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STABILITY THEOREMS OF PERTURBED LINEAR SYSTEMS WITH IMPULSE EFFECT

Jurang Yan

Abstract: In this paper, some sufficient conditions for stability of the zero solution of perturbed linear ordinary differential equations with impulse effect are given. The approach presented is based on the variation of constants formula and the comparison theorem of differential inequalities with impulse effect.

1 – Introduction

In recent years the theory of differential equations with impulse effect has been the subject of many investigations [1–6] due to the wide application of these systems to the control theory, biology, electronics, etc., and various interesting results have been reported. Some of them studied systems with impulse effect of the form

$$(*) \qquad \begin{aligned} &\frac{dx}{dt} = f(t,x)\,, \qquad t \neq t_k\ , \\ &\Delta x = I_k(x)\,, \qquad t = t_k\ , \end{aligned}$$

where $\{t_k\}$ is an unbounded increasing sequence and $\Delta x = x(t^+) - x(t)$. The purpose of this paper is to investigate sufficient conditions of the stability of the zero solution of the perturbed linear differential systems with impulse effect at fixed moments. The investigations are carried out by means of the variation of constant formula and the comparison theorem of integral inequalities with impulse effect.

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Let $0 < t_1 < t_2 < ... < t_k < ...$, $\lim_{k\to\infty} t_k = \infty$, be a given sequence of real numbers. Consider the linear system with impulse effect at fixed moments

$$\text{(1)} \qquad \begin{aligned} &\frac{dx}{dt} = A(t)\,x\,, \qquad t \neq t_k\;, \\ &\Delta x = B_k\,x\,, \qquad t = t_k, \quad k = 1, 2, ...\;, \end{aligned}$$

and the perturbed system of (1)

$$\text{(2)} \qquad \begin{aligned} &\frac{dx}{dt} = A(t)\,x + f(t,x)\,, \qquad t \neq t_k\;, \\ &\Delta x = B_k\,x + I_k(x)\,, \qquad t = t_k, \quad k = 1, 2, ...\;. \end{aligned}$$

where $x\colon R_+ = [0,\infty) \to R^n$, $f\colon R_+ \times \Omega \to R^n$, $I_k\colon \Omega \to R^n$, $A(t)$ is $n\times n$ matrix defined on R_+, B_k are $n \times n$ constant matrices, $k = 1, 2, ...$; $\Delta x = x(t^+) - x(t)$; R^n is n-dimensional Euclidean space with a usual norm, $\Omega = \{x \in R^n;\ |x| < h\}$, $h > 0$. Moreover, we shall use the notation $B(\alpha) = \{x \in \Omega;\ |x| < \alpha\}$, $0 < \alpha < h$.

Let $t_0 \in R_+$ and $x_0 \in \Omega$. Denote by $x(t; t_0, x_0)$ the solution of (2) satisfying the initial condition $x(t_0^+; t_0, x_0) = x_0$. The solution $x(t) \equiv x(t; t_0, x_0)$ of (2) defined on $[t_0, \infty)$, $t_0 \geq 0$, are continuously differentiable for $t \neq t_k$ with points of discontinuity of the first kind at $t = t_k$ and $x(t_k) = x(t_k^-)$, $k = 1, 2, ...$.

2 – Preliminary notes

Let us consider the linear system

$$\frac{dx}{dt} = A(t)\,x\;,$$

where the matrix $A(t)$ is piecewise continuous in the interval $[0,\infty)$ with points of discontinuous of the first kind at $t = t_k$ and $A(t_k) = A(t_k^-)$, $k = 1, 2, ...$. If we denote by $U_k(t,s)$ the fundamental matrix of the linear system without impulse effect

$$\text{(3)} \qquad \frac{dx}{dt} = A(t)\,x\,, \qquad t_{k-1} < t \leq t_k\;,$$

then the solution $x(t)$ of (1) satisfying the initial condition $x(t_0^+) = x_0$ can be written in the form

$$\text{(4)} \qquad x(t) \equiv x(t; t_0, x_0) = W(t, t_0^+)\,x_0\;,$$

where $W(t_0, t_0^+) = I$ and

$$(5) \qquad W(t,s) = \begin{cases} U_k(t,s), & \text{for } t,s \in (t_{k-1}, t_k]\ , \\ U_{k+1}(t,t_k)\,(I+B_k)\,U_k(t_k,s), & \\ \qquad \text{for } t_{k-1} < s \le t_k < t \le t_{k+1}\ , & \\ U_{k+1}(t,t_k) \prod_{j=k}^{i+1} (I+B_j)\,U_j(t_j,t_{j+1})\,(I+B_i)\,U_i(t_i,s), & \\ \qquad \text{for } t_{i-1} < s \le t_i < t_k < t \le t_{k+1}\ , & \end{cases}$$

I being the unit matrix.

We introduce the following set of conditions (A):

(**A**$_1$) The matrix $A(t)$ is piecewise continuous with points of discontinuity of the first kind at $t = t_k$ and $A(t)$ is left continuous at $t = t_k$, $k = 1, 2, \ldots$;

(**A**$_2$) B_k, $k = 1, 2, \ldots$, are $n \times n$ constant matrices;

(**A**$_3$) $f(t,x)\colon R_+ \times \Omega \to R^n$ is continuous in $(t_{k-1}, t_k] \times \Omega$, and nondecreasing in x, for every $x \in R^n$,

$$\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^+, x)$$

exist $k = 1, 2, \ldots$;

(**A**$_4$) The functions $I_k(x)\colon \Omega \to R^n$, $k = 1, 2, \ldots$, are continuous in Ω;

(**A**$_5$) $f(t,0) \equiv 0$, $t \in R_+$, $I_k(0) \equiv 0$, $k = 1, 2, \ldots$.

Now we give definitions for stability of systems with impulse effect which shall be further used.

Definitions. The zero solution of system $(*)$ is called:

(**a**) stable, if for each $\varepsilon > 0$ and $t_0 \in R_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $|x_0| < \delta$ implies $|x(t; t_0, x_0)| < \varepsilon$ for $t \ge t_0$;

(**b**) uniformly stable, if δ in (a) is independent of t_0;

(**c**) attractive, if for each $\varepsilon > 0$ and $t_0 \in R_+$ there exist $\delta = \delta(t_0) > 0$ and a $T = T(t_0, \varepsilon)$ such that $|x_0| < \delta$ implies $|x(t; t_0, x_0)| < \varepsilon$ for $t \ge t_0 + T$;

(**d**) uniformly attractive if δ and T in (c) are independent of t_0;

(**e**) asymptotically stable, if it is stable and attractive;

(**f**) uniformly asymptotically stable if it is uniformly stable and uniformly attractive.

3 – Main results

In this section we state the main results of this paper. The first lemma was proved in [5, Theorem 8.1].

Lemma 1. *Assume that* (A_1) *holds and (5) is the fundamental matrix of (1). Then the zero solution of (1) is uniformly stable if and only if there exists a constant* N *such that*

$$|W(t,s)| \le N \quad \text{for } \ 0 \le s \le t < \infty \ . \tag{6}$$

Lemma 2. *Let the following conditions be satisfied:*

- **i**) *For each* $k = 1, 2, ...,$ $g(t,u)\colon\ R_+ \times [0,h) \to R_+$ *is continuous in* $(t_{k-1}, t_k] \times [0,h)$ *and nondecreasing in* u*;*
- **ii**) *For each* $k = 1, 2, ...,$

$$\lim_{(t,v)\to(t_k^+,u)} g(t,v) = g(t_k^+, u)$$

and $g(t,0) \equiv 0$, $t \in R_+$*;*

- **iii**) *For each* $k = 1, 2, ...,$ $\psi_k(u)\colon\ R_+ \to R_+$ *is continuous and nondecreasing in* u.

Assume that $m(t) \in C(t_{k-1}, t_k]$, $k = 1, 2, ...,$ *satisfies the following inequality*

$$m(t) \le m(t_0) + \int_{t_0}^{t} g(s, m(s))\, ds + \sum_{t_0<t_k<t} \psi_k(m(t_k))$$

and $r(t)$ *is the maximal solution of*

$$\begin{aligned} &u' = g(t,u), \quad t \ne t_k\,, \qquad u(t_0) = u_0 \ge 0 \ , \\ &u(t_k^+) = \psi_k^*(u(t_k))\,, \qquad k = 1, 2, ... \ . \end{aligned}$$

existing on $[t_0, \infty)$, *where* $\psi_k^*(u) = u + \psi_k(u)$. *Then for* $t \ge t_0$,

$$m(t) \le r(t) \ ,$$

provided $m(t_0) \le u_0$.

Proof: Let

$$v(t) = m(t_0) + \int_{t_0}^{t} g(s, m(s))\, ds + \sum_{t_0 < t_k < t} \psi_k(m(t_k)) \ .$$

Then, by using the facts that $g(t,u)$ is nondecreasing in u and $m(t) \le v(t)$, $t \ge t_0$, we obtain

$$\begin{aligned} &v'(t) \le g(t, v(t)), \quad t \ne t_k\,, \qquad v(t_0) = m(t_0) \ge 0\ , \\ &v(t_k^+) \le v(t_k) + \psi_k(v(t_k)) = \psi_k^*(v(t_k))\ . \end{aligned}$$

By using Theorem 1.4.3 in [3], we get

$$m(t) \le r(t)\,, \qquad t \ge t_0\ .$$

The proof is complete. ∎

We shall say that condition (B) is satisfied if the following condition holds.

(**B**) The zero solution of system (1) is uniformly stable, that is, the condition (6) is satisfied.

Theorem 1. *Assume that conditions (A) and (B) hold and the following conditions are satisfied:*

i) *There is a constant η, $0 < \eta < h$, such that for any $t \in R_+$ and $x \in B(\eta)$*

$$|f(t,x)| \le g(t, |x|)\ ,$$

where $g(t,u)$ satisfies the conditions i) and ii) of Lemma 1.

ii) *For each $k = 1, 2, \ldots$ and $x \in B(\eta)$*

$$|I_k(x)| \le \psi_k(|x|)\ ,$$

where $\psi_k(u)$ satisfies the condition iii) of Lemma 1.

Then the stability properties of the zero solution of the differential equation

$$\begin{aligned} &\frac{du}{dt} = N g(t,u), \quad t \ne t_k\,, \qquad u(t_0) = u_0 \ge 0\ , \\ &u(t_k^+) = \overline{\psi}_k(u(t_k))\,, \qquad k = 1, 2, \ldots\ , \end{aligned} \tag{7}$$

where $\overline{\psi}_k(u) = u + N\psi_k(u)$, implies the corresponding stability properties of the zero solution of (2).

Proof: Let $t_0 \in R_+$, $x_0 \in B(\eta)$ and $x(t) = x(t; t_0, x_0)$ be the solution of (2). Then $x(t)$ satisfies the integral equation with impulse effect

$$x(t) = W(t, t_0^+)\, x_0 + \int_{t_0}^{t} W(t, s)\, f(s, x(s))\, ds + \sum_{t_0 < t_k < t} W(t, t_k^+)\, I_k(x(t_k))\,, \qquad t \geq t_0\ , \tag{8}$$

where $W(t, s)$, $t \geq s$, is defined by (5). By the condition (B) and (8), we obtain

$$|x(t)| \leq N|x_0| + \int_{t_0}^{t} N g(s, |x(s)|)\, ds + \sum_{t_0 < t_k < t} N \psi_k(|x(t_k)|)\ .$$

Let $u_0 \geq N|x_0|$ and $u(t) = u(t; t_0, u_0)$ be the solution of (7). From Lemma 1 it follows that

$$|x(t)| \leq u(t)\,, \qquad t \geq t_0\ . \tag{9}$$

We only prove the conclusion that if the zero solution of (7) is uniformly asymptotically stable, then the zero solution of (2) is also. In an analogous way the other stability properties of the zero solution of (2) can be proved.

First, we prove the zero solution of (2) is uniformly stable if the zero solution of (7) is uniformly stable. Let $\varepsilon > 0$ be given. It follows that there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that for any $t_0 \in R_+$ and $0 \leq u_0 < \delta_1$ we have

$$0 \leq u(t) < \varepsilon, \qquad \text{for all } \ t \geq t_0\ .$$

Set $\delta = \min(\delta_1/N, \eta)$. For any $|x_0| < \delta$ choose u_0 with $N|x_0| \leq u_0 < \delta_1$, then from (9) we obtain

$$|x(t)| < \varepsilon, \qquad \text{for all } \ t \geq t_0\ ,$$

which implies that the zero solution of (2) is uniformly stable.

Using similar arguments we can prove that if the zero solution of (7) is uniformly attractive, then the zero solution of (2) is also uniformly attractive.

The proof is complete. ∎

Theorem 2. *Assume that conditions (A) and (B) hold and the following conditions are satisfied:*

i) *There exists $\eta > 0$ ($\eta < h$) such that for any $t \in R_+$ and $x \in B(\eta)$*

$$|f(t, x)| \leq a(t)\, g(|x|)\ ,$$

where

a) *$a(t)\colon R_+ \to R_+$, is piecewise continuous on R_+ with points of discontinuity of the first kind at $t = t_k$ and $a(t_k) = a(t_k^-)$, $k = 1, 2, \ldots$;*

b) *$g(u)\colon R_+ \to R_+$, is continuous and nondecreasing, positive, submultiplicative on $(0,\infty)$, and $g(0) \equiv 0$.*

ii) *There exists the sequence $\{\beta_k\}$ such that*

$$|I_k(x)| \le \beta_k |x| \,, \qquad k = 1, 2, \ldots \,.$$

iii) *There is a constant M such that for all $t_0 \in R_+$ and $u_0 \in (0,\eta)$*

$$G^{-1}\Big[G\Big(\prod_{t_0<t_k<\infty}(1{+}N\beta_k)\Big) + \frac{g(u_0)}{u_0}\int_{t_0}^{\infty} N \prod_{s<t_k<\infty} \frac{1{+}N\beta_k}{g(1{+}N\beta_k)}\, a(s)\, ds\Big] < M \,, \tag{10}$$

where $G(u) = \int_{u_0}^{u} \frac{ds}{g(s)}$, $u_0 > 0$ and $G(\infty) = \infty$, $G(0^+) = -\infty$, $G^{-1}(u)$ is the inverse of $G(u)$.

Then the zero solution of (2) is uniformly stable.

Proof: First, we show that the zero solution of the differential equation with impulse effect

$$\begin{aligned} &\frac{du}{dt} = N a(t)\, g(u), \quad t \ne t_k \,, \qquad u(t_0) = u_0 > 0 \,, \\ &\Delta u(t_k) = N\beta_k\, u(t_k) \,, \qquad k = 1, 2, \ldots \,, \end{aligned} \tag{11}$$

is uniformly stable. Let $0 < u_0 < \eta$ and $u(t) \equiv u(t; t_0, u_0)$ be a solution of (11). It then follows from standard arguments that $u(t)$ exists on $[t_0, \infty)$. Consequently

$$u(t) = u_0 + \int_{t_0}^{t} N a(s)\, g(u(s))\, ds + \sum_{t_0<t_k<t} N\beta_k\, u(t_k) \,.$$

Hence for $t > t_0$

$$\frac{u(t)}{u_0} \le 1 + \int_{t_0}^{t} N a(s)\, \frac{g(u_0)}{u_0}\, g\Big(\frac{u(s)}{u_0}\Big) ds + \sum_{t_0<t_k<t} N\beta_k\, \frac{u(t_k)}{u_0} \,.$$

By using Theorem 1.5.5 in [3] we obtain that

$$\frac{u(t)}{u_0} \le G^{-1}\Big[G\Big(\prod_{t_0<t_k<t}(1 + N\beta_k)\Big) + \int_{t_0}^{t}\Big(\prod_{s<t_k<t} \frac{1 + N\beta_k}{g(1 + N\beta_k)}\Big) N a(s)\, \frac{g(u_0)}{u_0}\, ds\Big] \,.$$

Using condition iii) we have

$$u(t) \le u_0\, M \,, \qquad t \ge t_0 \,,$$

which implies that the zero solution of (11) is uniformly stable. Then by using Theorem 1 we conclude that the zero solution of (2) is uniformly stable. ∎

Corollary. *Assume that the conditions of Theorem 2 be fulfilled, condition (10) being replaced by the conditions*

i) $\displaystyle\prod_{u=1}^{\infty}(1+N\beta_k)<\infty,$

ii) $\displaystyle\prod_{k=1}^{\infty}\frac{(1+N\beta_k)}{g(1+N\beta_k)}<\infty,$

iii) $\displaystyle\int_0^{\infty}a(s)\,ds<\infty,$

and for $u\in(0,\eta)$, $g(u)\le u$.

Then the zero solution of (2) is uniformly stable.

Proof: From conditions i)–iii) and $g(u)\le u$ it follows that condition (10) holds. Thus by Theorem 2 we find that the zero solution of (2) is uniformly stable. ∎

Example: Consider the Ricatti scalar equation with impulse effect

$$\begin{aligned}&\frac{dx}{dt}=p(t)\,x(t)+a(t)\,x^2(t),\quad t\neq t_k\,,\qquad x(t_0)=x_0\,,\\&\Delta x=\beta_k\,x(t)\,,\qquad t=t_k,\quad k=1,2,\ldots\,,\end{aligned}\tag{12}$$

where $p\colon R_+\to R$, $a\colon R_+\to R$, $\beta_k\ge 0$; p and a are continuous on $(t_{k-1},t_k]$, $k=1,2,\ldots$, with points of discontinuity of first kind at $t=t_k$, $k=1,2,\ldots$; $\{t_k\}$ is increasing sequence and $\lim_{k\to\infty}t_k=\infty$. If $\sum_{k=1}^{\infty}\beta_k<\infty$ and there is a constant N such that for any $t_0\in R_+$, $|\int_{t_0}^{\infty}p(s)\,ds|\le N$, and $\int_{t_0}^{\infty}|a(s)|\,ds<\infty$, then by using Corollary it follows that the zero solution of (12) is uniformly stable.

In fact, here $g(u)=u^2\le u$ if $0\le u<1$. It is easy to show that $\sum_{k=1}^{\infty}\beta_k<\infty$ implies that $\prod_{k=1}^{\infty}(1+N\beta_k)<\infty$ and $\prod_{k=1}^{\infty}\frac{1}{1+N\beta_k}=\prod_{k=1}^{\infty}(1-\frac{N\beta_k}{1+N\beta_k})<\infty$. Thus all conditions of Corollary satisfied.

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Jurang Yan,
Department of Mathematics, Shanxi University,
Taiyuan, Shanxi, 030006 – P.R. CHINA