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REDUCTION OF COMPLEX POISSON MANIFOLDS

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Abstract: In this paper we define the reduction of complex Poisson manifolds and we present a reduction theorem. We give an example of reduction on the dual of a complex Lie algebra with its complex Lie–Poisson structure. In this example the reduction is obtained by the action of a complex Lie subgroup of $SL(2,\mathbb{C})$ on $sl^*(2,\mathbb{C})$. Finally, we establish a relationship between complex and real Poisson reduction.

Introduction

The notion of reduction has been formulated in terms of the modern differential geometry by J. Sniatycki and W. Tulczyjew [11], in the case of real symplectic manifolds. J. Marsden and A. Weinstein [7] stated a famous theorem concerning the real symplectic reduction, when a Lie group acts on a symplectic manifold with a Hamiltonian action, having an equivariant momentum map.

In the case of real differential manifolds, reduction methods have been established for Poisson manifolds by J. Marsden and T. Ratiu [6], for contact and cosymplectic manifolds by C. Albert [1] and for Jacobi manifolds by K. Mikami [8] and J.M. Nunes da Costa [9].

The notion of complex Poisson structure, defined on a complex manifold, was introduced by A. Lichnerowicz [5].

The aim of this paper is to show that we can also establish a notion of reduction for the complex Poisson manifolds.

In sections 1 and 2 we recall some definitions and properties concerning the almost complex structure of a manifold, the Schouten bracket [10] and the complex Poisson manifolds.

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Section 3 is devoted to the reduction of complex Poisson manifolds. We prove a reduction theorem containing the necessary and sufficient condition for a submanifold of a complex Poisson manifold to inherit a reduced complex Poisson structure.

Using the same procedure as in the real case, A. Lichnerowicz [5] has shown that the dual of a finite-dimensional complex Lie algebra carries a complex Poisson structure. In section 4 we present an example where the complex Poisson manifold is $sl^*(2,\mathbb{C})$ and the reduction is due to the action of a complex Lie group on this complex Lie algebra.

In the last section we study the relationship between the real and the complex Poisson reduction.

1 – Almost complex structure of a manifold

We recall briefly some definitions concerning the almost complex structure of an analytic complex manifold (see [2], [3], [4] and [5]).

Let M be an analytic complex manifold of (complex) dimension m and let J be its operator of almost complex structure. We consider an analytic complex atlas on M such that if $(z^\alpha)_{\alpha=1,\ldots,m}$ is a system of local complex coordinates, then

$$z^\alpha = \frac{1}{\sqrt{2}}(x^\alpha + i\,x^{\overline{\alpha}})\,, \qquad \overline{\alpha} = \alpha + m\ ,$$

where the $2m$ real numbers (x^k) are the real local coordinates associated with the complex coordinates (z^α).

We denote by $(T_xM)^c$ (resp. $(T^*_xM)^c$) the complexification of the tangent space (resp. cotangent) to M on $x \in M$. Let

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial x^{\overline{\alpha}}}\right) \qquad \left(\text{resp. } dz^\alpha = \frac{1}{\sqrt{2}}(dx^\alpha + i\,dx^{\overline{\alpha}})\right)$$

and denote by $\frac{\partial}{\partial z^{\overline{\alpha}}}$ (resp. $dz^{\overline{\alpha}}$) the complex conjugate. For every $x \in M$, the m vectors $\frac{\partial}{\partial z^\alpha}(x)$ (resp. $dz^\alpha(x)$) generate a m-dimensional complex vector subspace of $(T_xM)^c$ (resp. $(T^*_xM)^c$) which we denote by $(T_xM)^{(1,0)}$ (resp. $(T^*_xM)^{(1,0)}$). On the other hand, the m vectors $\frac{\partial}{\partial z^{\overline{\alpha}}}(x)$ (resp. $dz^{\overline{\alpha}}(x)$) generate a m-dimensional complex vector subspace of $(T_xM)^c$ (resp. $(T^*_xM)^c$) which we denote by $(T_xM)^{(0,1)}$ (resp. $(T^*_xM)^{(0,1)}$). So, we have the following direct sums

$$(T_xM)^c = (T_xM)^{(1,0)} \oplus (T_xM)^{(0,1)} \quad \text{and} \quad (T^*_xM)^c = (T^*_xM)^{(1,0)} \oplus (T^*_xM)^{(0,1)}\ .$$

The linear operator J_x on T_xM can be extended to the whole $(T_xM)^c$ and has the eigenvalues i and $-i$. The vector subspace $(T_xM)^{(1,0)}$ (resp. $(T_xM)^{(0,1)}$) of $(T_xM)^c$ is generated by the eigenvectors associated with the eigenvalue i (resp. $-i$).

A vector $v \in (T_xM)^c$ of components $(v^\alpha, v^{\overline{\alpha}})$, with $v^{\overline{\alpha}} = \overline{v^\alpha}$, is called a *real vector* and, if v is a real vector, $J_x(v)$ is also a real vector.

2 – Schouten bracket and complex Poisson manifolds

For each $x \in M$, the decompositions $(T_xM)^c = (T_xM)^{(1,0)} \oplus (T_xM)^{(0,1)}$ and $(T_x^*M)^c = (T_x^*M)^{(1,0)} \oplus (T_x^*M)^{(0,1)}$ of $(T_xM)^c$ and $(T_x^*M)^c$, allow us to introduce the notion of *type* for the complex tensor fields. So, given a skew symmetric contravariant complex tensor field T of order t (briefly a t-tensor) on M we can write T as a sum of t-tensors $T^{(p,q)}$ of type (p,q), with $p+q=t$. A t-tensor of type $(t,0)$ is called a *holomorphic* t-tensor if its components are holomorphic functions in all local complex charts.

If Λ is a real 2-tensor on M, the following decomposition stands:

$$\Lambda = \Lambda^{(2,0)} + \Lambda^{(1,1)} + \Lambda^{(0,2)} ,$$

with $\Lambda^{(0,2)} = \overline{\Lambda^{(2,0)}}$. Besides, we suppose that $\Lambda^{(1,1)} = 0$, which is equivalent to the fact that the Poisson bracket of a holomorphic function on M and an antiholomorphic function on M locally vanishes (cf. [5]).

On the space of skew symmetric contravariant complex tensor fields on M, we consider the Schouten bracket [10], whose properties are similar to the real case. If R is a r-tensor and T is a t-tensor, then the Schouten bracket $[R,T]$ of R and T is a $(r+t-1)$-tensor. In particular, if R and T are holomorphic, then $[R,T]$ is also holomorphic.

Definition 1 ([5]). Let M be a connected paracompact analytic complex manifold and let $\Lambda^{(2,0)}$ be a 2-tensor on M with $\Lambda^{(0,2)} = \overline{\Lambda^{(2,0)}}$. The couple $(M, \Lambda^{(2,0)})$ is called a *complex Poisson manifold* if

$$[\Lambda^{(2,0)}, \Lambda^{(2,0)}] = 0 \quad \text{and} \quad [\Lambda^{(2,0)}, \Lambda^{(0,2)}] = 0 .$$

Remark 1. If $\Lambda^{(2,0)}$ is a holomorphic 2-tensor, we always have $[\Lambda^{(2,0)}, \Lambda^{(0,2)}] = 0$.

Let $(M, \Lambda^{(2,0)})$ be a complex Poisson manifold. If we consider the differentiable structure of M, underlying to its analytic complex structure, the real 2-tensor

$\Lambda = \Lambda^{(2,0)} + \Lambda^{(0,2)}$ (with $\Lambda^{(0,2)} = \overline{\Lambda^{(2,0)}}$) defines a real Poisson manifold structure on M.

Conversely, if the real 2-tensor Λ defines a real Poisson structure on the analytic complex manifold M and if $\Lambda = \Lambda^{(2,0)} + \Lambda^{(0,2)}$, then $(M, \Lambda^{(2,0)})$ is a complex Poisson manifold.

Associated with the real 2-tensor Λ, there exists a morphism of (real) vector bundles,

$$\Lambda^{\#}\colon\, T^*M \to T\,M \ ,$$

given by $\langle \beta, \Lambda^{\#}(\alpha)\rangle = \Lambda(\alpha, \beta)$, with α and β elements of the same fibre of T^*M. Analogously, there exists a morphism of complex vector bundles, associated with $\Lambda^{(2,0)}$,

$$(\Lambda^{(2,0)})^{\#}\colon\, (T^*M)^{(1,0)} \to (T\,M)^{(1,0)} \ ,$$

defined in a similar way.

3 – Reduction of complex Poisson manifolds

Let N be a paracompact analytic complex submanifold of the analytic complex manifold M, with complex dimension n ($n \leq m$). We denote by $T_N M$ the (real) tangent bundle $T\,M$ of M restricted to N.

Let F be a vector subbundle of $T_N M$, that verifies the two following properties:

i) for all $x \in N$, $J_x(F_x) \subset F_x$;

ii) $F \cap T\,N$ is a completely integrable (real) vector subbundle of the tangent bundle of N, which defines a simple foliation of N; the set $\widehat{N}$ of the leaves determined by $F \cap T\,N$ is a differentiable manifold and the canonical projection $\pi\colon\, N \to \widehat{N}$ is a submersion.

Proposition 1. *If conditions i) and ii) hold, then $\widehat{N}$ has the structure of a complex manifold.*

Proof: We only have to show that $\widehat{N}$ is an almost complex manifold whose torsion vanishes.

Let S be a leaf of the foliation of N determined by $F \cap T\,N$. By condition i), for every $x \in S$, $J_x(T_x S) \subset T_x S$ and so, we may define a map

$$\widehat{J}_{\pi(x)}\colon\, T_{\pi(x)}\widehat{N} \to T_{\pi(x)}\widehat{N} \ ,$$

such that $\widehat{J}_{\pi(x)} \circ T_x\pi = T_x\pi \circ J_x$. This map establishes an almost complex structure on $\widehat{N}$.

Since N is a paracompact analytic complex manifold, its torsion vanishes. The torsion of $\widehat{N}$ also vanishes because it is the projection of the torsion of N. ∎

Remark 2. Condition i) means that the subbundle F is invariant under J. Then, for all $x \in N$, J_x defines a complex structure on the (real) vector space F_x and we have the direct sum

$$F_x^c = F_x^{(1,0)} \oplus F_x^{(0,1)} ,$$

with $F_x^{(1,0)} \subset (T_xM)^{(1,0)}$ and $F_x^{(0,1)} \subset (T_xM)^{(0,1)}$.

Definition 2. Suppose that conditions i) and ii) above hold and assume the following:

- **iii**) if f and g are complex functions defined on M, with differentials df and dg vanishing on F^c, then $d(\{f,g\}^{(2,0)})$ also vanishes on F^c, where $\{\cdot,\cdot\}^{(2,0)}$ denotes the Poisson bracket on $(M, \Lambda^{(2,0)})$.

We say the triple (M, N, F) is *complex Poisson reducible* if $\widehat{N}$ has the structure of a complex Poisson manifold such that, if $\widehat{f}$ and $\widehat{g}$ are complex functions on $\widehat{N}$ and if f and g are complex functions on M, which are extensions of $\widehat{f} \circ \pi$ and $\widehat{g} \circ \pi$ respectively, with df and dg vanishing on F^c, then

$$\{\widehat{f}, \widehat{g}\}_{\widehat{N}}^{(2,0)} \circ \pi = \{f, g\}^{(2,0)} \circ j ,$$

where $j\colon N \to M$ is the canonical injection.

Remark 3. For all complex functions f on M, we set $df = (df)^{(1,0)} + (df)^{(0,1)}$, with $(df)^{(1,0)} \in (T^*M)^{(1,0)}$ and $(df)^{(0,1)} \in (T^*M)^{(0,1)}$. It is then obvious that df vanishes on F^c if and only if $(df)^{(1,0)}$ vanishes on $F^{(1,0)}$ and $(df)^{(0,1)}$ vanishes on $F^{(0,1)}$.

Reduction Theorem. *Suppose that conditions i), ii) and iii) hold. The triple (M, N, F) is complex Poisson reducible if and only if*

$$(\Lambda^{(2,0)})^{\#}\,(F^{(1,0)})^0 \subset F^{(1,0)} + (TN)^{(1,0)} ,$$

where $(F^{(1,0)})^0$ is the subbundle of $(T^*M)^{(1,0)}$ with fibre

$$(F_x^{(1,0)})^0 = \left\{\alpha \in (T_x^*M)^{(1,0)} \colon \langle \alpha, v\rangle = 0,\ \forall\, v \in F_x^{(1,0)}\right\} .$$

Proof: Assume that (M, N, F) is complex Poisson reducible. Let x be an arbitrary element of N. If $\beta_x^{(1,0)} \in (F^{(1,0)})^0$, one can find a complex map f on M such that df vanishes on F^c and $(df)^{(1,0)}(x) = \beta_x^{(1,0)}$. By Remark 3, $(df)^{(1,0)}$ vanishes on $F^{(1,0)}$. Let $\alpha_x^{(1,0)}$ be an arbitrary element of $(F_x^{(1,0)} + (T_x N)^{(1,0)})^0$. Let us choose an extension g of the complex zero function on N such that $(dg)^{(1,0)}(x) = \alpha_x^{(1,0)}$ and dg vanishes on F^c (so, $(dg)^{(1,0)}$ vanishes on $F^{(1,0)}$). Then,

$$\left\langle \alpha_x^{(1,0)}, \, (\Lambda_x^{(2,0)})^{\#}\,(\beta_x^{(1,0)}) \right\rangle = \{f,g\}^{(2,0)}(j(x)) = \{\widehat{f}, 0\}_{\widehat{N}}^{(2,0)} = 0 \; ,$$

where $\widehat{f}\colon \widehat{N} \to \mathbb{C}$ with $\widehat{f} \circ \pi = f \circ j$, and we get

$$(\Lambda_x^{(2,0)})^{\#}\,(F_x^{(1,0)})^0 \subset F_x^{(1,0)} + (T_x N)^{(1,0)} \; .$$

Suppose now that for all $x \in N$, we have $(\Lambda_x^{(2,0)})^{\#}(F_x^{(1,0)})^0 \subset F_x^{(1,0)} + (T_x N)^{(1,0)}$. Let $\widehat{f}$ and $\widehat{g}$ be two complex functions on $\widehat{N}$ and let f and g be extensions of $\widehat{f} \circ \pi$ and $\widehat{g} \circ \pi$, respectively, with df and dg vanishing on F^c. From iii), $d(\{f,g\}^{(2,0)})$ vanishes on F^c; then, $\{f,g\}^{(2,0)}$ is constant on the leaves of N and induces a map on $\widehat{N}$ that we denote by $\{\widehat{f}, \widehat{g}\}_{\widehat{N}}^{(2,0)}$. One can show that this map does not depend on the choice of the extensions of $\widehat{f} \circ \pi$ and $\widehat{g} \circ \pi$. It only remains to check that the bracket $\{\cdot,\cdot\}_{\widehat{N}}^{(2,0)}$, defined in this way is in fact a complex Poisson bracket. But this is easy to do because each one of its properties is a consequence of the corresponding property of the Poisson bracket $\{\cdot,\cdot\}^{(2,0)}$ on M. ∎

4 – Complex Lie group actions on a complex Poisson manifold. Complex Poisson reduction of $sl^*(2,\mathbb{C})$

Let $(M, \Lambda^{(2,0)})$ be a complex Poisson manifold and let G be a complex Lie group acting on M with an action ϕ. We say that ϕ is a *complex Poisson action* if for every $g \in G$, the map $\phi_g\colon x \in M \to \phi(g,x) \in M$ is a complex holomorphic Poisson morphism.

For each X in the Lie algebra $\boldsymbol{G}$ of G, we denote by $X_M^{(1,0)}$ the fundamental vector field associated with X for the action ϕ. It is a holomorphic vector field of type $(1,0)$. If we take a connected complex Lie group G, the action ϕ of G on M is a complex Poisson action if and only if the vector field $X_M^{(1,0)}$ is a *complex Poisson infinitesimal automorphism*, this means, if and only if $[\Lambda^{(2,0)}, X_M^{(1,0)}] = 0$.

Remark 4. Since for every $g \in G$, the map $\phi_g \colon M \to M$ is holomorphic, it is also an almost complex map. This means that the following equality holds, for all $x \in M$:

$$T_x\phi_g \circ J_x = J_{\phi_g(x)} \circ T_x\phi_g \ .$$

If we take a Poisson action of a complex Lie group G on a complex Poisson manifold, such that the set of orbits is a complex manifold, this set has the structure of a reduced complex Poisson manifold. We are going to consider the case where the complex Poisson manifold is the dual $\boldsymbol{G}^*$ of a complex Lie algebra of the complex Lie group G and the complex Poisson action is a restriction of the complex coadjoint action of G on $\boldsymbol{G}^*$.

We take the connected complex Lie group $SL(2,\mathbb{C})$. Its Lie algebra $sl(2,\mathbb{C})$ may be identified with the complexification of the real Lie algebra $sl(2,\mathbb{R})$ and consists of all 2×2 traceless complex matrices. It is a complex Lie subalgebra of $gl(2,\mathbb{C})$, with complex dimension 3, that is, real dimension 6.

The set $\{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\}$ with

$$\alpha_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad \alpha_3 = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$
$$\beta_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \qquad \beta_2 = \begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix}, \qquad \beta_3 = \begin{pmatrix}0 & 0\\ i & 0\end{pmatrix},$$

is a (real) basis of $sl(2,\mathbb{C})$. Thus, the set $\{A_1,A_2,A_3\}$, with $A_j = \alpha_j+\beta_j$, $j=1,2,3$, is a complex basis of $sl(2,\mathbb{C})$.

Let $\{\frac{\partial}{\partial A_1},\frac{\partial}{\partial A_2},\frac{\partial}{\partial A_3}\}$ be the basis of $sl^*(2,\mathbb{C})$, dual of the basis $\{A_1,A_2,A_3\}$. (We consider the dual product of $sl(2,\mathbb{C})$ and $sl^*(2,\mathbb{C})$ given by $\frac{\partial}{\partial A_k}(A_j) = \frac{1}{2}\,\mathrm{tr}({}^{\mathrm{t}}\overline{A_k})\,A_j)$, $j,k\in\{1,2,3\}$).

If J is the (canonical) operator of almost complex structure of $sl^*(2,\mathbb{C})$, we have

$$J\Big(\frac{\partial}{\partial A_j}\Big) = i\,\frac{\partial}{\partial A_j}\,, \qquad j\in\{1,2,3\}\ ,$$

and every $\frac{\partial}{\partial A_j}$ is a vector field of type $(1,0)$ of the complex manifold $sl^*(2,\mathbb{C})$.

Let us compute the following brackets in $sl(2,\mathbb{C})$:

$$[A_1,A_2] = A_1A_2 - A_2A_1 = (\sqrt{2}+i\,\sqrt{2})\,A_2\ ,$$
$$[A_1,A_3] = A_1A_3 - A_3A_1 = -(\sqrt{2}+i\,\sqrt{2})\,A_3\ ,$$

and

$$[A_2,A_3] = A_2A_3 - A_3A_2 = (\sqrt{2}+i\,\sqrt{2})\,A_1\ .$$

The 2-tensor

$$\Lambda^{(2,0)} = (\sqrt{2} + i\,\sqrt{2})\,A_2\,\frac{\partial}{\partial A_1}\wedge\frac{\partial}{\partial A_2} - (\sqrt{2} + i\,\sqrt{2})\,A_3\,\frac{\partial}{\partial A_1}\wedge\frac{\partial}{\partial A_3} + (\sqrt{2} + i\,\sqrt{2})\,A_1\,\frac{\partial}{\partial A_2}\wedge\frac{\partial}{\partial A_3}$$

is of type $(2,0)$ and defines a complex Poisson manifold structure on $sl^*(2,\mathbb{C})$, because the following equalities hold

$$[\Lambda^{(2,0)},\Lambda^{(2,0)}] = 0 \quad \text{and} \quad [\Lambda^{(2,0)},\Lambda^{(0,2)}] = 0\ .$$

This is the so called complex Lie-Poisson structure of $sl^*(2,\mathbb{C})$.

Let H be the subgroup of $SL(2,\mathbb{C})$ of complex dimension 1, whose Lie algebra is generated by A_1. We consider the following action of H on the complex Poisson manifold $(sl^*(2,\mathbb{C}),\Lambda^{(2,0)})$,

$$\psi\colon\ (h,\xi)\in H\times sl^*(2,\mathbb{C}) \to {}^{t}(h^{-1})\,\xi\,{}^{t}h\ \in\ sl^*(2,\mathbb{C})\ ,$$

which is the restriction of the coadjoint action of $SL(2,\mathbb{C})$ on $sl^*(2,\mathbb{C})$, to the Lie subgroup H. Since H is connected and its Lie algebra is generated by A_1, for showing that ψ is a complex Poisson action, we only have to show that $[(A_1)_M^{(1,0)},\Lambda^{(2,0)}] = 0$. But, since ψ is a restriction of the coadjoint action, we have $(A_1)_M^{(1,0)}(\xi) = -ad^*_{A_1}(\xi)$, for all $\xi\in sl^*(2,\mathbb{C})$. (For every $X,Y\in sl(2,\mathbb{C})$, $\langle ad^*_X(\xi),Y\rangle = -\langle\xi,[X,Y]\rangle$).

Thus,

$$A_M^{(1,0)} = (\sqrt{2} + i\,\sqrt{2})\,A_2\,\frac{\partial}{\partial A_2} - (\sqrt{2} + i\,\sqrt{2})\,A_3\,\frac{\partial}{\partial A_3}$$

and a straightforward calculation leads to

$$[(A_1)_M^{(1,0)},\Lambda^{(2,0)}] = 0\ .$$

If $F_\xi^{(1,0)}$ denotes the complex vector space generated by $(A_1)_M^{(1,0)}(\xi)$, by Remark 4, we can deduce that $J_\xi(F_\xi)\subset F_\xi$, for all $\xi\in sl^*(2,\mathbb{C})$, where F_ξ is the vector space of the real vectors of $F_\xi^c = F_\xi^{(1,0)}\oplus F_\xi^{0,1}$. Then, the triple $(sl^*(2,\mathbb{C}), sl^*(2,\mathbb{C}), F)$ is complex Poisson reducible and the manifold of the leaves of $sl^*(2,\mathbb{C})$ is a reduced complex Poisson manifold of complex dimension 2.

5 – The relationship between real and complex Poisson reduction

Let M be an analytic complex manifold and let Λ be a real 2-tensor Poisson on M with $\Lambda = \Lambda^{(2,0)} + \Lambda^{(0,2)}$, where $\Lambda^{(0,2)} = \overline{\Lambda^{(2,0)}}$. The real 2-tensor Λ determines a real Poisson structure on M while $\Lambda^{(2,0)}$ determines a complex Poisson structure. We can state the following.

Proposition 2. *Let N be a paracompact analytic complex submanifold of M and let F be a real subbundle of that verifies conditions i), ii) and iii). Then, the triple (M, N, F) is complex Poisson reducible if and only if it is real Poisson reducible.*

Proof: By Poisson reduction theorems in the real and complex cases, we only have to show that the two following conditions are equivalent:

1) $\Lambda^{\#}(F^0) \subset F + T\,N$;

2) $(\Lambda^{(2,0)})^{\#}(F^{(1,0)})^0 \subset F^{(1,0)} + (TN)^{(1,0)}$,

where F^0 is the annihilator of F in T^*M.

Let $\alpha_x^{(1,0)}$ be an arbitrary element of $(F_x^{(1,0)})^0$ and let $\alpha_x^{(0,1)}$ be its complex conjugate. Then, $\alpha_x = \alpha_x^{(1,0)} + \alpha_x^{(0,1)}$ is a (real) element of F_x^0. If we assume that $\Lambda_x^{\#}(\alpha_x) \in F_x + T_x N$, we can deduce that $(\Lambda_x^{(2,0)})^{\#}(\alpha_x^{(1,0)}) \in F_x^{(1,0)} + (T_x N)^{(1,0)}$ and 1)⇒2). For showing that 2)⇒1), we have to remark that if $\alpha \in T^*M$ (α real) we can write $\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}$, where $\alpha^{(1,0)} \in (T^*M)^{(1,0)}$ and $\alpha^{(0,1)} \in (T^*M)^{(0,1)}$, with $\alpha^{(0,1)} = \overline{\alpha^{(1,0)}}$. If $\alpha \in F^0$, then $\alpha^{(1,0)} \in (F^{(1,0)})^0$ and also $\alpha^{(0,1)} \in (F^{(0,1)})^0$. ∎

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