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A STATIONARY STEFAN PROBLEM WITH CONVECTION AND NONLINEAR DIFFUSION

José Miguel Urbano

Abstract: We consider a stationary two-phase Stefan problem with prescribed convection and prove existence of bounded solutions. The main features of this problem are a nonlinear constitutive law of diffusion involving the p-Laplacian and a discontinuous nonlinearity in the convection term due to the change of phase. The basic approach consists of using monotonicity techniques and an extended weak maximum principle.

1 – Introduction

The Stefan problem has been extensively studied by many authors and is still a fruitful area of research in mathematical physics. In general terms, it consists of determining a temperature field and the phase change boundaries in a pure material (see [4] or [6] for an introduction to the Stefan problem). In this work we study a stationary two-phase Stefan problem with prescribed convection from the point of view of the existence of solution.

We consider an incompressible material ocupying a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or $N = 3$), with two phases, a solid phase corresponding to a region Σ and a liquid phase corresponding to a region Λ. The two phases are separated by a solidification front Φ, which is *a priori* unknown, thus a free boundary, and bounded by the fixed boundary of the domain $\partial\Omega$. We restrict ourselves to the stationary case and will make use of the Boussinesq approximation, assuming a constant density which will be taken $\rho \equiv 1$, for simplicity. We also prescribe the velocity field $\mathbf{v}$, which must then satisfy the incompressibility condition $\nabla \cdot \mathbf{v} = 0$.

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The temperature of solidification at Φ is assumed to be constant and after renormalization we can consider it to be zero.

In this setting, the equation of conservation of energy, which is the appropriate conservation law to consider here, reduces to

$$\mathbf{v}\cdot\nabla e = r - \nabla\cdot\mathbf{q}\ , \tag{1}$$

where e is the specific internal energy, r a density of heat, including the dissipation effects, and $\mathbf{q}$ the heat flux.

Introducing the specific constitutive relations of the material, we relate the energy e and the temperature θ, by

$$e(\theta) = b(\theta) + \lambda\, h(\theta) \quad \text{for}\ \ \theta \neq 0\ , \tag{2}$$

where b is a given continuous function, $\lambda = [e]_-^+ > 0$ is the latent heat of phase transition, with $[\cdot]_-^+$ denoting the jump across the free boundary and h is the Heaviside function.

The relation between the heat flux and the temperature is described by a generalized Fourier law,

$$\mathbf{q} = -k\,\nabla\theta = -|\nabla\theta|^{p-2}\,\nabla\theta\,, \qquad 1 < p < \infty\ , \tag{3}$$

with $k = k(\nabla\theta)$ representing the thermal conductivity. For $p = 2$, (3) reduces to the usual Fourier law.

We then have that, in the solid region $\Sigma(\theta) = \{\theta < 0\}$ and the liquid region $\Lambda(\theta) = \{\theta > 0\}$, equation (1) takes the form

$$\mathbf{v}\cdot\nabla b(\theta) = r + \Delta_p\theta \quad \text{in}\ \ \Sigma(\theta)\cup\Lambda(\theta)\ , \tag{4}$$

which is the stationary heat equation with convection, where $\Delta_p\theta = \nabla\cdot(|\nabla\theta|^{p-2}\,\nabla\theta)$ is the p-Laplacian.

On the free boundary Φ, in addition to the condition $\theta = 0$, we have the Stefan condition, which represents the balance of heat fluxes

$$[\mathbf{q}]_-^+\cdot\mathbf{n} = \Big[-|\nabla\theta|^{p-2}\,\nabla\theta\Big]_-^+\cdot\mathbf{n} = -\lambda\mathbf{v}\cdot\mathbf{n} \quad \text{on}\ \ \Phi = \{\theta = 0\}\ , \tag{5}$$

where $\mathbf{n}$ is the unit normal to Φ, pointing to the liquid region.

This formulation of the problem can be generalized if we condensate (4) and (5) into the single equation

$$\mathbf{v}\cdot\nabla e(\theta) = r + \Delta_p\theta \quad \text{in}\ \ \mathcal{D}'(\Omega)\ ,$$

in which all references to the free boundary have disappeared. For details, see [6].

Finally, we must specify the boundary conditions for the temperature on $\partial\Omega$. We will consider $\partial\Omega$ divided into two components Γ_D and Γ_N, and take mixed boundary conditions:

$$\theta = \theta_D \quad \text{on } \Gamma_D , \tag{6}$$

which is a Dirichlet condition, and

$$-\mathbf{q}\cdot\mathbf{n} = \Big(|\nabla\theta|^{p-2}\,\nabla\theta\Big)\cdot\mathbf{n} = g(x,\theta) \quad \text{on } \Gamma_N , \tag{7}$$

that specifies the value of the conormal derivative, with $g(x,\theta)$ a given function on Γ_N and, here, $\mathbf{n}$ the unit outward normal to Γ_N.

For technical reasons, we shall assume that the velocity field satisfies an additional geometrical restriction:

$$\mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_N . \tag{8}$$

The paper is organized as follows. In section 2 we define a concept of weak solution, via the variational formulation, since classical solutions are not expected. The proof of the existence result is the object of section 4 where we approximate the problem, regularizing some data, and then pass to the limit to obtain the solution. We basically use monotonicity techniques.

Section 3 is devoted to the study of the existence and uniqueness for a class of convection-diffusion problems that includes the approximated problem as a particular case. We generalize for any $1<p<\infty$, the result presented in §5 of [5], corresponding to the case $p=2$. The existence theorem is based on an *a priori* estimate, classical results for monotone operators and the application of Schauder fixed point theorem. For the uniqueness, we combine the results of [2] for $p\geq 2$ with the results of [1] for $1<p\leq 2$.

2 – The variational formulation and the existence result

We will assume Ω to be a bounded domain in $\mathbb{R}^N$, with $\partial\Omega\in C^{0,1}$ such that $\partial\Omega=\overline{\Gamma_N}\cup\overline{\Gamma_D}$, with Γ_N and Γ_D relatively open in $\partial\Omega$ and $\int_{\Gamma_D} \mathrm{d}\sigma>0$, where σ represents the surface measure over $\partial\Omega$. We also take $1<p<\infty$.

Since classical solutions are not expected, we introduce a notion of weak solution through the variational formulation of the problem. We considered mixed boundary conditions, so the appropriate space of test functions is

$$V_p = \Big\{ \xi \in W^{1,p}(\Omega) \colon\; \xi = 0 \text{ on } \Gamma_D \Big\}$$

with the norm $\|\xi\|_{V_p} = \|\nabla\xi\|_{[L^p(\Omega)]^N}$. As a closed subspace of $W^{1,p}(\Omega)$, V_p is a reflexive Banach space for this norm, which is equivalent to the usual norm of $W^{1,p}(\Omega)$, due to the following extension of Poincaré inequality, valid for $1 \le p < \infty$:

$$\exists\, c_0 > 0 \colon \quad \|\xi\|_{L^p(\Omega)} \le c_0 \, \|\nabla\xi\|_{[L^p(\Omega)]^N} \,, \quad \forall\, \xi \in V_p \,. \tag{9}$$

Integrating formally by parts equation (4), with $\xi \in V_p$, and taking into account (5) and (7), we get

$$\begin{aligned} \int_\Omega |\nabla\theta|^{p-2}\, \nabla\theta \cdot \nabla\xi + \int_\Omega [\mathbf{v} \cdot \nabla b(\theta)]\, \xi - \int_\Omega r(x,\theta)\, \xi = {} & \\ = \int_\Phi \Big(\Big[|\nabla\theta|^{p-2}\, \nabla\theta \Big]_-^+ \cdot \mathbf{n} \Big)\, \xi + \int_{\Gamma_N} \Big(|\nabla\theta|^{p-2}\, \nabla\theta \cdot \mathbf{n} \Big)\, \xi & \\ = \int_\Phi (\lambda \mathbf{v} \cdot \mathbf{n})\, \xi + \int_{\Gamma_N} g(x,\theta)\, \xi \,. & \end{aligned} \tag{10}$$

Recalling that, by assumption, $\nabla \cdot \mathbf{v} = 0$ in Ω and $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_N, and denoting by $\chi_{\Lambda(\theta)}$ the characteristic function of the liquid zone, we can write

$$\int_\Omega \chi_{\Lambda(\theta)} (\mathbf{v} \cdot \nabla\xi) = \int_{\Lambda(\theta)} \nabla \cdot (\xi\, \mathbf{v}) = \int_{\partial\Lambda(\theta)} (\xi\, \mathbf{v}) \cdot \mathbf{n} = \int_\Phi (\mathbf{v} \cdot \mathbf{n})\, \xi \,,$$

and also,

$$\int_\Omega [\mathbf{v} \cdot \nabla b(\theta)]\, \xi = \int_\Omega [\nabla \cdot (b(\theta)\mathbf{v})]\, \xi = -\int_\Omega [b(\theta)\mathbf{v}] \cdot \nabla\xi \,.$$

Then (10) becomes, $\forall\, \xi \in V_p$,

$$\int_\Omega \Big\{ |\nabla\theta|^{p-2}\, \nabla\theta - [b(\theta) + \lambda \chi_{\Lambda(\theta)}]\, \mathbf{v} \Big\} \cdot \nabla\xi - \int_\Omega r(x,\theta)\, \xi = \int_{\Gamma_N} g(x,\theta)\, \xi \,. \tag{11}$$

As usually in the weak formulation for the Stefan problem, we replace the characteristic function $\chi_{\Lambda(\theta)}$, in the above equation, by a function χ which is in the maximal monotone graph $H(\theta)$, associated with the Heaviside function, i.e., $\chi = 1$ in $\Lambda(\theta) = \{\theta > 0\}$ and $\chi = 0$ in $\Sigma(\theta) = \{\theta < 0\}$. We allow, in this way, a possible mushy region at $\{\theta = 0\}$.

Definition 1. We say that (θ, χ) is a weak solution of (4)–(5), with boundary conditions (6)–(7), for $\Lambda(\theta) = \{x \in \Omega\colon\ \theta(x) > 0\}$ and $\Sigma(\theta) = \{x \in \Omega\colon\ \theta(x) < 0\}$, if

$$\theta \in W^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \qquad \theta = \theta_D \ \text{ on } \Gamma_D\ ; \tag{12}$$

$$\chi \in L^\infty(\Omega)\,, \qquad 0 \le \chi_{\Lambda(\theta)} \le \chi \le 1 - \chi_{\Sigma(\theta)} \le 1 \ \text{ a.e. in } \Omega\ ; \tag{13}$$

$$\int_\Omega \Big\{ |\nabla\theta|^{p-2}\, \nabla\theta - [b(\theta) + \lambda\,\chi]\,\mathbf{v} \Big\} \cdot \nabla\xi - \int_\Omega r(\theta)\,\xi = \int_{\Gamma_N} g(\theta)\,\xi\,, \qquad \forall\,\xi \in V_p\ . \tag{14}$$

Remark 1. The free boundary Φ is absent from this weak formulation but can be recovered *a posteriori* as the level set $\Phi = \{x \in \Omega\colon\ \theta(x) = 0\} = \partial\Lambda \cap \partial\Sigma$. This is a measurable subset of Ω, which is closed if $p > N$, since by Sobolev inclusion, θ is then a continuous function.

In order to obtain existence of solution we need the following set of assumptions on the functions and for the data:

(**A1**) $b\colon\ \mathbb{R} \to \mathbb{R}$ is a continuous function;

(**A2**) $r\colon\ \Omega \times \mathbb{R} \to \mathbb{R}$ is such that $r(x, \cdot)$ is continuous and decreasing for each $x \in \Omega$;

(**A3**) $r(\cdot, t) \in L^{p'}(\Omega)$, for each $t \in \mathbb{R}$;

(**A4**) $\exists\, M > 0\colon\ r(x, t)\, t \le 0$ for $|t| \ge M$, a.e. $x \in \Omega$;

(**A5**) $g\colon\ \Gamma_N \times \mathbb{R} \to \mathbb{R}$ is such that $g(x, \cdot)$ is continuous and decreasing for each $x \in \Gamma_N$;

(**A6**) $g(\cdot, t) \in L^{p'}(\Gamma_N)$, for each $t \in \mathbb{R}$;

(**A7**) $g(x, t)\, t \le 0$ for $|t| \ge M$, a.e. $x \in \Gamma_N$;

(**A8**) $\mathbf{v} \in [L^{p'}(\Omega)]^N$, $\displaystyle\int_\Omega \mathbf{v} \cdot \nabla\xi = 0$, $\forall\, \xi \in V_p$;

(**A9**) $\theta_D \in W^{1,p}(\Omega)$ and $\|\theta_D\|_{L^\infty(\Omega)} \le M$.

We note that (A8) is the weak form of the condition for the divergence free velocity vector field satisfying (8).

Theorem 1. *Under the assumptions (A1)–(A9) and (8), there exists at least one weak solution for the Stefan problem, in the sense of Definition 1, such that the temperature satisfies the estimate*

$$\|\theta\|_{L^\infty(\Omega)} \le M\ . \tag{15}$$

The proof, that we postpone to section 4, consists of passing to the limit in an approximated problem that belongs to a class of problems that we study in the next section from the point of view of the existence and uniqueness of solution. For simplicity we will consider, from now on, $\theta_D = 0$, remarking the necessary changes for the non homogeneous case.

3 – A class of related convection-diffusion problems

Here we prove, under appropriate assumptions, existence and uniqueness of solution for a class of nonlinear convection-diffusion problems. This result generalizes for any $1 < p < \infty$, the result presented in §5 of [5], corresponding to the case $p = 2$. The problem we will consider is the following:

Problem (P): Find $\theta \in V_p$ such that

$$\begin{aligned}(16)\qquad \int_\Omega |\nabla\theta|^{p-2}\,\nabla\theta\cdot\nabla\xi - \int_\Omega r(x,\theta)\,\xi - \int_{\Gamma_N} g(x,\theta)\,\xi &= \\ = \int_\Omega W(x,\theta)\cdot\nabla\xi\,, \qquad \forall\,\xi\in V_p\,.\end{aligned}$$

The assumptions are the ones of the previous section with respect to r and g and for the convective term W we assume that

(**B1**) $W\colon\ \Omega\times\mathbb{R}\to\mathbb{R}^N$ is a Carathéodory function;

(**B2**) $\operatorname{sign}(t)\displaystyle\int_\Omega W(y,t)\cdot\nabla\xi\,dy \le 0\,,\ \forall\,\xi\in V_p\colon\ \xi\ge 0$ in Ω, $\forall\,t\in\mathbb{R}\colon\ |t|\ge M$;

(**B3**) $\exists\,w_0\in L^{p'}(\Omega)$ such that $|W(y,t)|\le w_0(y)$, a.e. $y\in\Omega$, $\forall\,t\in[-M,M]$;

(**B4**) $\exists\,k\in L^{p'}(\Omega)$ such that $|W(y,t)-W(y,s)|\le k(y)\,|t-s|$, a.e. $y\in\Omega$, $\forall\,t,s\in[-M,M]$.

Remark 2. As observed in [5], if for $|t|\ge M$, $W(\cdot,t)\in[W^{1,p'}(\Omega)]^N$, assumption (B2) is equivalent to the double condition

$$\operatorname{sign}(t)\,\nabla_y\cdot W(y,t)\ge 0 \quad \text{a.e. } y\in\Omega$$

and

$$\operatorname{sign}(t)\,W(y,t)\cdot\mathbf{n}(y)\le 0 \ \text{ on } \Gamma_N, \quad |t|\ge M\ .$$

We start by defining an auxiliary operator for what follows, $A\colon\ V_p\to V_p'$, and showing that it possesses an interesting set of properties. For any $\sigma,\xi\in V_p$, we put

$$\langle A\sigma, \xi\rangle = \int_\Omega |\nabla\sigma|^{p-2}\,\nabla\sigma\cdot\nabla\xi - \int_\Omega r^M(x,\sigma)\,\xi - \int_{\Gamma_N} g^M(x,\sigma)\,\xi\ , \tag{17}$$

where the truncated function of a function ϕ is defined by

$$\phi^M(x,t) = \phi\Big(x, \min\{M, \max(-M,t)\}\Big)\ .$$

Lemma. *The operator A is bounded, hemicontinuous, strictly monotone and coercive.*

Proof: To show that A is bounded, use Hölder inequality to get

$$\begin{aligned}|\langle A\sigma,\xi\rangle| &\le \int_\Omega |\nabla\sigma|^{p-1}\,|\nabla\xi| + \int_\Omega |r^M(\sigma)|\,|\xi| + \int_{\Gamma_N} |g^M(\sigma)|\,|\xi| \\ &\le \Big(\|\sigma\|_{V_p}^{p-1} + C_1\,\|r_0\|_{L^{p'}(\Omega)} + C_2\,\|g_0\|_{L^{p'}(\Gamma_N)}\Big)\,\|\xi\|_{V_p}\ ,\end{aligned}$$

due to the trace theorem, the inequality of Poincaré and defining

$$r_0(x) \equiv \max\Big\{|r(x,-M)|\,,\ |r(x,M)|\Big\} \in L^{p'}(\Omega)\ ,$$

by (A3), and similarly g_0 that belongs to $L^{p'}(\Gamma_N)$, by (A6).

Next we show that A is continuous from V_p strong to V_p' weak, which implies the hemicontinuity. Given a sequence u_n in V_p we want to prove that

$$u_n \xrightarrow{V_p} u \quad\Longrightarrow\quad \langle Au_n, w\rangle \longrightarrow \langle Au, w\rangle, \quad \forall\, w\in V_p\ .$$

By the properties of the Nemytskii operators

$$|\nabla u_n|^{p-2}\,\nabla u_n \longrightarrow |\nabla u|^{p-2}\,\nabla u \quad \text{in}\ \ [L^{p'}(\Omega)]^N\ ,$$

and now the conclusion is imediate since we can deal with the other terms using Lebesgue theorem, because $u_n(x)\to u(x)$, a.e. $x\in\Omega$, r^M and g^M are continuous and $|r^M(u_n)\,w| \le r_0\,|w| \in L^1(\Omega)$ and $|g^M(u_n)\,w| \le g_0\,|w| \in L^1(\Gamma_N)$.

The strict monotonicity is a simple consequence of the assumptions (A2) and (A5) and the following well known inequality, valid for $1<p<\infty$ and $x,y\in\mathbb{R}^N$:

$$\exists\, C_p>0\colon\ \ C_p\,|x-y|^p \le \Big[\Big(|x|^{p-2}\,x-|y|^{p-2}\,y\Big)\cdot(x-y)\Big]^{\alpha/2}\,\Big[|x|^p+|y|^p\Big]^{1-\alpha/2}\ , \tag{18}$$

with $\alpha=p$ if $1<p\le 2$ and $\alpha=2$ if $p\ge 2$.

For example, for $1 < p \le 2$, from (18), we obtain

$$\int_\Omega \Big(|\nabla u|^{p-2}\,\nabla u - |\nabla v|^{p-2}\,\nabla v\Big)\cdot(\nabla u - \nabla v) \ge C^*\,\frac{\|u-v\|^2_{V_p}}{\|u\|^{2-p}_{V_p} + \|v\|^{2-p}_{V_p}}\ .$$

Finally, to show that A is coercive, we see, using the estimate obtained before, that

$$\frac{\langle A\sigma,\sigma\rangle}{\|\sigma\|_{V_p}} = \|\sigma\|^{p-1}_{V_p} - \frac{\displaystyle\int_\Omega r^M(\sigma)\,\sigma + \int_{\Gamma_N} g^M(\sigma)\,\sigma}{\|\sigma\|_{V_p}} \ge \|\sigma\|^{p-1}_{V_p} - C \;\longrightarrow\; +\infty$$

when $\|\sigma\|_{V_p} \to \infty$. ∎

The next result deals with the existence of solution.

Theorem 2. *In the previous setting, there exists a solution for the problem (P), that satisfies*

$$\|\theta\|_{L^\infty(\Omega)} \le M\ . \tag{19}$$

Proof: The result is based on the *a priori* estimate (19) and on the application of Schauder fixed point theorem to a convenient operator defined in $L^1(\Omega)$.

• L^∞ *a priori estimate*: Taking $\xi = (\theta+M)^-$ in (16), we obtain

$$\int_\Omega |\nabla\theta|^{p-2}\,\big|\nabla(\theta+M)^-\big|^2 = \tag{20}$$

$$= -\int_\Omega W(x,\theta)\cdot\nabla(\theta+M)^- - \int_\Omega r(x,\theta)\,(\theta+M)^- - \int_{\Gamma_N} g(x,\theta)\,(\theta+M)^- \le 0\ .$$

In fact, denoting the set $\{x\in\overline{\Omega}\colon\, \theta(x) < -M\}$ by $\{\theta < -M\}$, we have

$$-\int_{\Gamma_N} g(x,\theta)\,(\theta+M)^- = -\int_{\Gamma_N\cap\{\theta<-M\}} g(x,\theta)\,(-\theta-M) \le 0\ ,$$

since, due to the monotonicity of g, $\theta < -M \Rightarrow g(x,\theta) \ge g(x,-M) \ge 0$ a.e. $x\in\Gamma_N$, because $g(x,-M)(-M) \le 0$, a.e. $x \in \Gamma_N$. The same holds for r. On the other hand, defining the function $\overline{W}_M\colon\, \Omega\times\mathbb{R} \to \mathbb{R}^N$ by

$$\overline{W}_M(y,t) = \begin{cases} \displaystyle\int_t^{-M} W(y,s)\,ds & \text{if} \quad t \le -M \\ 0 & \text{if} \quad t \ge -M \end{cases}\ ,$$

we have, in $\{\theta < -M\}$,

$$\nabla_x \cdot \overline{W}_M(x,\theta(x)) = \int_{\theta(x)}^{-M} (\nabla_y \cdot W)(x,s)\, ds - W(x,\theta(x)) \cdot \nabla\theta \ .$$

Integrating over this set, and taking into account the definition of $\overline{W}_M$ and Remark 2, we get:

$$\begin{aligned}
(21) \qquad & -\int_\Omega W(x,\theta(x)) \cdot \nabla(\theta+M)^- \, dx = \int_{\Omega\cap\{\theta<-M\}} W(x,\theta(x)) \cdot \nabla\theta \, dx = \\
&= -\int_{\Omega\cap\{\theta<-M\}} \nabla_x \cdot \overline{W}_M(x,\theta(x))\, dx + \int_{\Omega\cap\{\theta<-M\}} \Big(\int_{\theta(x)}^{-M} (\nabla_y \cdot W)(x,s)\, ds\Big)\, dx \le \\
&\le -\int_\Omega \nabla_x \cdot \overline{W}_M(x,\theta(x))\, dx = -\int_{\Gamma_N} \overline{W}_M(x,\theta(x)) \cdot \mathbf{n}(x)\, d\sigma = \\
&= -\int_{\Gamma_N\cap\{\theta<-M\}} \Big(\int_{\theta(x)}^{-M} W(x,s) \cdot \mathbf{n}(x)\, ds\Big)\, d\sigma \le 0 \ .
\end{aligned}$$

If $W(y,t)$ is not sufficiently regular in y, we need to argue with its regularization by convolution with a mollifier in y, for each t fixed, $W_\delta(y,t) = (W(\cdot,t)*\rho_\delta)(y) \to W(y,t)$, $\delta \to 0$, and then pass to the limit in the inequality corresponding to (21). Therefore, noting that

$$\begin{aligned}
\int_\Omega |\nabla\theta|^{p-2}\, |\nabla(\theta+M)^-|^2 &= \int_{\Omega\cap\{\theta<-M\}} |\nabla\theta|^p \\
&= \int_\Omega |\nabla(\theta+M)^-|^p = \|(\theta+M)^-\|_{V_p}^p \ ,
\end{aligned}$$

from (20) we get $\|(\theta+M)^-\|_{V_p} = 0$ and so $(\theta+M)^- = 0$ and $\theta \ge -M$. The inequality $\theta \le M$ is obtained similarly, taking $\xi = (\theta-M)^+$ in (16).

• *An auxiliar problem*: Using a classical result of [4, p. 171], we conclude from the Lemma that A is a one-to-one mapping from V_p onto V_p'. Therefore, the nonlinear problem: given $\tau \in L^1(\Omega)$, find $\sigma \in V_p$ such that

$$\begin{aligned}
(22) \qquad \int_\Omega |\nabla\sigma|^{p-2}\, \nabla\sigma \cdot \nabla\xi - \int_\Omega r^M(x,\sigma)\, \xi - \int_{\Gamma_N} g^M(x,\sigma)\, \xi = \\
= \int_\Omega W^M(x,\tau) \cdot \nabla\xi\,, \qquad \forall\, \xi \in V_p \ ,
\end{aligned}$$

has a unique solution, because if $\langle f_\tau, \xi\rangle := \int_\Omega W^M(x,\tau) \cdot \nabla\xi$, we have $f_\tau \in V_p'$. Since (22) can then be written as $A\sigma = f_\tau$, the conclusion is obvious.

• *Schauder fixed point*: Define the mapping

$$S\colon\, B_\rho \to B_\rho \quad \text{ with } \ B_\rho = \Big\{\tau \in L^1(\Omega)\colon\, \|\tau\|_{L^1(\Omega)} \le \rho\Big\}$$

such that, to each $\tau \in L^1(\Omega)$ corresponds $\sigma = S(\tau)$, the unique solution of problem (22). The constant ρ is given by a suitable *a priori* estimate for that solution. It is obvious that a fixed point of S, $\theta^* = S(\theta^*)$, is a solution of (P), since it also verifies the estimate (19). To obtain the *a priori* estimate put $\xi = \sigma$ in (22) to get

$$\begin{aligned}\|\sigma\|_{V_p}^p = \int_\Omega |\nabla\sigma|^p = \int_\Omega W^M(x,\tau)\cdot\nabla\sigma + \int_\Omega r^M(x,\sigma)\,\sigma + \int_{\Gamma_N} g^M(x,\sigma)\,\sigma \leq \\ \leq \|w_0\|_{L^{p'}(\Omega)}\,\|\sigma\|_{V_p} + \|r_0\|_{L^{p'}(\Omega)}\,\|\sigma\|_{L^p(\Omega)} + \|g_0\|_{L^{p'}(\Gamma_N)}\,\|\sigma\|_{L^p(\Gamma_N)} \leq C\,\|\sigma\|_{V_p}\end{aligned}$$

and consequently $\|\sigma\|_{V_p} \leq C^{1/p-1} \equiv C^*$. Finally,

$$\|\sigma\|_{L^1(\Omega)} \leq C'\,\|\sigma\|_{V_p} \leq C'\,C^* \equiv \rho\ ,$$

and $S(B_\rho) \subset B_\rho$.

It remains to show that S is compact, but since, for each $\tau \in L^1(\Omega)$, we have $S(\tau) \in W^{1,p}(\Omega)$ and the imbedding $W^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, it is enough to prove that it is continuous. We first show that

$$\tau_n \xrightarrow{L^1} \tau \quad\Longrightarrow\quad f_n \equiv f_{\tau_n} \xrightarrow{V_p'} f_\tau \equiv f\ . \tag{23}$$

Due to the fact that W is a Carathéodory function ((B1)), we define the Nemytskii operator

$$\begin{array}{rccc} G\colon & L^1(\Omega) & \longrightarrow & [L^{p'}(\Omega)]^N \\ & \tau & \longmapsto & G\tau \end{array} \qquad \text{with} \quad G\tau(x) = W^M(x,\tau(x))\ ,$$

which is continuous by (B3). Then:

$$\|f_n - f\|_{V_p'} \leq \|W^M(\tau_n) - W^M(\tau)\|_{[L^{p'}]^N} \ \longrightarrow\ 0\ .$$

To conclude, we prove that

$$f_n \xrightarrow{V_p'} f \quad\Longrightarrow\quad S(\tau_n) \equiv \sigma_n \xrightarrow{V_p} \sigma \equiv S(\tau)\ .$$

We start by observing that $A\sigma_n = f_n$ and $A\sigma = f$ and consequently

$$\frac{\langle A\sigma_n, \sigma_n\rangle}{\|\sigma_n\|_{V_p}} = \frac{\langle f_n, \sigma_n\rangle}{\|\sigma_n\|_{V_p}} \leq \|f_n\|_{V_p'} \leq C\ .$$

But A is coercive, so fately $\|\sigma_n\|_{V_p} \leq C'$ and there exists $\sigma^* \in V_p$ such that $\sigma_n \xrightarrow{V_p} \sigma^*$. Then, $\langle A\sigma_n - A\sigma\,,\ \sigma_n - \sigma\rangle \to \langle f - A\sigma\,,\ \sigma^* - \sigma\rangle = 0$. Now, the strict monotonicity of A yields, for $p \geq 2$, $\langle A\sigma_n - A\sigma\,,\ \sigma_n - \sigma\rangle \geq C_p\,\|\sigma_n - \sigma\|_{V_p}^p$ and for $1 < p \leq 2$,

$$\langle A\sigma_n - A\sigma\,,\ \sigma_n - \sigma\rangle \geq C^*\,\frac{\|\sigma_n - \sigma\|_{V_p}^2}{\|\sigma_n\|_{V_p}^{2-p} + \|\sigma\|_{V_p}^{2-p}} \geq C\,\|\sigma_n - \sigma\|_{V_p}^2\ ,$$

since $\|\sigma_n\|_{V_p} \leq C'$. Therefore, $\|\sigma_n - \sigma\|_{V_p} \to 0$ and the proof is complete. ∎

Remark 3. In the non homogeneous case, we can still get an existence result applying the fixed point to the operator defined by the solution of the problem: given $\tau \in L^1(\Omega)$, find $\sigma \in W^{1,p}(\Omega)$ such that $\sigma = \theta_D$ on Γ_D and $A\sigma = f_\tau$, with A defined in $W^{1,p}(\Omega)$. We easily reduce this problem to the one we studied. In fact, for $\psi = \sigma - \theta_D \in V_p$, the problem can be written in the form

$$\psi \in V_p\colon\ \ \widetilde{A}\psi = f_\tau\ ,$$

where $\widetilde{A}\psi := A(\psi + \theta_D)$ is an operator defined from V_p to V_p' for which the Lemma is true. After determining ψ we recover $\sigma = \psi + \theta_D$.

Remark 4. For $p = 2$, we additionaly obtain the Hölder continuity of the temperature. In this case, $|\nabla\theta|^{p-2}\,\nabla\theta = \nabla\theta$, and the problem is essentialy the studied in [5]. There, Shauder fixed point theorem is applied to an operator

$$S\colon B_k \to B_k \quad \text{ with }\ B_k = \Big\{\tau \in C^0(\overline{\Omega})\colon\ \|\tau\|_{C^0(\overline{\Omega})} \leq k\Big\}\ ,$$

defined by the unique solution of the linear problem: find $\sigma \in V_2$ such that

$$\int_\Omega \nabla\sigma\cdot\nabla\xi = \int_\Omega r^M(\tau)\,\xi + \int_{\Gamma_N} g^M(\tau)\,\xi + \int_\Omega W^M(\tau)\cdot\nabla\xi\,, \quad \forall\,\xi \in V_2\ .$$

The constant k is given by an estimate, due to Stampacchia [7], for the solution of this problem:

$$\|\sigma\|_{C^{0,\gamma}(\overline{\Omega})} \leq C_\gamma\Big(\|r_0\|_{L^{\frac{q}{2}}(\Omega)} + \|w_0\|_{L^q(\Omega)} + \|g_0\|_{L^s(\Gamma_N)}\Big) \equiv k\ , \tag{24}$$

for an exponent $0 < \gamma < 1$ and assuming in addition $r_0 \in L^{\frac{q}{2}}(\Omega)$, $w_0 \in L^q(\Omega)$ and $g_0 \in L^s(\Gamma_N)$ with $q > N \geq 2$ and $s > N - 1$. For the non homogeneous case we still need to assume $\theta_D \in C^{0,1}(\overline{\Omega})$, with $\|\theta_D\|_{C^{0,1}(\overline{\Omega})}$ taking part in the previous estimate.

Concerning uniqueness, we deal with the cases $1 < p \leq 2$ and $p \geq 2$ separately. For the latter, we can apply the results of [2], also used in [5]. The former case, $1 < p \leq 2$, demands a different approach and we use some recent results from [1].

Theorem 3. *Problem (P) has a unique solution for $1 < p \le 2$ and also for $p > 2$ if we suppose r strictly decreasing in the latter case.*

Proof: For $p \ge 2$, with the notations of [3, §3, p. 148], we have here $K = V = V_p$, $f \equiv 0 \in V_p'$ and we rewrite the problem in the form

$$\theta \in V_p\colon \ \ \langle \mathbf{B}(x,\theta,\nabla\theta), \xi\rangle = 0\,, \qquad \forall\, \xi \in V_p\ ,$$

with the nonlinear operator $\mathbf{B}\colon V_p \to V_p'$ given by

$$\langle \mathbf{B}(x,\theta,\nabla\theta),\xi\rangle = \int_\Omega \Big(\Big[|\nabla\theta|^{p-2}\,\nabla\theta - W^M(x,\theta)\Big]\cdot\nabla\xi - r(x,\theta)\,\xi\Big)\,dx - \int_{\Gamma_N} g(x,\theta)\,\xi\,d\sigma.$$

We easily find ourselves in the conditions of [2] since we have the strict coercivity of $\mathbf{B}$ as an immediate consequence of inequality (18) for this case, and the strong continuity property of $\mathbf{B}$ (assumption (1.9) of [1]) as a consequence of (B4). Therefore, and since we are assuming $t \to r(\cdot,t)$ strictly decreasing, the result is now imediate. Due to the *a priori* estimate (19), it is still valid with W^M replaced by W.

For $1 < p \le 2$, we follow the approach of [1], defining, for each $\epsilon > 0$, the function $T_\epsilon\colon \mathbb{R} \to \mathbb{R}$, given by:

$$T_\epsilon(s) = \begin{cases} s & \text{if} \quad |s| \le \epsilon \\ \epsilon\,\dfrac{s}{|s|} & \text{if} \quad |s| > \epsilon \end{cases} .$$

Since $T_\epsilon \in C(\mathbb{R})$, $T_\epsilon{}' \in L^\infty(\mathbb{R})$ and $T_\epsilon(0) = 0$, we know that $\forall\, u \in V_p$, $T_\epsilon \circ u \in V_p$ and that

$$\nabla(T_\epsilon \circ u) = T_\epsilon{}'(u)\,\nabla u = \begin{cases} \nabla u & \text{if} \quad |u| \le \epsilon \\ 0 & \text{if} \quad |u| > \epsilon \end{cases}, \quad \text{a.e. in } \Omega\ . \tag{25}$$

If θ_1 and θ_2 are two solutions of (P) and $\vartheta = \theta_1 - \theta_2$, we put $\xi = T_\epsilon \circ \vartheta$ in the equations (16) corresponding to θ_1 and θ_2, which is possible since $T_\epsilon \circ \vartheta \in V_p$. Subtracting, we get

$$\begin{aligned} \int_\Omega \Big(|\nabla\theta_1|^{p-2}\,\nabla\theta_1 - |\nabla\theta_2|^{p-2}\,\nabla\theta_2\Big)\cdot\nabla T_\epsilon(\vartheta) = \\ = \int_\Omega [W(x,\theta_1) - W(x,\theta_2)]\cdot\nabla T_\epsilon(\vartheta) + \int_\Omega [r(x,\theta_1) - r(x,\theta_2)]\,T_\epsilon(\vartheta) \\ + \int_{\Gamma_N} [g(x,\theta_1) - g(x,\theta_2)]\,T_\epsilon(\vartheta)\ . \end{aligned} \tag{26}$$

Since $g(x,\cdot)$ is decreasing ((A5)), $[g(x,\theta_1)-g(x,\theta_2)]\,T_\epsilon(\vartheta)\le 0$, which is obvious if $|\vartheta|\le\epsilon$ and also true for $|\vartheta|>\epsilon$ because, in that case, if $\vartheta>\epsilon>0$, then $\theta_1>\theta_2$ and $T_\epsilon(\vartheta)=\epsilon$ and if $\vartheta<-\epsilon<0$, then $\theta_1<\theta_2$ and $T_\epsilon(\vartheta)=-\epsilon$. A similar reasoning holds for r. Defining the set $A_\epsilon=\{x\in\Omega\colon\ 0<|\vartheta(x)|\le\epsilon\}$, where, from (25), $\nabla T_\epsilon(\vartheta)=\nabla\vartheta$, (26) yields

$$\text{(27)}\quad \int_{A_\epsilon}\Big(|\nabla\theta_1|^{p-2}\,\nabla\theta_1-|\nabla\theta_2|^{p-2}\,\nabla\theta_2\big)\cdot\nabla\vartheta\le\int_{A_\epsilon}[W(x,\theta_1)-W(x,\theta_2)]\cdot\nabla\vartheta\ .$$

The inequality (18), for $1<p\le 2$, and assumption (B4) allow us to conclude, from (27), that

$$\text{(28)}\qquad c\int_{A_\epsilon}\frac{|\nabla\vartheta|^2}{|\nabla\theta_1|^{2-p}+|\nabla\theta_2|^{2-p}}\le\int_{A_\epsilon}k\,|\vartheta|\,|\nabla\vartheta|\le$$
$$\le\int_{A_\epsilon}\frac{c}{|\nabla\theta_1|^{2-p}+|\nabla\theta_2|^{2-p}}\,\frac{|\nabla\vartheta|^2}{2}+\int_{A_\epsilon}\frac{|\nabla\theta_1|^{2-p}+|\nabla\theta_2|^{2-p}}{c}\,\frac{k^2\,|\vartheta|^2}{2}\ ,$$

with the last inequality being a consequence of the inequality of Young, $ab\le\rho^p\,\frac{a^p}{p}+\rho^{-q}\,\frac{b^q}{q}$, with $p=q=2$ and $\rho=\sqrt{c}\,(|\nabla\theta_1|^{2-p}+|\nabla\theta_2|^{2-p})^{-1/2}$. To simplify, put $\Psi_1=|\nabla\theta_1|^{2-p}+|\nabla\theta_2|^{2-p}$ and $\Psi_2=k^2\,\Psi_1$. We show that Ψ_1 and Ψ_2 belong to $L^1(\Omega)$. In fact, since $\theta_i\in W^{1,p}(\Omega)$, $i=1,2$, we have $|\nabla\theta_i|^p\in L^1(\Omega)$ and so $|\nabla\theta_i|^{2-p}\in L^{p/(2-p)}(\Omega)\subset L^1(\Omega)$, since $p/(2-p)>1$ because $1<p\le 2$. Therefore, $\Psi_1\in L^1(\Omega)$. For Ψ_2, it is enough to remark that $k^2\in L^{p'/2}(\Omega)$ and that $\frac{2}{p'}+\frac{2-p}{p}=1$. Back to (28), we obtain

$$\text{(29)}\qquad \int_{A_\epsilon}\frac{|\nabla\vartheta|^2}{\Psi_1}\le\frac{1}{c^2}\int_{A_\epsilon}\Psi_2\,|\vartheta|^2\le\frac{\epsilon^2}{c^2}\int_{A_\epsilon}\Psi_2\ ,$$

since $|\vartheta|\le\epsilon$ in A_ϵ. Using the inequality of Cauchy–Schwarz, from (29), we obtain

$$\text{(30)}\qquad \int_\Omega|\nabla T_\epsilon(\vartheta)|=\int_{A_\epsilon}|\nabla\vartheta|=\int_{A_\epsilon}\frac{|\nabla\vartheta|}{(\Psi_1)^{1/2}}\,(\Psi_1)^{1/2}\le$$
$$\le\Big(\int_{A_\epsilon}\frac{|\nabla\vartheta|^2}{\Psi_1}\Big)^{1/2}\Big(\int_{A_\epsilon}\Psi_1\Big)^{1/2}\le\frac{\epsilon}{c}\Big(\int_{A_\epsilon}\Psi_2\Big)^{1/2}\Big(\int_{A_\epsilon}\Psi_1\Big)^{1/2}\ .$$

But $(A_\epsilon)_\epsilon$ is decreasing and $\bigcap_{\epsilon>0}A_\epsilon=\emptyset$, so $|A_\epsilon|\to 0$ $(\epsilon\to 0)$, and $\int_{A_\epsilon}\Psi_i\to 0$ $(\epsilon\to 0)$, $i=1,2$. To complete the proof, fix $\delta>0$. For $0<\epsilon<\delta$,

$$\Big|\Big\{x\in\Omega\colon\ |\vartheta(x)|\ge\delta\Big\}\Big|\le\Big|\Big\{x\in\Omega\colon\ |\vartheta(x)|>\epsilon\Big\}\Big|=\frac{1}{\epsilon}\Big(\int_\Omega|T_\epsilon(\vartheta)|-\int_{\{|\vartheta|\le\epsilon\}}|\vartheta|\Big)\le$$
$$\le\frac{1}{\epsilon}\int_\Omega|T_\epsilon(\vartheta)|\le\frac{c_0}{\epsilon}\int_\Omega|\nabla T_\epsilon(\vartheta)|\le\frac{c_0}{c}\Big(\int_{A_\epsilon}\Psi_2\Big)^{1/2}\Big(\int_{A_\epsilon}\Psi_1\Big)^{1/2}\ \longrightarrow 0\ \ (\epsilon\to 0)\ ,$$

using (9) and (30). Finally, since $\{x \in \Omega\colon\, |\vartheta(x)| \neq 0\} = \bigcup_{\delta>0} \{x \in \Omega\colon\, |\vartheta(x)| \geq \delta\}$, we have

$$\left|\Big\{x \in \Omega\colon\, |\vartheta(x)| \neq 0\Big\}\right| = \lim_{\delta\to 0}\left|\Big\{x \in \Omega\colon\, |\vartheta(x)| \geq \delta\Big\}\right| = 0$$

and $\vartheta(x) = 0$ a.e. $x \in \Omega$, as desired. ∎

Remark 5. The assumption that r is strictly decreasing for the case $p > 2$ is necessary due to a counter example of [1].

4 – Proof of the existence of a solution

The proof of Theorem 1 consists of passing to the limit in an appropriate approximated problem obtained after regularization of the data.

For each $\epsilon > 0$, we define the continuous function $\chi_\epsilon\colon\, \mathbb{R} \to \mathbb{R}$, given by

$$\chi_\epsilon(\tau) = \begin{cases} 0 & \text{if} \quad \tau < -\epsilon \\ 2 + \dfrac{2\tau}{\epsilon} & \text{if} \quad -\epsilon \leq \tau \leq -\epsilon/2 \\ 1 & \text{if} \quad \tau > -\epsilon/2 \end{cases} ,$$

set $r_\epsilon(t) = r(t) - \epsilon\, t$ and choose a function $b_\epsilon \in C^{0,1}(\mathbb{R})$ such that $b_\epsilon \to b$ uniformly on compact sets as $\epsilon \to 0$. Then, for each ϵ, the following approximated problem has a unique solution.

Problem (P_ϵ): Find $\theta_\epsilon \in V_p$, such that

$$\int_\Omega \Big\{ |\nabla\theta_\epsilon|^{p-2}\, \nabla\theta_\epsilon - [b_\epsilon(\theta_\epsilon) + \lambda\, \chi_\epsilon(\theta_\epsilon)]\, \mathbf{v} \Big\} \cdot \nabla\xi - \int_\Omega r_\epsilon(\theta_\epsilon)\, \xi = \\ = \int_{\Gamma_N} g(\theta_\epsilon)\, \xi, \quad \forall\, \xi \in V_p \,. \tag{31}$$

In fact it belongs to the class of problems studied in the previous section with

$$W(x,t) = [b_\epsilon(t) + \lambda\, \chi_\epsilon(t)]\, \mathbf{v}(x) \ ,$$

that trivially verifies assumptions (B1)–(B4). In addition we obtain the following estimates, independently of ϵ:

$$\|\theta_\epsilon\|_{V_p} \leq k_1 \,, \qquad \|\theta_\epsilon\|_{L^\infty(\Omega)} \leq M \ .$$

Using the compactness properties of the functional spaces envolved, from these estimates, we obtain subsequences such that, for $\epsilon \to 0$, and $s \leq \dfrac{Np-p}{N-p}$ if $p < N$,

$$\begin{aligned} &(32) \qquad \theta_\epsilon \longrightarrow \theta \quad \text{in } V_p\text{-weak, } L^p(\Omega)\text{-strong, pointwise a.e. } x \in \Omega \\ &\qquad\qquad\qquad\qquad \text{and the corresponding traces in } L^s(\Gamma_N)\text{-strong ;} \end{aligned}$$

$$(33) \qquad \chi_\epsilon(\theta_\epsilon) \rightharpoonup \chi \quad \text{in } L^\infty(\Omega)\text{-weak} * ;$$

for some limit functions $\theta \in V_p$ and $\chi \in L^\infty(\Omega)$. We use the same index for the subsequences as usually and for simplicity.

To prove (13), we observe that since $0 \leq \chi_\epsilon(\theta_\epsilon) \leq 1$, in the limit we also obtain $0 \leq \chi \leq 1$ a.e. in Ω. From (33) and the fact that $(\theta_\epsilon + \epsilon)^- \to \theta^-$ in $L^1(\Omega)$, due to (32), we obtain

$$\int_\Omega \chi_\epsilon(\theta_\epsilon)\,(\theta_\epsilon + \epsilon)^- \longrightarrow \int_\Omega \chi\,\theta^- \ .$$

But $\chi_\epsilon(\theta_\epsilon)\,(\theta_\epsilon + \epsilon)^- = 0$ and then $\chi\,\theta^- = 0$ a.e. in Ω. We conclude with

$$\theta^- > 0 \ \text{ in } \Sigma(\theta) \quad \Longrightarrow \quad \chi = 0 \ \text{ a.e. in } \Sigma(\theta) \ ,$$

so $\chi \leq 1 - \chi_{\Sigma(\theta)}$ a.e. in Ω. To obtain $\chi \geq \chi_{\Lambda(\theta)}$, we repeat the above reasoning with ${\theta_\epsilon}^+$ and $1 - \chi_\epsilon(\theta_\epsilon)$.

To pass to the limit in the equation for the temperature, we define $H_\epsilon(\theta_\epsilon), H(\theta) \in V_p'$ by

$$(34) \qquad \langle H_\epsilon(\theta_\epsilon), \xi\rangle = \int_\Omega [\, b_\epsilon(\theta_\epsilon) + \lambda\,\chi_\epsilon(\theta_\epsilon)\,]\,\mathbf{v}\cdot\nabla\xi - \epsilon\int_\Omega \theta_\epsilon\,\xi \ ,$$

$$(35) \qquad \langle H(\theta), \xi\rangle = \int_\Omega [\, b(\theta) + \lambda\,\chi\,]\,\mathbf{v}\cdot\nabla\xi \ .$$

Equation (31) can be rewritten in the form

$$\langle A\theta_\epsilon, \xi\rangle = \langle H_\epsilon(\theta_\epsilon), \xi\rangle \,, \qquad \forall\,\xi \in V_p \ ,$$

with the notations of the preceding section and ignoring the truncations of r e g, because of the *a priori* estimate in L^∞ for θ_ϵ. Since A is monotone, for any $v \in V_p$ we get

$$\langle A\theta_\epsilon - Av,\, \theta_\epsilon - v\rangle \geq 0 \quad \Longleftrightarrow \quad \langle H_\epsilon(\theta_\epsilon) - Av,\, \theta_\epsilon - v\rangle \geq 0 \ .$$

To pass to the limit in this inequality we observe that due to the *a priori* estimate $\|\theta_\epsilon\|_{L^\infty(\Omega)} \leq M$ and the assumed uniform convergence on compact sets $b_\epsilon \to b$,

$b_\epsilon(\theta_\epsilon)$ is uniformly bounded and converges pointwise a.e. to $b(\theta)$ in Ω. From (A8) we get

$$b_\epsilon(\theta_\epsilon)\,\mathbf{v} \to b(\theta)\,\mathbf{v} \quad \text{ in } \ [L^{p'}(\Omega)]^N \ .$$

But we only have $\lambda\,\chi_\epsilon(\theta_\epsilon)\,\mathbf{v} \rightharpoonup \lambda\,\chi\,\mathbf{v}$ in $[L^\infty(\Omega)]^N$-weak $*$, so in $[L^q(\Omega)]^N$-weak, $\forall\, q<\infty$, and $H_\epsilon(\theta_\epsilon) \rightharpoonup H(\theta)$ but not strongly. The problem is then to deal with $\langle H_\epsilon(\theta_\epsilon),\theta_\epsilon\rangle$, since here both convergences are weak. To overcome this difficulty, we pick a function $K_\epsilon \in C^1(\mathbb{R})$ such that $K'_\epsilon = \chi_\epsilon$, and use the fact that $K_\epsilon(t) \to t^+$ uniformly in $\mathbb{R}$ to get

$$\begin{aligned}(36)\qquad \int_\Omega \chi_\epsilon(\theta_\epsilon)\,\mathbf{v}\cdot\nabla\theta_\epsilon = \int_\Omega \mathbf{v}\cdot\nabla[K_\epsilon(\theta_\epsilon)] = \int_{\partial\Omega} (\mathbf{v}\cdot\mathbf{n})\,K_\epsilon(\theta_\epsilon) \quad &\longrightarrow \\ \longrightarrow \quad \int_{\partial\Omega}(\mathbf{v}\cdot\mathbf{n})\,\theta^+ = \int_\Omega \mathbf{v}\cdot\nabla\theta^+ = \int_\Omega \chi\,\mathbf{v}\cdot\nabla\theta\ ,&\end{aligned}$$

because $\chi\,\nabla\theta = \nabla\theta^+$, a.e. in Ω, since $\nabla\theta^+ = \chi_{\{\theta>0\}}\,\nabla\theta$ and $\chi_{\{\theta>0\}} = \chi$ if $\theta \neq 0$ due to (13) and $\nabla\theta = \nabla\theta^+ = 0$ if $\theta = 0$. Then $\langle H_\epsilon(\theta_\epsilon),\theta_\epsilon\rangle \to \langle H(\theta),\theta\rangle$, and passing to the limit in the inequality, we obtain

$$\langle H(\theta) - Av,\, \theta - v\rangle \geq 0\ .$$

Choosing $v = \theta - \delta\,u$, for $u \in V_p$ and $\delta \in \mathbb{R}$ arbitrary, we get $\langle H(\theta) - A(\theta - \delta u), \delta u\rangle \geq 0$, and letting $\delta \to 0$, the hemicontinuity of A gives

$$\langle H(\theta), u\rangle = \langle A\theta, u\rangle\,, \qquad \forall\, u \in V_p\ ,$$

so $A\theta = H(\theta)$ in V'_p, and this is equation (14). ∎

Remark 6. In the case $p = 2$, we obtain $\theta \in C(\overline{\Omega})$ from the estimate (24) relative to the approximated problem. For $p > N$, we also get continuity of θ, via Sobolev embedding theorem.

Remark 7. In the non homogeneous case, we can take a similar approach with $\psi_\epsilon = \theta_\epsilon - \theta_D \in V_p$ and the operator $\widetilde{A}\psi := A(\psi + \theta_D)$, putting

$$\left\langle \widetilde{A}\psi_\epsilon - \widetilde{A}(v - \theta_D),\ \psi_\epsilon + \theta_D - v \right\rangle \,\geq\, 0\ ,$$

for any $v \in W^{1,p}(\Omega)$ such that $v = \theta_D$ on Γ_D.

Remark 8. Concerning uniqueness, we remark that, even in the case $p = 2$, the full stationary problem is still open. Some partial results have been obtained in [10], in a special case. See also [6] where the evolution problem with prescribed convection is discussed, in particular with respect to the assymptotic behaviour as $t \to \infty$, where the stationary problem is obtained in the case $p = 2$.

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José Miguel Urbano,
Departamento de Matemática, Universidade de Coimbra
3000 Coimbra – PORTUGAL