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ON A RESULT OF WILLIAMSON

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Abstract: In this paper we generalize a result of Williamson on the structure of the kernel of a symmetrizer and also obtain some other related results.

1 – Introduction

Let S_m be the full symmetric group of degree m and $c(\sigma)$ an arbitrary nonzero function from S_m into the complex field $\mathbb{C}$. Given an $m \times m$ matrix $X = [x_{ij}]$ we define its generalized matrix function $d_c(X)$ by

$$d_c(X) = \sum_{\sigma \in S_m} c(\sigma) \prod_{i=1}^{m} x_{i\sigma(i)} \ .$$

When $c = \lambda$ is a character of a subgroup G of S_m, we will write d_c as d_λ^G. We denote by $\Gamma_{m,n}$ the set of maps from $\{1, ..., m\}$ into $\{1, ..., n\}$. If $\alpha \in \Gamma_{m,n}$, we identify it with the m-tuple $(\alpha(1), ..., \alpha(m))$. For an $n \times n$ matrix $A = [a_{ij}]$ and $\alpha, \beta \in \Gamma_{m,n}$, $A[\alpha|\beta]$ will denote the $m \times m$ matrix whose (i,j) element is $a_{\alpha(i),\beta(j)}$. For $\alpha \in \Gamma_{m,n}$, $\sigma \in S_m$, we write $\alpha\,\sigma = (\alpha(\sigma(1)), ..., \alpha(\sigma(m)))$. We also write $e = (1, ..., m)$.

Let V be an n-dimensional unitary vector space over $\mathbb{C}$, and $\bigotimes^m V$ be its m-th tensor power. If $\sigma \in S_m$, there exists a unique linear operator $P(\sigma)$ on $\bigotimes^m V$ such that

$$P(\sigma)\, x_1 \otimes ... \otimes x_m = x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(m)} \ .$$

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The linear mapping

$$T_c = \sum_{\sigma \in S_m} c(\sigma)\, P(\sigma)$$

will be called a symmetrizer. The star product $x_1 * ... * x_m$ is, by definition, $T_c(x_1 \otimes ... \otimes x_m)$.

As we know, the characterization of the kernel of the symmetrizer T_c is equivalent to that of the following set (see [2] and [4])

$$\mathcal{N}(c) = \left\{ A \in M_m(\mathbb{C}) \mid d_c(AX) = 0,\ \forall\, X \right\} . \tag{1.1}$$

In [7], Williamson proves a fundamental combinatorial property of cyclic permutations of finite sequences of integers and considers an application of this result to the characterization of the kernel of T_c when c is a homomorphism from G into $\mathbb{C}$.

If Δ is an orbit of the subgroup G, let G^{Δ} be the subgroup of G restricted to Δ. Following Williamson, we denote by $\mathcal{G}$ the class of all subgroups G of S_m such that if Δ is any orbit of G the G^{Δ} is cyclic. For $\alpha \in \Gamma_{m,n}$ we shall denote by G_α that subgroup of G defined by

$$G_\alpha = \left\{ \sigma \in G \mid \alpha(\sigma(i)) = \alpha(i),\ i = 1, ..., m \right\} .$$

For any homogeneous tensor $w = y_1 \otimes ... \otimes y_m$, α is called an indicator of w if $\alpha(i) = \alpha(j)$ if and only if y_i and y_j are linearly dependent. Now we are able to state the following two results of Williamson [7]:

Theorem 1.1. *Let $G \in \mathcal{G}$. For any $\gamma \in \Gamma_{m,n}$ such that γ has at least two elements, there exists $\omega \in \Gamma_{m,n}$ such that:*

i) range $\omega \subseteq$ range γ;

ii) $\gamma(i) \neq \omega(i)$, $i = 1, ..., m$;

iii) *For each $\sigma \notin G_\gamma$ there is an integer j, $1 \leq j \leq m$, such that $\gamma(\sigma(j)) = \omega(j)$.*

Theorem 1.2. *Let $G \in \mathcal{G}$ and $w = y_1 \otimes ... \otimes y_m$. If λ is any homomorphism of G into $\mathbb{C}$ and γ an indicator of w, then $y_1 \otimes ... \otimes y_m$ is in the kernel of T_λ iff $\sum_{\sigma \in G_\gamma} \lambda(\sigma) = 0$.*

In this paper, we generalize Theorem 1.2 to arbitrary functions c. We also obtain some other related results.

2 – Results

Let $E=\{e_1,...,e_n\}$ be an orthonormal basis of V. For $\alpha\in\Gamma_{m,n}$, let $x^\otimes_\alpha = x_{\alpha(1)}\otimes ... \otimes x_{\alpha(m)}$ and $T_c(x^\otimes_\alpha)=x^*_\alpha$. Define as in [3].

$$b(\pi)=\sum_{\sigma\in S_m} c(\sigma\,\pi)\,\overline{c(\sigma)}\,, \qquad c\in S_m\ .$$

Particularly, when $c=\lambda$ is a character of the subgroup G, we have

$$b(\pi)=\sum_{\sigma\in G}\lambda(\sigma\,\pi)\,\overline{\lambda(\sigma)}=\frac{|G|}{\lambda(id)}\,\lambda(\pi)\ . \tag{2.1}$$

If $v^*=v_1*...*v_m$ with $v_i=\sum_{l=1}^n a_{il}\,e_l\in V$, then

$$v^*=\sum_{\alpha\in\Gamma_{m,n}} a_\alpha\,T_c\Big(e_{\alpha(1)}\otimes...\otimes e_{\alpha(m)}\Big)=\sum_{\alpha\in\Gamma_{m,n}} a_\alpha\,e^*_\alpha$$

with $a_\alpha=a_{1\alpha(1)}\cdots a_{m\alpha(m)}$, and $\|e^*_\alpha\|^2=\sum_{\sigma\in G_\alpha} b(\sigma)$.

We have already known that (see [1], [6]):

$$e^*_\alpha=0 \quad \text{iff} \quad \sum_{\pi\in G_\alpha} c(\sigma\,\pi)=0, \quad \forall\,\sigma\in S_m\ . \tag{2.2}$$

When $c=\lambda$ is a character of the subgroup $G\subseteq S_m$, a stronger result can be obtained. In fact we have

Proposition 2.1. *Let λ be a character of the subgroup G. Then $e^*_\alpha=0$ iff $\sum_{\sigma\in G_\alpha}\lambda(\pi\,\sigma\,\tau)=0$ for $\forall\,\pi,\tau\in G$.*

Proof: At first, we have

$$(e^*_{\alpha\tau},e^*_{\beta\pi})=\frac{\lambda(id)}{|G|}\sum_{\sigma\in G}\lambda(\sigma)\,\delta_{\alpha,\beta\pi\sigma\tau^{-1}}$$

and

$$\|e^*_\alpha\|^2=\frac{\lambda(id)}{|G|}\sum_{\sigma\in G_\alpha}\lambda(\sigma)\ .$$

The "if" part is easy.

The "only if" part: When $e^*_\alpha=0$, then for arbitrary $\sigma\in G$, $e^*_{\alpha\sigma}=0$ and for $\forall\,\pi,\tau\in G$,

$$\begin{aligned}(e^*_{\alpha\tau},e^*_{\alpha\pi})&=\frac{\lambda(id)}{|G|}\sum_{\sigma\in G}\lambda(\sigma)\,\delta_{\alpha,\alpha\pi\sigma\tau^{-1}}\\&=\frac{\lambda(id)}{|G|}\sum_{\sigma\in G_\alpha}\lambda(\pi^{-1}\,\sigma\,\tau)=0\ .\ \blacksquare\end{aligned}$$

With this result, we can prove

Proposition 2.2. *Let λ be a character of the subgroup G. Then $\sum_{\sigma\in G_\alpha}\lambda(\sigma)=0$ iff $\sum_{\sigma\in G}\lambda(\sigma)\,\phi(\alpha\sigma)=0$ for arbitrary $\phi\colon\ \Gamma_{m,n}\to C$.*

Proof: The "if" part: Since $\sum_{\sigma\in G}\lambda(\sigma)\,\phi(\alpha\sigma)=0$ for arbitrary $\phi\colon \Gamma_{m,n}\to C$, let

$$\phi(\beta)=\begin{cases}1, & \text{if } \beta=\alpha,\\ 0, & \text{otherwise}\ .\end{cases}$$

Then we have

$$\sum_{\sigma\in G}\lambda(\sigma)\,\phi(\alpha\,\sigma)=\sum_{\sigma\in G_\alpha}\lambda(\sigma)=0\ .$$

The "only if" part: Let $\tau_1,...,\tau_r$ be a system of right coset representatives of G_α in G. Using Proposition 2.1,

$$\begin{aligned}\sum_{\sigma\in G}\lambda(\sigma)\,\phi(\alpha\,\sigma)&=\sum_{j=1}^{r}\sum_{\sigma\in G_\alpha}\lambda(\sigma\,\tau_j)\,\phi(\alpha\,\sigma\,\tau_j)\\&=\sum_{j=1}^{r}\Big(\sum_{\sigma\in G_\alpha}\lambda(\sigma\,\tau_j)\Big)\,\phi(\alpha\,\tau_j)=0\ .\ \blacksquare\end{aligned}$$

The next result can be similarly proved:

Corollary 2.3. *Let λ be a character of the subgroup G. Then $\sum_{\sigma\in G_\alpha}\lambda(\sigma)=0$ iff $\sum_{\sigma\in G}\lambda(\sigma^{-1})\,\phi(\alpha\sigma)=0$ for arbitrary $\phi\colon\ \Gamma_{m,n}\to C$.*

Now we come to discuss the case when c is an arbitrary function. With (2.2), we can prove

Proposition 2.4. *If $e^*_\alpha=0$, then $A[\alpha|e]\in\mathcal{N}(c)$.*

Proof: Let $\tau_1,...,\tau_r$ be a system of left coset representatives of G_α in S_m, using (2.2), for arbitrary $\beta \in \Gamma_{m,n}$, we have

$$\begin{aligned} d_c A[\alpha|\beta] &= \sum_{\sigma\in S_m} c(\sigma) \prod_{i=1}^m a_{\alpha(i),\beta(\sigma(i))} \\ &= \sum_{\sigma\in S_m} c(\sigma) \prod_{i=1}^m a_{\alpha(\sigma^{-1}(i)),\beta(i)} \\ &= \sum_{j=1}^r \sum_{\sigma\in G_\alpha} c(\tau_j\,\sigma) \prod_{i=1}^m a_{\alpha(\sigma^{-1}\tau_j^{-1}(i)),\beta(i)} \\ &= \sum_{j=1}^r \Big(\sum_{\sigma\in G_\alpha} c(\tau_j\,\sigma)\Big) \prod_{i=1}^m a_{\alpha(\tau_j^{-1}(i)),\beta(i)} = 0 \ . \end{aligned}$$

Noting that (see [2] or [5])

$$d_c(A[\alpha|e]\,X) = \sum_{\beta\in\Gamma_{m,n}} d_c A[\alpha|\beta] \prod_{i=1}^m x_{\beta(i)i} \ ,$$

we arrive at $d_c(A[\alpha|e]\,X) = 0$ for arbitrary X. ∎

Bearing in mind the definition of the indicator of a homogeneous tensor, recalling the remark preceding (1.1) and using Proposition 2.4, we can easily prove the following

Corollary 2.5. *Let γ be an indicator of $w = y_1 \otimes ... \otimes y_m$. If $e^*_\gamma = 0$, then $y_1 \otimes ... \otimes y_m$ is in the kernel of T_c.*

Now we are in a position to prove the main result of this paper.

Proposition 2.6. *Let G be in $\mathcal{G}$, c an arbitrary function from G into $\mathbb{C}$ and $b(\pi) = \sum_{\sigma\in G} c(\sigma\,\pi)\,\overline{c(\sigma)}$. Let γ be an indicator of $v^\otimes = v_1 \otimes ... \otimes v_m$. Then $v^* = 0$ if and only if $\sum_{\sigma\in G_\gamma} b(\sigma) = \nu(\gamma) = 0$.*

Proof: The "if" part follows immediately from Corollary 2.5. For the "only if" part, our proof parallels that of [7], with some slight modifications. As in [7], we assume that $\nu(\gamma) \neq 0$. If $\gamma(1) = ... = \gamma(m)$, then $G_\gamma = G$ and

$$\begin{aligned} \nu(\gamma) &= \sum_{\sigma\in G_\gamma} b(\sigma) = \sum_{\sigma\in G} b(\sigma) = \sum_{\sigma\in G}\sum_{\pi\in G} c(\pi\,\sigma)\,\overline{c(\pi)} \\ &= \sum_{\pi\in G}\Big(\sum_{\sigma\in G} c(\pi\,\sigma)\Big)\,\overline{c(\pi)} = \Big\|\sum_{\pi\in G} c(\pi)\Big\|^2 \neq 0 \ . \end{aligned}$$

Hence

$$v^* = T_c(v_1 \otimes ... \otimes v_m) = K v_\gamma^\otimes \sum_{\sigma \in G} c(\sigma) \neq 0 \ ,$$

where K is a nonzero constant.

Assume γ has at least two elements. Let $\tau_1 = id, ..., \tau_r$ be a system of left coset representatives of G_γ in G. Then

$$\begin{aligned}\sum_{\sigma\in G} b(\sigma)\,P(\sigma)\,v^\otimes &= \sum_{\sigma\in G} b(\sigma)\,P(\sigma)\,(v_1\otimes ... \otimes v_m)\\ &= K\sum_{\sigma\in G} b(\sigma)\,P(\sigma)\,v_\gamma^\otimes\\ &= K\sum_{i=1}^{r}\sum_{\sigma\in G_\gamma} b(\tau_i\,\sigma)\,P(\tau_i\,\sigma)\,v_\gamma^\otimes\\ &= K\sum_{i=1}^{r}\sum_{\sigma\in G_\gamma} b(\tau_i\,\sigma)\,P(\tau_i)\,v_\gamma^\otimes\ .\end{aligned}\tag{2.3}$$

Let $z^\otimes = z_1 \otimes ... \otimes z_m$. Then from (2.3), we have

$$\begin{aligned}(v^*, z^*) &= \Big(\sum_{\sigma\in G} b(\sigma)\,P(\sigma)\,v^\otimes,\ z^\otimes\Big)\\ &= K\sum_{i=1}^{r}\sum_{\sigma\in G_\gamma} b(\tau_i\,\sigma)\,(P(\tau_i)\,v_\gamma^\otimes,\ z^\otimes)\ .\end{aligned}\tag{2.4}$$

By Theorem 1.1, there exists an ω such that

i) range $\omega \subseteq$ range γ;

ii) $\gamma(i) \neq \omega(i)$, $i = 1, ..., m$;

iii) For each σ not in G_γ there is an i such that $\gamma(\sigma(i)) = \omega(i)$.

Now we may choose $z^\otimes = z_1 \otimes ... \otimes z_m$ such that:

$$(v_{\gamma(i)}, z_i) = 1 \quad\text{and}\quad (v_{\omega(i)}, z_i) = 0 \qquad\text{for}\quad i = 1, ..., m\ . \tag{2.5}$$

This is possible since γ is an indicator for $v^\otimes$, and also because range $\omega \subseteq$ range γ, $\gamma(i) \neq \omega(i)$ implies $v_{\gamma(i)}$ and $v_{\omega(i)}$ are linearly independent.

For any i we have

$$(P(\tau_i) v_\gamma^\otimes,\ z^\otimes) = \prod_{t=1}^{m} (v_{\gamma(\tau_i^{-1}(t))}, z_t)\ .$$

When $i \geq 2$, using iii) of Theorem 1.1, there exists j such that $\gamma(\tau_i^{-1}(j)) = \omega(j)$, and the term

$$(v_{\gamma(\tau_i^{-1}(j))}, z_j) = (v_{\omega(j)}, z_j) = 0 \ .$$

So for any $i \geq 2$, $(P(\tau_i) v_\gamma^\otimes, z^\otimes) = 0$.

For $i = 1$, according to (2.5), $(v_\gamma^\otimes, z^\otimes) = 1$ and from (2.4),

$$(v^*, z^*) = K \sum_{\sigma \in G_\gamma} b(\sigma) = K\, \nu(\gamma) \neq 0 \ .$$

Therefore, $T_c(v^\otimes) = v^* \neq 0$. This ends the proof. ∎

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