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ON THE LOCALLY m-CONVEX ALGEBRA $L_\Gamma(E)$ AND A DIFFERENTIAL-GEOMETRIC INTERPRETATION OF IT

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0 – Introduction

The purpose of the present paper is to supply the algebra $L_\Gamma(E)$ of continuous endomorphisms of a given locally convex space E with a suitable locally m-convex topology (Theorem 1.1), a theme already treated by E.A. Michael [4] in a, say, geometric manner by looking at an appropriate local basis of the algebra at issue. We take here, instead, a rather "arithmetic" point of view, by giving a pertinent family $\overline{\Gamma} = \{\overline{p}\}$ of submultiplicative seminorms on $L(E)$, starting from a standard family $\Gamma = \{p\}$ of seminorms for the given locally convex space E. Apart from the obvious practical utility of the latter point of view, one finds a further justification by getting an improvement of Michael's result, referring to an inverse of the main theorem, as before (cf. Theorem 1.2).

On the other hand, by using this locally m-convex algebra structure of $L(E)$, one can define for complete E the exponential function (see e.g. A. Mallios [2]), which can be, of course, at the basis of the differential-geometric interpretation of $L(E) \equiv L_\Gamma(E)$, alluded to at the heading, as well as, of any locally m-convex algebra, in general (see Remark 4.1).

1 – The algebra $L_\Gamma(E)$

To start with, suppose that E is a $\mathbb{C}$-*vector space* and let $p\colon E \to \mathbb{R}_+$ be a seminorm on E. So denoting by $L(E)$ the $\mathbb{C}$-vector space of $\mathbb{C}$-*linear endomor-*

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phisms of E, one defines the *p-bound* of an element $T \in L(E)$ by the relation

$$\overline{p}(T) := \sup\left\{\frac{p(T(x))}{p(x)} : p(x) \neq 0\right\} \tag{1.1}$$

(whenever this exists). We also call $T \in L(E)$, for which $\overline{p}(T) < +\infty$, *p-bounded*. Thus, we first have the following lemma, whose proof is standard.

Lemma 1.1. *Keeping the above notation, suppose that* $S, T \in L(E)$, *with* $\overline{P}(S)$, $\overline{p}(T) < +\infty$. *Then, one has the following relations:*

$$(2) \qquad \begin{aligned} &(1)\ \ \overline{p}(Tx) \leq \overline{p}(T) \cdot \overline{p}(x)\,, \qquad x \in E\ , \\ &(2)\ \ \overline{p}(\lambda T) = |\lambda| \cdot \overline{p}(T)\,, \qquad \lambda \in \mathbb{C}, \quad p \in \Gamma\ , \\ &(3)\ \ \overline{p}(T+S) \leq \overline{p}(T) + \overline{p}(S)\ , \\ &(4)\ \ \overline{p}(T \circ S) \leq \overline{p}(T)\,\overline{p}(S)\ . \end{aligned}$$

That is, $\overline{p}$ *yields a submultiplicative seminorm on that part of* $L(E)$, *for which* $\overline{p} < +\infty$, *satisfying then (1) as well.*

Proof: The only thing we have to vindicate is prop. (1) in the case that, for some $x \in E$, $p(x) = 0$. Then, *one proves that* $p(Tx) = 0$, as well; indeed, if $p(y) \neq 0$, for some $y \in E$, one has (by considering $z_n := x + \frac{1}{n}\,y$, $n \in \mathbb{N}$):

$$p(Tx) \leq 2\frac{1}{n}\,\overline{p}(T)\,p(y)\,, \qquad n \in \mathbb{N}\ , \tag{1.2}$$

which proves the assertion. ∎

Of course, prop. (1) in the previous lemma *characterizes the p-continuity* of a *p-bounded* operator $T \in L(E)$. Thus, denoting by

$$L(E,p) \equiv L_p(E) \tag{1.3}$$

the *p-bounded* elements of $L(E)$, one has, by Lemma 1.1, that

(1.4) $L_p(E)$ *is a locally* m*-convex algebra,*
in fact, a seminormed one.

Hence, one obtains.

Theorem 1.1. *Let* $(E, \Gamma \equiv \{p\})$ *be a given locally convex space. Then,*

$$L(E) \equiv L_\Gamma(E) := \bigcap_{p \in \Gamma} L_p(E) \subseteq L(E) \tag{1.5}$$

is a unital locally m-convex algebra in the topology defined by the family of submultiplicative seminorms (cf. (1.1))

$$\overline{\Gamma} \equiv \{\overline{p}\}_{p\in\Gamma} \; . \tag{1.6}$$

In particular, $L_\Gamma(E)$ is (Hausdorff) complete, if E is a complete (Hausdorff) locally convex space.

Proof: The first part of the assertion is obviously true, according to (1.4). On the other hand, *if E is Hausdorff* and $\overline{p}(T) = 0$, for some $T \in L(E)$, for every $p \in \Gamma$, then by (1.1), $p(Tx) = 0$, *for any $p \in \Gamma$ and $x \in E$*, so that by hypothesis $Tx = 0$, hence, *$L_\Gamma(E)$ is Hausdorff* too. Now, suppose that

$$(T_\delta)_{\delta\in I} \subseteq L_\Gamma(E) \equiv L(E) \tag{1.7}$$

is a *Cauchy net*; viz. for every $\varepsilon > 0$ and $p \in \Gamma$ (cf. (1.6)), there exists $\delta_0(\varepsilon, p) \equiv \delta_0 \in I$, such that

$$\overline{p}(T_\delta - T_{\delta'}) \le \varepsilon, \quad \text{with} \;\; \delta, \delta' \ge \delta_0 \; . \tag{1.8}$$

Thus, *for every $x \in E$*, one has

$$p(T_\delta x - T_{\delta'} x) = p((T_\delta - T_{\delta'})\, x) \le \overline{p}(T_\delta - T_{\delta'}) \cdot p(x) \le \varepsilon\, p(x) \tag{1.9}$$

(cf. also (1.1) and Lemma 1.1), that is, $(T_\delta x) \subseteq E$ *is a Cauchy net*, so that, by the hypothesis for E,

$$T(x) \equiv \lim_\delta T_\delta x \;\in\; E \; . \tag{1.10}$$

We prove that *the operator T*, as given by (1.10), *is p-bounded, for every $p \in \Gamma$*, viz. $T \in L_\Gamma(E)$ such that $T = \lim_\delta T_\delta$.

Indeed, given $p \in \Gamma$, one has

$$\begin{aligned} p(T_{\delta_0} x - Tx) &= p(T_{\delta_0} x - \lim_\delta T_\delta x) = \lim_\delta \Big(p(T_{\delta_0} x - T_\delta x)\Big) \equiv \\ &\equiv \lim_\delta \Big(p(T_{\delta_0} - T_\delta) x\Big) \le \lim_\delta \Big(\overline{p}(T_{\delta_0} - T_\delta) \cdot p(x)\Big) = p(x) \cdot \lim_\delta \overline{p}(T_{\delta_0} - T_\delta) \le \varepsilon \cdot p(x) \; , \end{aligned} \tag{1.11}$$

for any $\delta \ge \delta_0(\varepsilon, p)$, as in (1.8). Thus, due to the arbitrariness of $x \in E$ in (1.11), by (1.1),

$$\overline{p}(T_{\delta_0} - T) \le \varepsilon \; , \tag{1.12}$$

hence, $T_{\delta_0} - T \in L_\Gamma(E)$, while the same relation (1.12) proves that $T = \lim_\delta T_\delta$, as desired, and this terminates the proof. (The relation (1.12) remains true if, in place of δ_0, we put any $\delta' \geq \delta_0$). ■

Thus, suppose now that E is a *(complete) locally convex space* so that, according to Theorem 1.1, $L_\Gamma(E) \equiv L(E)$ is a *(complete) locally m-convex algebra.* So it admits the so-called *Arens–Michael decomposition* (cf., for instance, A. Mallios [1, p. 88, Theorem 3.1]), that is *one has*

$$L(E) = \varprojlim L(E)_p \ , \tag{1.13}$$

such that one has

$$L(E)_p := \widehat{L(E)/\ker \overline{p}} \ , \tag{1.14}$$

for every $p \in \Gamma$ (see (1.1), (1.6)). On the other hand, the *complete locally convex space* E is of the form

$$E = \varprojlim E_p \ , \tag{1.15}$$

where

$$E_p := \widehat{E/\ker p} \ , \tag{1.16}$$

for every $p \in \Gamma$. Thus, the latter space being, by definition, a *Banach space*, one can further consider the *Banach algebra*

$$L(E_p) \, , \qquad p \in \Gamma \ . \tag{1.17}$$

In this context, we further remark that one has

$$L(E)_p \underset{\phi_p}{\underset{\rightarrow}{\subseteq}} L(E_p) \, , \qquad p \in \Gamma \ , \tag{1.18}$$

within a *Banach algebra isomorphism* (into). Indeed, the said isomorphism (being also a *topological* one for the respective uncompleted spaces (cf. (1.14) and (1.16)) is given by the relation

$$\widetilde{u}_p(x + \ker p) := u(x) + \ker p \ , \tag{1.19}$$

for any $u \in L(E)$ *and* $p \in \Gamma$ (see also Lemma 1.1).

In sum, one has the relation (cf. also (1.13) and (1.18))

$$L(E) = \varprojlim L(E)_p = \varprojlim \phi_p(L(E)_p) \underset{\rightarrow}{\subseteq} \prod_{p \in \Gamma} L(E_p) \ . \tag{1.20}$$

For convenience and for later applications, we still note that if

$$u \in L(E) \ , \tag{1.21}$$

then, by virtue of (1.20), one has

$$u = \varprojlim u_p \ , \tag{1.22}$$

where we set (cf. (1.14))

$$u_p = \phi_p(u + \ker \overline{p}) \, , \qquad p \in \Gamma \ . \tag{1.23}$$

Scholium 1.1. Concerning the above topologization of the algebra $L_\Gamma(E)$, which is already contained in the author's thesis [6], it was actually conceived independently of E. Michael's treatment [4] of the same notion, the latter author having given a "geometric" aspect of the same topology, in terms, namely, of neighborhoods of zero. The analytic terminology adopted herewith, as in (1.1), had among other things the advantage of giving a genuine inverse of Theorem 1.1 improving, thus, the analogous result of Michael (ibid. p. 19, Proposition 4.4). See also *Theorem 1.2* below.

As already said, we have the following inverse of Theorem 1.1. Namely, one gets at the next.

Theorem 1.2. *Let $(A, \Gamma = (p))$ be a locally m-convex algebra. Then, there exists a topological algebraic isomorphism of A into an algebra of the form $L_\Gamma(E)$, for a locally convex space E.*

Proof: By considering the *unitization of* A, $A^+ \equiv A \oplus \mathbb{C}$, we first prove our assertion for A^+. Thus, taking now $E = A^+$, we further look at the (canonical) left representation of A^+, $x \mapsto \ell(x) \equiv \ell_x$, $x \in A^+$, such that (cf. (1.1))

$$\overline{p}(\ell_x) = p(x) \, , \qquad x \in A^+ \ . \tag{1.24}$$

Hence, $\ell_x \in L(A^+) \equiv L(E)$ (see also (1.5)). On the other hand, one has, of course, the relation

$$A \underset{\rightarrow}{\subseteq} A^+ \underset{\rightarrow}{\subseteq} L(A^+) \ , \tag{1.25}$$

within *topological algebra isomorphisms*, and this completes the proof. ∎

2 – The exponential function

We have already established in the previous section that $L(E)$ is a *locally m-convex algebra*, for any given locally convex space E, in the topology defined by (1.1), which, moreover, is *complete*, whenever E is so.

In this context, it is enough to suppose, for the applications we have in mind, that E is just *σ-complete*. Thus, applying an analogous argument as before (cf. Theorem 1.1), one proves that $L(E)$ is a *σ-complete locally m-convex algebra*. Hence, one defines in $L(E)$ *the exponential function*, by the relation

$$\exp u := \sum_{n=0}^{\infty} \frac{1}{n!} \cdot u^n \ , \tag{2.1}$$

for every $u \in L(E)$. (In this regard, see also A.Mallios [3: p.196; (8.5)]).

On the other hand, by considering the locally convex space $(E, \Gamma \equiv \{p\})$, as before, each one of the respective *Banach algebras* $L(E_p)$, $p \in \Gamma$ (see (1.17)) has a corresponding *exponential function* $\exp_p$, given by

$$\exp_p(u_p) := \sum_{n=0}^{\infty} \frac{1}{n!} \cdot u_p^n \ , \tag{2.2}$$

which also, for simplicity and by a slight abuse of notation, we denote occasionally, just by $\exp(u_p)$, for $u_p \in L(E_p)$. Thus, our next objective is to show that one has (supposing always that E is *σ-complete*)

$$\exp = \varprojlim \exp_p \ . \tag{2.3}$$

Indeed, by the very definitions, one proves first that $(\exp_p)_{p\in\Gamma}$, as given by (2.2), is a *projective system of maps*, which thus guarantees the existence of the second member of (2.3). Furthermore, the later map is actually exp, as this is given by (2.1), according to the relation

$$\exp_p \circ \rho_p = \rho_p \circ \exp_p \ , \tag{2.4}$$

where $\rho_p \colon L(E) \to L(E)_p$, $p \in \Gamma$, stands for the canonical "projection", derived from the corresponding *Arens–Michael decomposition of* $L(E)$.

3 – Differentiable curves

Suppose we have a *locally convex space* $(E, \Gamma \equiv (p))$ and let

$$\alpha \colon I \to E \tag{3.1}$$

be a *curve* in E, where I is an (open) neighborhood of 0 in $\mathbb{R}$.

Now, we say that α is *differentiable at* $t_0 \in I$, whenever the following limit exists (in E)

$$\dot{\alpha}(t_0) \equiv \lim_{t \to t_0} \frac{1}{t - t_0} \Big(\alpha(t) - \alpha(t_0)\Big) \; . \tag{3.2}$$

We call (3.2) the *derivative of* α *at* t_0, while we say that α is *differentiable in* I, if this happens for every $t \in I$, as before, in which case we denote by $\dot{\alpha}\colon I \to E$ the corresponding *derivative of* α. Of course, as follows from the very definitions, *every differentiable curve* α, as above, *is continuous.*

In this context, the following is already an easy consequence of the very definitions.

Lemma 3.1. *Given a locally convex space* $(E, \Gamma \equiv (p))$, *a curve* $\alpha\colon I \to E$ *is differentiable if, and only if, this is the case for every curve (cf. (1.16))*

$$\alpha_p \equiv \pi_p \circ \alpha\colon\; I \to E_p\,, \qquad p \in \Gamma \;, \tag{3.3}$$

where $\pi_p\colon E \to E_p$ *stands for the canonical projection map.* ∎

As a consequence of the proof of Lemma 3.1, one sees that (cf. (3.3))

$$\dot{\alpha} = (\dot{\alpha}_p \equiv \pi_p \circ \dot{\alpha})_{p \in \Gamma} \; . \tag{3.4}$$

Thus, we are now in the position to give our main result of this section, that runs as follows.

Theorem 3.1. *Let* $(E, \Gamma \equiv (p))$ *be a* σ*-complete locally convex space and* $L_\Gamma(E)$ *the corresponding* σ*-complete locally* m*-convex algebra, as given by (1.5). Then, for every* $u \in L_\Gamma(E)$, *one finds a solution (curve* α*, cf. (3.1)) of the differential equation*

$$\dot{\alpha} = u\,\alpha \tag{3.5}$$

(where one gets $\dot{\alpha}(t) = u\,\alpha(t) \equiv u(\alpha(t))$, $t \in I$*). This solution is unique, under the requirement that* $\alpha(0) = x \in E$.

Proof: Based on the Arens–Michael deconposition of the locally m-convex algebra $L_\Gamma(E)$ (cf. (1.20)), the problem is reduced to the analogous one for the corresponding (for each $p \in \Gamma$) factor Banach algebra $L(E)_p \underset{\to \phi_p}{\subseteq} L(E_p)$

(cf. (1.18)); now the solution is obtained in a standard way, through the *exponential function* (see, for instance, J. Nieto [5: p.29, (I)]). Then, one obtains (see also (3.1))

$$\alpha(t) = (\exp tu)(x)\,, \qquad t \in I\ , \tag{3.6}$$

where, of course, $\alpha(0) = x \in E$. (In this regard, one applies the continuity of the (canonical) projection maps in (1.15), along with (3.2) and the fact that $\dot\alpha(t) \in E \underset{\rightarrow}{\subseteq} \underset{\longleftarrow}{\lim} E_p$ (see (3.4)). The uniqueness is now easily reduced (coordinate-wise) to the factors of the respective Arens–Michael decomposition, where one can then follow J. Nieto [5]. ∎

Scholium 3.1. Following an analogous argument, as before, one can further extend, within the present context, several other relevant results of J. Nieto [5]. Thus, we can also discuss solutions of the equation

$$\dot\alpha(t) = u(t)\cdot\alpha(t)\,, \qquad t \in I\ , \tag{3.7}$$

(cf. also (3.6)). Such type of results are given in [6].

4 – A differential-geometric interpretation of $L(E)$

Suppose we have again a *σ-complete locally convex space* E, so that $L(E)$ (see (1.5)) is a *σ-complete locally m-convex algebra*. As the heading indicates, our objective in the sequel is to express the algebra $L(E)$ as the *orbit space* of suitable group actions, according to standard differential-geometric arguments.

So we denote by

$$GL(E) \equiv L(E)^* \subseteq L(E) \tag{4.1}$$

the *group of invertible elements of* $L(E)$, viz. the set of topological vector space automorphisms of the locally convex space E, the same set being, in view of the sort of the topology of $L(E)$, a *topological group*. Now, one can consider three types of *action* of $GL(E)$; that is, first one has

$$GL(E) \times GL(E) \to GL(E)\ , \tag{4.2}$$

such that

$$(u, v) \mapsto v \circ u \equiv v\,u = r_u(v)\ , \tag{4.3}$$

for any u, v *in* $GL(E)$, while (4.3) satisfies, of course, the usual axioms for a *group action*. We thus obtain a *topological group action of* $GL(E)$ *on itself* (by *right* "translations"; similarly, one defines a "*left action*"). Furthermore, one defines the following *topological* (viz. *continuous*) *actions of* $GL(E)$

$$GL(E) \times L(E) \to L(E) \ , \tag{4.4}$$

such that

$$(\alpha, u) \mapsto \alpha \circ u \equiv \alpha \cdot u \ , \tag{4.5}$$

for any $\alpha \in GL(E)$ *and* $u \in L(E)$, and finally

$$GL(E) \times E \to E \ , \tag{4.6}$$

where we define

$$(\alpha, x) \mapsto \alpha(x) \ , \tag{4.7}$$

for any $\alpha \in GL(E)$ *and* $x \in E$. (It is clear that (4.4) *is continuous*, being a particular case of (4.2), while the *continuity of* (4.7) follows easily from Lemma 1.1, prop.(1)).

Now given an element $u \in L(E)$, one obtains the *curve* in $GL(E) \subseteq L(E)$,

$$\alpha_u \colon \mathbb{R} \to GL(E) \quad : \quad t \mapsto \alpha_u(t) := \exp(t\,u) \ , \tag{4.8}$$

such that, by considering it in $L(E)$, its derivative is given by

$$\dot{\alpha}_u(t) = u \cdot \exp(tu) = u \cdot \alpha_u(t) \ , \tag{4.9}$$

with $t \in \mathbb{R}$, or even (formally)

$$\dot{\alpha}_u = u \cdot \alpha_u \ . \tag{4.10}$$

Remark 4.1. The above argument concerning the algebra $L(E)$, (cf. Theorem 3.1), can be formulated, more generally, for any *σ-complete locally m-convex algebra E* and its *group of invertible elements E^**.

Thus, based on the preceding Remark 4.1 and the above, we can give the following differential-geometric interpretation of the last argument. That is, one has the next.

Proposition 4.1. *Suppose we have a unital σ-complete locally m-convex algebra E, with E^* its group of units. Moreover, let*

$$\mathrm{Hom}^\infty(\mathbb{R}, E^*) \tag{4.11}$$

be the set of differentiable curves in E^*, that are also morphisms of the groups concerned. Then one has

$$E = \mathrm{Hom}^\infty(\mathbb{R}, E^*) \equiv T(E^*, 1) \ , \tag{4.12}$$

within a bijection (the last item in (4.12) denoting the "tangent space" of E^* at $1 \in E^*$).

Proof: Given $x \in E$, consider the differentiable curve

$$\alpha_x \colon \mathbb{R} \to E^* \quad : \quad t \mapsto \alpha_x(t) := \exp(tx) \tag{4.13}$$

(see also, for instance, (4.9)), which is also a morphism of the groups concerned. Furthermore, since $\dot{\alpha}_x(0) = x$ (the same rel. (4.9)), one easily proves that *the correspondence $x \mapsto \alpha_x$*, as given by (4.13), *is one-to-one*. On the other hand, based on the *uniqueness of the solution curves of* (3.5) (see also Remark 4.1, or even (1.25)), one finally proves that *the same correspondence is onto*, as well, and the proof is complete. ∎

Concerning the above result, we further remark that *the same set, as in (4.11), can also be viewed as that one of "left invariant vector fields"* on E^*. Thus to every $x \in E$, one associates, through the (canonical) *right regular representation* of E, the operator $r_x \in L_\Gamma(E)$, which can be thought of as *left invariant*, according to the relation,

$$r_x \circ \ell_y = \ell_y \circ r_x \ , \tag{4.14}$$

for any $x, y \in E$. (In this context, we finally note that an analogous study to the preceding one can also be given for *non-unital algebras*, by employing an appropriate interpretation of the *exponential function* via the *circle operation*; see thus, for instance, [7], [8]).

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