+1)$, $D(2(2l+1)m+1)$, $D(8lm+1)$, $D(2(6l-1)m+(4l-1)^2)$, $D(2(l+1)\,(3l+1)m+(2l+1)^2)$, respectively.

It is shown in [4] that Diophantine quadruple with the property $D(n)$ does not exist for an integer n, $n\equiv 2 \pmod 4$ (see also [2]). In the same paper, it is proved that if an integer n is: $n\not\equiv 2 \pmod 4$ and $n\notin S=\{3,5,8,12,20,-1,-3,-4\}$ then, there exists at least one Diophantine quadruple with the property $D(n)$ and if $n\notin S\cup t$, where $T=\{7,13,15,21,24,28,32,48,52,60,84,-7,-12,-15\}$ then there exist at least two different Diophantine quadruples with the property $D(n)$. However, number 52 can be omitted from the set T regarding the fact that the set $\{1,12,477,23052\}$ has the property $D(52)$.

Let us return to the consideration of the set $A=\{a_im+b_i\colon\ i=1,2,3,4\}$ with the property $D(am+b)$. By multiplying the elements of the set A by n and by substitution $mn\leftrightarrow m$, we get that the set $A'=\{a_im+b_in\colon\ i=1,2,3,4\}$ has the property $D(amn+bn^2)$. Let $l=am+bn$. We conclude that the set $A''=\{\frac{a_i(l-bn)}{a}+b_in\colon\ i=1,2,3,4\}$ has the property that the product of its any two distinct elements increased by ln is a square of a rational number. To insure that the elements of A'' are integers, we can proceed as follows.

For an integer a, let us assign the index of the least Fibonacci number divisible by a with $h(a)$. It is easy to show that $h(a)$ exists and that $h(a)\le a^2-1$ (see

[13, p. 27]). It can also be shown (see [3]) that $h(a) \le 2a$ and that $h(a) = 2a$ iff $a = 6 \cdot 5^q$, $q \ge 0$.

The conclusion is that the set

$$\left\{\frac{a_i\,F_{h(a)p} + (a\,b_i - a_i\,b)\,F_{h(a)r}}{a} :\ i = 1,2,3,4\right\}$$

has the property $D(F_{h(a)p}F_{h(a)r})$.

Example 1: Putting $l = 8$ in (4) we get the set $\{m, 625m + 16,\ 676m + 18,\ 2601m + 68\}$ with the property $D(34m + 1)$. Considering the fact that $F_9 = 34$, i.e., $h(34) = 9$ and applying the above construction, we get the set

$$\left\{\frac{F_{9p} - F_{9r}}{34}, \frac{625F_{9p} - 81F_{9r}}{34}, \frac{676F_{9p} - 64F_{9r}}{34}, \frac{2601F_{9p} - 289F_{9r}}{34}\right\} \tag{10}$$

with the property $D(F_{9p}F_{9r})$. Putting e.g. $p = 2$, $q = 1$ in (10) we get the set $\{75, 47419, 51312, 197387\}$ with the property $D(87856)$.

It is obvious that there exist formulas of a different type from the above (4)–(9). It is provable that the set

$$\left\{1,\ a^2 - 4,\ \frac{a}{4}(a^3 - 2a^2 - 3a + 8),\ \frac{a}{4}(a^3 + 2a^2 - 3a - 8)\right\} \tag{11}$$

has the property $D(4 - a^2)$. Putting $a = L_{2n}$ in (11) and using the fact that $L_{2n}^2 - 5F_{2n}^2 = 4$ (see [12, p. 29]), leads to the set

$$\left\{1,\ 5F_{2n}^2,\ \frac{L_{2n}}{4}(25F_n^4L_n^2 + L_{2n}),\ \frac{L_{2n}}{4}(5F_n^2L_n^4 + L_{2n})\right\}$$

with the property $D(-5F_{2n}^2)$.

7 – Concluding remarks

Remark 2. The question whether any of the Diophantine quadruples from Theorems 1, 2, 3 or 4 can be extended to the Diophantine quintuple is still in abeyance. The results from [6] imply that if the set $\{a, b, c, d\}$ with the property $D(n)$ is any of quadruples from Theorems 1, 2, 3 or 4, then there exists a rational number r so that the set $\{a, b, c, d, r\}$ has the property that the product of its any two distinct elements increased by n is the square of a rational number.

For example, using the fourth set from the Theorem 1, we can get that the product of any two distinct elements of the set

$$\Big\{2F_{n-1},\, 2F_{n+1},\, 2F_{n-2}F_{n-1}F_n^3,\, 2F_n^3F_{n+1}F_{n+2},\, \frac{4L_n(F_n^4-F_{n-1}^2F_{n-2}^2)\,(F_{n+1}^2F_{n+2}^2-F_n^4)\,(2F_n^4-1)}{(16F_{n-2}F_{n-1}^2F_n^2F_{n+1}^2F_{n+2}-1)^2}\Big\} \tag{12}$$

increased by F_n^2 is the square of a rational number. Putting $n=3$ and $n=4$ in (12) leads to the set {777480, 8288641, 24865923, 66309128, 994636920} with the property $D(2879^4)$ and the set {219604, 22108804, 55272010, 596937708, 11938754160} with the property $D(9\cdot 2351^4)$.

Remark 3. There is a natural problem of generalizing the results in this paper concerning the sequence defined by $w_n=w_n(a,b;p,q)$, $w_0=a$, $w_1=b$, $w_n=p\,w_{n-1}-q\,w_{n-2}$ $(n\geq 2)$; the sequence was considered by Horadam [8]. It is provable (see [5]) that for $u_n=w_n(0,1;p,-1)$ the sets $\{2u_{n-1},2u_{n+1}\}$, $\{u_{n-1},4u_{n+1}\}$, $\{4u_{n-1},u_{n+1}\}$ with the property $D(u_n^2)$ can be extended to the quadruples with the property $D(u_n^2)$ and that the analogue of Theorem 3 is valid for the sequence (u_n).

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