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VARIETIES OF DISTRIBUTIVE LATTICES WITH UNARY OPERATIONS II

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Abstract: This paper extends to the general setting of [11], [25] procedures presented earlier for varieties of Ockham algebras. Given a suitable finitely generated variety $\mathcal{A}$ of distributive-lattice-ordered algebras with unary operations, in which the subdirectly irreducible algebras are assumed to have been pre-determined, a natural duality, free algebras and coproducts can be obtained algorithmically for any prescribed subvariety of $\mathcal{A}$. Further, the meet-irreducible members of the lattice of equational theories of $\mathcal{A}$ can be written down. The theory is illustrated by carrying this programme through for varieties of double MS-algebras.

1 – Introduction

Our purpose, in this paper and its predecessor [25], is to unify a corpus of existing literature, to strip away the particular features to reveal the underlying structure, and to provide a uniform and algorithmic method for solving algebraic problems relating to a wide class of equational theories arising in non-classical propositional logic. We have chosen to illustrate the theory by considering the variety of double **MS**-algebras. This variety **DMS** includes among its 21 join-irreducible subvarieties de Morgan algebras, Kleene algebras, double Stone algebras and a variety whose members are the 3-valued Łukasiewicz algebras (see [8], Theorem 2.9, or [3], Theorem 5.1). It is therefore of intrinsic algebraic interest, and is an excellent target for our methods.

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Given a finitely generated variety $\mathcal{A}$ of distributive-lattice-ordered algebras we may seek to construct:

(**1**) a natural duality for each subvariety $\mathcal{B}$ of $\mathcal{A}$;

(**2**) the (Priestley duals of) the free algebras $F\mathcal{B}(s)$ ($s \geqslant 1$) for each subvariety $\mathcal{B}$ of $\mathcal{A}$;

(**3**) identities defining the meet-irreducible elements of the lattice of subvarieties of $\mathcal{A}$.

In [25] we presented a framework into which it is possible to fit a large class of varieties $\mathcal{A}$ for which the operations are unary. This earlier paper set up the basic machinery, and developed the theory far enough to present a duality for any quasivariety generated by a subdirectly irreducible algebra. Here we address (1)–(3) above. The first, fragmentary, contributions in this area dealt with individual varieties having small subvariety lattices: de Morgan algebras, Kleene algebras and other small varieties of Ockham algebras, double Stone algebras, ... A significant proportion of the examples investigated had a common feature: the algebraic operations, besides the lattice operations $\wedge$ and $\vee$ and nullary operations 0 and 1, were unary and satisfied de Morgan's laws. This reflected the fact that the varieties studied arose from algebraic logic, where a unary operation $\neg$ modelled a negation. Also, if $\neg$ interchanges $\wedge$ and $\vee$, then $\neg\neg$ preserves them. Therefore we embrace a spectrum of important examples by taking $\mathcal{A}$ to be generated by a finite algebra $(A; \wedge, \vee, 0, 1, \{f_\mu\}_{\mu \in N})$, where each operation f_μ is unary, and either an endomorphism or a dual endomorphism of the reduct $(A; \wedge, \vee, 0, 1)$. A detailed study of such varieties was begun by W.H. Cornish in [10], [11], with the assumption that the operations defined a monoid action on A. This investigation was pursued in [25]. There Cornish's framework was extended, and natural dualities derived in a uniform manner for many familiar varieties, including **DMS** — a simple but instructive example involving a 2-generated monoid.

This paper is a sequel to [25] and draws upon it. The material is divided between the two papers so as to make this one as self-contained as possible, while at the same time sparing the reader of [25] natural duality theory applied in its most general form. The next section presents a minimal summary of what we need from [25]. Section 3 sets up the natural dualities we require. The underlying theory here is taken from [15]. It has been outlined in several subsequent applications [23] and [1], and we assume familiarity with it, but advise that this section should be read in parallel with the highly pictorial **DMS** examples in Section 4. Our general theory makes much detailed work unnecessary. It supplies the dualities for join-irreducible (or indeed arbitrary) subvarieties with a minimum of calculation, obviating the need to derive ab initio in any particular case the endomorphism monoid of the generating algebra and, by hand or by computer, the subalgebras demanded by the Piggyback Duality Theorem (Theorem 3.2). Such calculations were carried out in [28] for the **DMS** case. Another advantage of working at the level of generality of [25] is that it can reveal what is really going on. To take an example, consider the natural duality for the Ockham varieties $\mathbf{P}_{m,n}$, where $m-n$ is odd, which was first presented in [19]. This duality has as one of its relations a partial order which, in isolation, looks very curious. Once the duality is cast as one of our monoidal dualities, this 'oddity' ceases to look at all odd; see [25].

Finally, in Section 5, we discuss equational bases, extending techniques originally devised for Ockham varieties (see [22], [1], [24]). As an application, we give a list of 21 identities sufficient to define all subvarieties of **DMS**. Previously only selected order-ideals in the subvariety lattice had been analysed (see [3] and [6]). A more detailed analysis of identities appears in [27].

2 – Preliminaries

In this section we summarise the definitions and results from [25] that we shall need.

We denote by H and K the hom-functors setting up Priestley duality between the categories $\mathbf{D}$ (bounded distributive lattices) and $\mathbf{P}$ (Priestley spaces), as surveyed, for example, in [23]. We may identify a lattice $L \in \mathbf{D}$ with its second dual $KH(L) = k_L(L)$, where $k_L : a \mapsto e_a$, $e_a(x) := x(a)$, for $a \in L$, $x \in H(L)$. The restrictions of H and K to non-full subcategories are also denoted by H and K.

An **ordered $\pm$-semigroup** is a structure $\mathbf{N} = (N; \cdot\,, e\,, N^+, N^-, \leqslant)$ such that

(**M1**) $(N; \cdot)$ is a finite semigroup, in which e is a right identity;

(**M2**) $N = N^+ \cup N^-$;

(**M3**) for $\sigma, \tau \in \{\pm\}$, $N^\sigma \cdot N^\tau \subseteq N^{\sigma\tau}$;

(**M4**) $N^+ \cap N^- = \emptyset$;

(**M5**) $\leqslant$ is a partial order on N such that, for all $\lambda, \mu, \nu \in N$,

$$\lambda \leqslant \mu \implies \begin{cases} \nu\lambda \leqslant \nu\mu & \text{if } \nu \in N^+, \\ \nu\lambda \geqslant \nu\mu & \text{if } \nu \in N^- . \end{cases}$$

If (M4) is omitted, then $\mathbf{N}$ is said to be a **weak ordered $\pm$-semigroup**. If further e is a 2-sided identity and the order is discrete (and hence can be suppressed) then $\mathbf{N}$ is a finite $\pm$-monoid as defined by Cornish in [11], Chapter 5. We let $\mathbf{N}_=$ be the structure obtained from $\mathbf{N}$ by replacing the given order by the discrete order. We denote the class of finite ordered $\pm$-monoids by $\mathcal{M}$ and the class of finite weak ordered $\pm$-semigroups by $\mathcal{N}$. We let $\mathcal{N}^*$ denote the subclass of $\mathcal{N}$ consisting of those monoids $\mathbf{N} \in \mathcal{N}$ which satisfy

(**M6**) for all $\mu, \nu_1, \nu_2 \in N$, $\nu_1 \leqslant \nu_2 \implies \nu_1\mu \leqslant \nu_2\mu$.

Of course, (M6) is satisfied automatically if $\leqslant$ is the discrete order. The reason for introducing this supplementary condition emerges below. Such stringent restrictions are not needed for every result (for example sometimes a semigroup suffices, instead of a monoid). Working with the full set of assumptions ensures that the entire theory goes through smoothly without the distraction of a proliferation of subclasses of $\mathcal{N}$ and without sacrificing any important examples.

Given any $\mathbf{N} \in \mathcal{N}$ we may associate a distributive-lattice-ordered algebra $\underline{P}$ (or where we wish to make $\mathbf{N}$ explicit, $\underline{P}^{\mathbf{N}}$) in the following way.

(**a**) $\underline{P}$ has a bounded distributive lattice reduct whose Priestley dual $H(\underline{P})$ is the ordered set $(N; \leqslant)$, with the discrete topology, τ.

(**b**) $\underline{P}$ has operations $f_{(\mu,\epsilon)}$ indexed by the disjoint union $(N^+ \times \{1\}) \cup (N^- \times \{-1\})$ of N^+ and N^-; we write f^+_μ for $f_{(\mu,+1)}$ and f^-_μ for $f_{(\mu,-1)}$. These operations are defined by

$$\begin{cases} f^+_\mu(a)(\nu) = a(\nu\mu) & \text{if } \mu \in N^+, \\ f^-_\mu(a)(\nu) = 1 - a(\nu\mu) & \text{if } \mu \in N^- \ , \end{cases}$$

for $a \in \underline{P}$ and $\nu \in N$, and are required to satisfy the identities

$$f^+_\mu(a) \vee f^-_\mu(a) = 1 \quad \text{and} \quad f^+_\mu(a) \wedge f^-_\mu(a) = 0 \ ,$$

whenever $\mu \in N^+ \cap N^-$.

If $\mathbf{N} \in \mathcal{M}$, we drop the superscripts from f^+_μ, f^-_μ. We let $\mathcal{A}^{\mathbf{N}} := \mathbb{HSP}(\underline{P}^{\mathbf{N}})$.

Given $\mathbf{N} \in \mathcal{N}$, we may use the functors H and K to set up a Priestley-type duality between $\mathcal{A}^{\mathbf{N}}$, which we also denote by **D**-**N**, and a category **N**-**P** of structures $(Y; \mathcal{T}, \leqslant, \{g_\mu\}_{\mu \in N})$, where $(Y; \mathcal{T}, \leqslant) \in \mathbf{P}$ and the maps g_μ define a continuous semigroup action on Y, with g_μ order-preserving for $\mu \in N^+$ and order-reversing for $\mu \in N^-$. Then $\mathbf{N}$, viewed as a structured Priestley space, itself belongs to **N**-**P**, with the operation g_μ given by left multiplication by μ. For further comments on left and right actions, and the reconciliation of our notation with Cornish's, see Section 2 of [25], where motivation for the above definitions is also given.

It is shown in [25] that every monoid $\mathbf{N}$ in $\mathcal{N}$ is the image of some $\mathbf{M} \in \mathcal{M}$, under a map which is both a monoid homomorphism and an $\mathbf{N}$-$\mathbf{P}$-morphism. This means that $\mathcal{A}^{\mathbf{N}}$ is a subvariety of $\mathcal{A}^{\mathbf{M}}$. We may also assume without loss of generality that $\mathbf{M}$ satisfies (M6), since this can always be arranged by giving $\mathbf{M}$ the discrete order. We shall take as the varieties $\mathcal{A}$ whose subvariety lattices we analyse to be those of the form $\mathcal{A}^{\mathbf{M}}$, where $\mathbf{M} \in \mathcal{N}^* \cap \mathcal{M}$.

If a given variety $\mathcal{V}$ is expressed in the form $\mathbb{HSP}(\underline{P}^{\mathbf{N}})$, we write $\mathbf{N}^{\mathcal{V}}$ for $\mathbf{N}$. The weak ordered $\pm$-monoid associated with the variety **DMS** is shown in Figure 1; see [25]. Note that this monoid satisfies (M6).

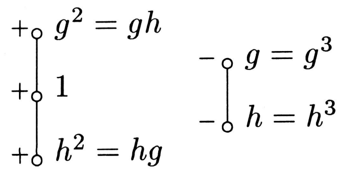


Fig. 1: $\mathbf{N}^{\mathbf{DMS}}$.

In Section 3 of [25] a discussion is given of subdirectly irreducible algebras, extending results of Cornish for the unordered case. The following lemmas give the properties we need here. These variously rely on, or are used in conjunction with, two basic algebraic facts, applicable to any variety $\mathcal{A}^{\mathbf{N}}$, for $\mathbf{N} \in \mathcal{N}$:

(**a**) the Congruence Extension Property (CEP) holds;

(**b**) Jónsson's Lemma is applicable.

Lemma 2.1 ([25], **Lemma 3.1**). *Assume* $\mathbf{N} \in \mathcal{N}$ *satisfies* (M6). *Then*

$$\operatorname{End} \underline{P}^{\mathbf{N}} = \Big\{ u_\mu := K(\eta_\mu) \mid \mu \in N \Big\} ,$$

where $\eta_\mu \colon \nu \mapsto \nu\mu$ $(\nu \in N)$.

Lemma 2.2 (from [25], Proposition 3.4). *Let* $\mathbf{N} \in \mathcal{N}$. *Then a finite algebra* $A \in \mathbf{D}$-$\mathbf{N}$ *is subdirectly irreducible if and only if there exists* $z \in H(A)$ *such that*

$$H(A) = \Big\{ \mu z \mid \mu \in N \Big\} .$$

In particular, the algebra $\underline{P}^{\mathbf{N}}$ *is subdirectly irreducible.*

Assume that $\mathbf{N} \in \mathcal{N}^*$. *Then the subdirectly irreducible algebras in* $\mathcal{A}^{\mathbf{N}}$ *are precisely* (*the isomorphic copies of*) *the subalgebras of* $\underline{P}^{\mathbf{N}}$.

Finally in this section we recall the structures used to build the duality for $\mathcal{A}^{\mathbf{N}}$. The underlying set of this schizophrenic object is a subset of 2^N. We write a typical element $a \in 2^N$ as $\langle a_\nu \rangle$, where $a_\nu = a(\nu)$ $(\nu \in N)$. We have two structures with underlying set 2^N:

$$\underline{\mathbf{2}}^N := (2^N; \wedge, \vee, 0, 1, \{\varphi_\mu^+\}_{\mu \in N^+}, \{\varphi_\mu^-\}_{\mu \in N^-}) \in \mathbf{D}\text{-}\mathbf{N} \ ,$$
$$\underset{\sim}{\mathbf{2}}^N := (2^N; \tau, \preccurlyeq, \{\gamma_\mu\}_{\mu \in N}) \in \mathbf{N}\text{-}\mathbf{P} \ .$$

We use distinctive symbols $\varphi_\mu^\pm$, γ_μ in preference to the generic symbols $f_\mu^\pm$, g_μ to stress the special role played by $\underline{\mathbf{2}}^N$ and $\underset{\sim}{\mathbf{2}}^N$. These structures are defined in the following way. The $\mathbf{D}$-reduct of $\underline{\mathbf{2}}^N$ is the Boolean lattice obtained as the pointwise product of copies of the 2-element chain $\underline{\mathbf{2}} \in \mathbf{D}$. We define operations $\varphi_\mu^\pm$ by the formulae:

$$\varphi_\mu^+(\langle a_\nu \rangle) = \langle a_{\mu\nu} \rangle \quad (\mu \in N^+) \ ,$$
$$\varphi_\mu^-(\langle a_\nu \rangle) = \langle \overline{a_{\mu\nu}} \rangle \quad (\mu \in N^-) \ ,$$

where $\overline{\delta} = 1 - \delta$ for $\delta = 0, 1$. Then (see [25]) $(2^N; \wedge, \vee, 0, 1, \{\varphi_\mu^+\}_{\mu \in N^+}, \{\varphi_\mu^-\}_{\mu \in N^-}) \in \mathbf{D}\text{-}\mathbf{N}_=$; indeed its dual is just $\mathbf{N}_=$.

We now define the alternative structure $\underset{\sim}{\mathbf{2}}^N$. The $\pm$-**order**, $\preccurlyeq$, is given by

$$a \preccurlyeq b \iff \begin{cases} a_\nu = 1 \implies b_\nu = 1 & \text{if } \nu \in N^+, \\ b_\nu = 1 \implies a_\nu = 1 & \text{if } \nu \in N^- \ , \end{cases}$$

and operations γ_μ $(\mu \in N)$ are defined by

$$\gamma_\mu(\langle a_\nu \rangle) := \langle a_{\nu\mu} \rangle \quad (\nu \in N) \ .$$

The topology τ is the discrete topology.

When the order on N is not discrete the algebra $\underline{P} := \underline{P}^{\mathbf{N}}$ properly sits inside $\underline{\mathbf{2}}^N$. Then (see [25], Lemma 4.7) the underlying set, P, is characterised as

$$P = \Big\{ a \colon N \to \mathbf{2} \mid \ \underset{\sim}{\mathbf{2}}^N \models \sigma_e(a) \Big\} \ ,$$

where

$$\sigma_e(a) := \bigwedge \Big\{ \nu_1 \leqslant \nu_2 \Rightarrow \gamma_{\nu_1}(a) \preccurlyeq \gamma_{\nu_2}(a) \mid \ \nu_1, \nu_2 \in N \Big\} \ .$$

Further, γ_μ maps P into P, for each $\mu \in N$. This final statement requires that $\mathbf{N}$ is a monoid rather than a semigroup; see Lemma 4.7 of [25].

Write $\mathcal{A} := \mathcal{A}^{\mathbf{N}}$, to simplify notation. In practice we may be given a subdirectly irreducible algebra $\underline{Q}$ in $\mathcal{A}$ by being given its dual $S := H(Q)$ — indeed,

$\mathrm{Si}(\mathcal{A})$ is often most easily determined by duality methods. It is therefore convenient to know how to re-interpret Q, *qua* subset of 2^S, explicitly in terms of 2^N. Let $\underline{Q}$ be a subdirectly irreducible algebra in $\mathcal{A}$, given as a homomorphic image of a subalgebra $\underline{R}$ of $\underline{P}$ under a homomorphism $u\colon \underline{R} \to \underline{Q}$. Dualising, there is a $\mathcal{Y}$-morphism $\psi := H(u)$ providing an order-embedding of S onto an N-closed subset of $H(\underline{R})$. Also, $H(\underline{R}) = \varphi(N)$, where φ is a $\mathcal{Y}$-morphism with domain N. Since $\underline{Q}$ is subdirectly irreducible, there exists $\lambda \in S$ such that $S = \{g_\nu(\lambda) \mid \nu \in N\}$. Since $\psi(S) \subseteq \varphi(N)$, there exists $\rho \in N$ such that $\psi(\lambda) = \varphi(\rho)$. Then

$$\psi(S) = \Big\{\, g_\nu(\varphi(\rho)) \mid \ \nu \in N \,\Big\} = \Big\{\, \varphi(\nu\rho) \mid \ \nu \in N \,\Big\} \ .$$

The required bijection from $Q \subseteq 2^S$ to $\rho R := \gamma_\rho(R) \subseteq 2^N$ is set up by

$$Z\colon u(a) \mapsto \gamma_\rho \circ K(\varphi) \circ a \ ,$$

where $K(\varphi)\colon \underline{R} \to \underline{P}$ is the natural embedding. To check that Z is a well-defined bijection, note that, for $\nu \in N$ and $b \in Q$, with $b = u(a)$ $(a \in R)$,

$$\begin{aligned} b(g_\nu(\lambda)) = u(a)(g_\nu(\lambda)) = a(\psi(g_\nu(\lambda))) = a(g_\nu(\psi(\lambda))) = \\ = a(\varphi(\nu\rho)) = K(\varphi)(a)(\nu\rho) = (\gamma_\rho(K(\varphi)(a)))(\nu) \ . \end{aligned}$$

3 – Natural dualities for subvarieties of a variety $\mathcal{A}^{\mathbf{N}}$ for $\mathbf{N} \in \mathcal{N}^*$

Let $\mathbf{N} \in \mathcal{N}^*$. It follows from the results of the preceding section that we have $\mathbb{HSP}(\underline{P}^{\mathbf{N}}) = \mathbb{ISP}(\underline{P}^{\mathbf{N}})$. Better still, it is possible to set up a natural duality for $\mathbb{HSP}(\underline{P}^{\mathbf{N}})$ based on the simple piggyback method developed in [18], [19], rather than on the generalised theory in [15]; see Proposition 4.6 and Theorem 4.8 of [25]. We shall combine the techniques of [15] with the duality theory obtained in [25] for $\mathcal{A}^{\mathbf{N}}$, to derive dualities for all subvarieties of $\mathcal{A}^{\mathbf{N}}$. All the varieties we wish to consider are thereby encompassed.

Theorem 3.1 ([25], **Theorem 4.8**). *Assume* $\mathbf{N} \in \mathcal{N}^*$. *Let* $\underline{P} := \underline{P}^{\mathbf{N}} \le \underline{\mathbf{2}}^N$ *and let* $\underset{\sim}{P}$ *be the substructure of* $\underset{\sim}{\mathbf{2}}^N$ *whose underlying set is* P. *Then the hom-functors*

$$\begin{aligned} D\colon A \mapsto \mathcal{A}(A, \underline{P}^{\mathbf{N}}) \le \underset{\sim}{P}^{\mathbf{N}} \in \mathcal{X} \ , \\ E\colon X \mapsto \mathcal{X}(X, \underset{\sim}{P}^{\mathbf{N}}) \le \underline{P}^{\mathbf{N}} \in \mathcal{A} \ , \end{aligned}$$

set up a natural duality between $\mathcal{A} := \mathbb{HSP}(\underline{P})$ *and* $\mathcal{X} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{P})$.

For an example see the beginning of Section 4, where the structures $\underline{P}$ and $\underset{\sim}{P}$ are shown for $\mathbf{N} = \mathbf{N}^{\mathbf{DMS}}$.

We shall now apply the Generalised Piggyback Duality Theorem, in the form given below. We summarise the basic definitions first, referring the reader to [15] for further information. Take $\mathcal{A} := \mathbb{ISP}(\underline{\Pi})$, where $\underline{\Pi}$ is a finite set of finite algebras all of the same type. Let

$$R \subseteq \bigcup \left\{ \mathbb{S}(\underline{P} \times \underline{Q}) \mid \underline{P}, \underline{Q} \in \underline{\Pi} \right\} ,$$

let τ be the discrete topology and let $\underset{\sim}{\Pi} = (\Pi; \tau, R)$. The power $\underset{\sim}{\Pi}^S$ is formed as follows: its underlying set is $\bigcup \{ P^S \mid \underline{P} \in \underline{\Pi} \}$, the topology is the union topology, where each component P^S has the product topology derived from the discrete topology on P, and the relations in R are lifted pointwise. More generally we consider structures $\mathbf{X} = (X; \mathcal{T}, R)$, such that

$$X := \dot{\bigcup} \left\{ X_{\underline{P}} \mid \underline{P} \in \underline{\Pi} \right\} ,$$

where each $X_{\underline{P}}$ carries a compact topology and X carries the sum topology, and where, for each relation $r \in R$, $r \in \mathbb{S}(\underline{P} \times \underline{Q})$, there is an associated relation

$$r \subseteq X_{\underline{P}} \times X_{\underline{Q}} .$$

Given two such $\underline{\Pi}$-*indexed structures*, $\mathbf{X}$ and $\mathbf{Y}$, a *morphism from* $\mathbf{X}$ *to* $\mathbf{Y}$ is a map which takes $X_{\underline{P}}$ into $Y_{\underline{P}}$ for each $\underline{P} \in \underline{\Pi}$ and which is structure-preserving in the obvious sense. Then $\mathcal{X}$ is defined to be the category of all $\underline{\Pi}$-indexed structures which take the form of an isomorphic copy of a substructure of some power $\underset{\sim}{\Pi}^S$ of $\underset{\sim}{\Pi}$ (in symbols, $\mathcal{X} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$).

For each $A \in \mathcal{A}$, the dual of A is defined to be

$$D(A) := \dot{\bigcup} \left\{ \mathcal{A}(A, \underline{P}) \mid \underline{P} \in \underline{\Pi} \right\} ,$$

viewed as an $\mathcal{X}$-substructure of $\underset{\sim}{\Pi}^A$. For each $\mathbf{X} \in \mathcal{X}$,

$$E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\Pi})$$

is an $\mathcal{A}$-subalgebra of $\prod \{ \underline{Q}_{\underline{P}} \mid \underline{P} \in \underline{\Pi} \}$ where $\underline{Q}_{\underline{P}}$ is $\underline{P}$ raised to the power $X_{\underline{P}}$. Just as in the case $|\underline{\Pi}| = 1$ we have well-defined contravariant functors

$$D \colon \mathcal{A} \to \mathcal{X} \quad \text{and} \quad E \colon \mathcal{X} \to \mathcal{A} .$$

Theorem 3.2 (**The Generalised Piggyback Duality Theorem, for distributive-lattice-ordered algebras**)**.** *Suppose that* $\mathcal{A} = \mathbb{ISP}(\underline{\Pi})$*, where* $\underline{\Pi}$ *is a finite set of finite algebras of a given fixed type each having a* $\mathbf{D}$*-reduct. For each* $\underline{Q}$ *in* $\underline{\Pi}$ *let* $\Omega_{\underline{Q}}$ *be a (possibly empty) subset of* $\mathbf{D}(\underline{Q}, \underline{\mathbf{2}})$*.*

Let $\underset{\sim}{\Pi} = (\Pi; \tau, R)$ *be the topological relational structure on* $\dot{\bigcup}\{\, P \mid \underline{P} \in \underline{\Pi} \,\}$ *in which*

(**i**) τ *is the discrete topology,*

(**ii**) $R = S \cup E$*, where*

(**a**) S *is the collection of maximal* $\mathcal{B}$*-subalgebras of sublattices of the form*

$$(\alpha, \beta)^{-1}(\leqslant) := \Big\{\, (a, b) \in \underline{B} \times \underline{C} \mid \ \alpha(a) \leqslant \beta(b) \,\Big\}\,,$$

where $\alpha \in \Omega_{\underline{P}}$*,* $\beta \in \Omega_{\underline{Q}}$ $(\underline{P}, \underline{Q} \in \underline{\Pi})$*, and*

(**b**) E *is the set of (graphs of) a set* $\mathcal{E} \subseteq \bigcup\{\, \mathcal{A}(\underline{P}, \underline{Q}) \mid \underline{P}, \underline{Q} \in \underline{\Pi} \,\}$ *of endomorphisms satisfying the following separation condition* (S)*: for all* $\underline{P} \in \underline{\Pi}$*, given* $a, b \in \underline{P}$ *with* $a \neq b$*, there exist* $\underline{Q} \in \underline{\Pi}$*,* $u \in \mathcal{A}(\underline{P}, \underline{Q}) \cap \mathcal{E}$ *and* $\alpha \in \Omega_{\underline{Q}}$ *such that* $\alpha(u(a)) \neq \alpha(u(b))$*.*

Then R *yields a duality on* $\mathcal{A}$*, that is, the functors* D *and* E *set up a dual equivalence between* $\mathcal{A}$ *and a full subcategory of* $\mathcal{X} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$*.*

When using the Generalised Piggyback Duality Theorem we shall, for notational convenience, identify an endomorphism with its graph. This sloppiness over types causes no problems in the context in which we are working (but see [14], Section 3). For any algebra A having a $\mathbf{D}$-reduct and whose additional operations are endomorphisms or dual endomorphisms of the reduct, any sublattice L of A is contained in a unique maximal subalgebra of A (Lemma 3.5 of [15]). Thus in our applications of Theorem 3.2 to a subvariety $\mathcal{C}$ of some variety of the form $\mathcal{A}^{\mathbf{N}}$, any lattice $(\alpha, \beta)^{-1}(\leqslant)$ as above will have a unique maximal $\mathcal{C}$-subalgebra, which we denote by $(\alpha, \beta)^{-1}(\leqslant)^o$.

For a given variety or quasivariety there may be several different ways in which the theorem above can be applied. The family $\underline{\Pi}$ is not uniquely prescribed. To illustrate, suppose we have some $\underline{B} \in \underline{\Pi}$ such that $\Omega_{\underline{B}}$ contains precisely two elements α, β. Then we may replace $\underline{B}$ by a pair of copies $\underline{B}_\alpha, \underline{B}_\beta$ of $\underline{B}$, and take $\Omega_{\underline{B}_\alpha} = \{\alpha\}$, $\Omega_{\underline{B}_\beta} = \{\beta\}$. The satisfaction of (S) is not altered by this manoeuvre. Thus there may be a trade-off of the size of $\underline{\Pi}$ against the size of $\bigcup_{\underline{Q} \in \underline{\Pi}} \Omega_{\underline{Q}}$ (the set of *carriers*, in the terminology of [15]). For a class $\mathbb{ISP}(\underline{Q})$ we can present a duality with $\underline{\Pi} = \{\underline{Q}\}$, taking sufficient carriers to ensure the separation condition is satisfied (note that $\Omega_{\underline{Q}} = H(\underline{Q})$ will certainly suffice). Alternatively, as in [25], we can go to the opposite extreme and take copies $\underline{Q}_\alpha$ of $\underline{Q}$ indexed by the points α of $H(\underline{Q})$, letting $\Omega_{\underline{Q}_\alpha} = \{\alpha\}$. The relative merits of the options 'single algebra, multiple carriers' and 'multiple algebras, single carrier' depend on what the objectives are. A duality based on a single algebra is ostensibly simpler than one based on multiple algebras. However the multiple algebra approach often gives a more direct translation to a Priestley-type duality. In the case of the variety generated by an algebra which is not subdirectly irreducible we may need both multiple algebras and multiple carriers; see Theorem 3.6. In any case, the more endomorphisms are available the fewer algebras and/or carriers are forced upon us.

From here on we take $\mathbf{N}$ to be a fixed monoid in $\mathcal{N}^*$. To simplify notation, we let $\underline{P} := \underline{P}^{\mathbf{N}}$ and $\mathcal{A} := \mathcal{A}^{\mathbf{N}}$ (= $\mathbf{D}$-$\mathbf{N}$). Recall (2.2) that up to isomorphism the subdirectly irreducible subalgebras are exactly the subalgebras of $\underline{P}$. To apply the Generalised Piggyback Duality Theorem to obtain a natural duality for $\mathbb{HSP}(\underline{Q})$, for $\underline{Q} \in \mathbb{S}(\underline{P})$, we first need to investigate the separation condition (S), by analysing the partial endomorphisms of $\underline{P}$.

Lemma 3.3. *Let $\underline{Q}$ be a subalgebra of $\underline{P}$. Then*

$$\operatorname{End}\underline{Q} = \Big\{\, u_\kappa{\restriction}_{\underline{Q}} \;\mid\; u_\kappa(\underline{Q}) \subseteq \underline{Q} \ (\kappa \in N) \Big\} \,.$$

Proof: Assume that $v\colon \underline{Q} \to \underline{Q}$ is an endomorphism, with dual $\psi := H(v)$. Denote the dual of the natural embedding $q\colon \underline{Q} \to \underline{P}$ by φ, so $\varphi\colon N \to H(\underline{Q})$ is surjective. Let $\psi(\varphi(e)) = \varphi(\kappa)$. Then, for all $\nu \in N$ and $b \in \underline{Q}$,

$$\begin{aligned}\nu(q(v(b))) &= \varphi(\nu)(v(b)) = (\psi(\varphi(\nu)))(b) = (\psi(g_\nu(\varphi(e))))(b) = \\ &= g_\nu(\psi(\varphi(e)))(b) = g_\nu(\varphi(\kappa))(b) = \varphi(\nu\kappa)(b) = \varphi(\eta_\kappa(\nu))(b) \\ &= \eta_\kappa(\nu)(q(b)) = \nu(u_\kappa(q(b))) = \nu(u_\kappa{\restriction}_{\underline{Q}}(b)) \,.\end{aligned}$$

We conclude that $v = u_\kappa{\restriction}_{\underline{Q}}$, and the image of this map must necessarily lie in $\underline{Q}$. Conversely, $u_\kappa{\restriction}_{\underline{Q}} \in \operatorname{End}\underline{Q}$ only if $u_\kappa(\underline{Q}) \subseteq \underline{Q}$. ∎

We are now ready to apply the Generalised Piggyback Duality Theorem to obtain an economical 'single algebra, multiple carrier' duality for subvarieties of $\mathcal{A}^{\mathbf{N}}$ of the form $\mathbb{ISP}(\underline{Q})$, where $\underline{Q}$ is subdirectly irreducible. Lemma 3.3 guides us in choosing our carriers. The result below may be compared with Proposition 4.6 of [25]. As already noted we may have to compensate for a paucity of endomorphisms by taking multiple carriers. In [25] we were concerned with a generating algebra $\underline{P}^{\mathbf{N}}$, and might lack the putative endomorphism u_κ because η_κ is not order-preserving. Here we have imposed conditions on $\mathbf{N}$ sufficient to guarantee this does not occur. However when we consider a subalgebra $\underline{Q}$ of $\underline{P}^{\mathbf{N}}$, we may have a reduced set of endomorphisms because the restriction to $\underline{Q}$ of the map u_κ may not have image lying in $\underline{Q}$. The following theorem gives a simple way to find a duality which will generally be more economical than one built using every element of $H(\underline{Q})$ as a carrier.

Theorem 3.4. *Let $\mathbf{N} \in \mathcal{N}^*$ and let $\underline{P} := \underline{P}^{\mathbf{N}}$. Assume that $\underline{Q}$ is a subdirectly irreducible subalgebra of $\underline{P}$, and that $\mathbb{H}(\underline{Q}) \subseteq \mathbb{IS}(\underline{Q})$. Let $\varphi := H(q)$, where $q\colon \underline{Q} \to \underline{P}$ is the natural embedding. Let*

$$N_0 := \left\{ \kappa \in N \mid \; u_\kappa(\underline{Q}) \subseteq \underline{Q} \right\} .$$

Define $\Omega_{\underline{Q}}$ to be a subset C of $\varphi(N)$ such that every element of $\varphi(N)$ is expressible in the form $\varphi(\mu\kappa)$, where $\kappa \in N_0$ and $\varphi(\mu) \in C$. Let

$$\mathcal{E} := \left\{ u_\kappa{\restriction}_{\underline{Q}} \mid \; \kappa \in N_0 \right\}$$

and

$$\mathcal{S} := \left\{ (\alpha, \beta)^{-1}(\leqslant)^o \mid \; \alpha, \beta \in C \right\} .$$

Then $\mathcal{E} \cup \mathcal{S}$ yields a duality on $\mathbb{HSP}(\underline{Q})$.

The set C may be taken to be $\{\alpha\} \cup (\varphi(N) \setminus \varphi(N_0))$, where $\alpha := \varphi(e)$.

Proof: Given $a \neq b$ in $\underline{Q}$, we can find $\varphi(\beta) \in H(\underline{Q})$ such that $\varphi(\beta)(a) \neq \varphi(\beta)(b)$. Take $\kappa \in N_0$ and $\beta = \varphi(\mu) \in C$ such that $\varphi(\mu\kappa)(a) \neq \varphi(\mu\kappa)(b)$. Then

$$\begin{aligned} \varphi(\mu)(u_\kappa(a)) = \mu(q(u_\kappa(a))) = \mu(u_\kappa(q(a))) = \\ = (\eta_\kappa(\mu))(q(a)) = (\mu\kappa)(q(a)) = (\varphi(\mu\kappa))(a) \end{aligned}$$

and likewise with b in place of a. Hence we have guaranteed that (S) holds by our choice of C. ∎

We now turn to the general case, first seeking information about the endomorphisms we shall want to use.

Suppose $\underline{Q}$ is a subalgebra of $\underline{P}$. For each $\kappa \in N$, let v_κ be the restriction of $u_\kappa := K(\eta_\kappa)$ to $\underline{Q}$. Then $\underline{Q}_\kappa := v_\kappa(\underline{Q})$ is a subalgebra of $\underline{P}$, and $\underline{Q}_e = \underline{Q}$.

Lemma 3.5. *Let $\underline{Q}$ be a subalgebra of $\underline{P}$. Then, for $\kappa \in N$ and for $a \neq b$ in $\underline{Q}_\kappa$, there exists $\lambda \in N$ such that $v_{\lambda,\kappa}(a) \neq v_{\lambda,\kappa}(b)$, where $v_{\lambda,\kappa} := u_\lambda{\restriction}_{\underline{Q}_\kappa} : \underline{Q}_\kappa \to \underline{Q}_{\lambda\kappa}$.*

Proof: For $\lambda, \kappa, \nu \in N$, and $a \in \underline{P}$,

$$\begin{aligned}(u_\lambda(u_\kappa(a))(\nu) = (u_\kappa(a))(\eta_\lambda(\nu)) = (u_\kappa(a))(\nu\lambda) = \\ = a(\eta_\kappa(\nu\lambda)) = a(\nu\lambda\kappa) = (u_{\lambda\kappa}(a))(\nu) \ .\end{aligned}$$

Hence $u_\lambda \circ u_\kappa = u_{\lambda\kappa}$, so that $v_{\lambda,\kappa}$ maps $\underline{Q}_\kappa$ into $\underline{Q}_{\lambda\kappa}$.

Now let q_κ be the natural embedding of $\underline{Q}_\kappa$ into $\underline{P}$ and $\varphi_\kappa \colon N \to S_\kappa := \varphi_\kappa(N)$ be its dual. Then the following diagrams commute.

$$\begin{tikzcd} \underline{P} \arrow[rr, "u_\lambda"] & & \underline{P} \\ & & \underline{Q}_\lambda \arrow[u, "q_\lambda"] \\ \underline{Q}_\kappa \arrow[uu, "q_\kappa"] \arrow[rr, "v_{\lambda,\kappa}"'] & & \underline{Q}_{\lambda\kappa} \arrow[u, "q_\kappa \restriction \underline{Q}_\lambda"] \arrow[uu, shift right=6, "q_{\lambda\kappa}"'] \end{tikzcd} \qquad \begin{tikzcd} N \arrow[dd, "\varphi_\kappa"'] & & N \arrow[ll, "\eta_\lambda"'] \arrow[d, "\varphi_\lambda"'] \arrow[dd, shift left=6, "\varphi_{\lambda\kappa}"] \\ & & \varphi_\lambda(N) \arrow[d] \\ S_\kappa & & S_{\lambda\kappa} \arrow[ll, "H(v_{\lambda,\kappa})"] \end{tikzcd}$$

Since $a \neq b$, we have $q_\kappa(a) \neq q_\kappa(b)$, and there exists $\lambda \in N$ such that $\lambda(q_\kappa(a)) \neq \lambda(q_\kappa(b))$. For $c \in \underline{Q}_\lambda$,

$$\begin{aligned}(\varphi_{\lambda\kappa}(e))(v_{\lambda,\kappa}(c)) &= ((H(v_{\lambda,\kappa})(\varphi_{\lambda\kappa})(e))(c) \\ &= ((\varphi_\kappa \circ \eta_\lambda)(e))(c) \\ &= (\varphi_\kappa(\lambda))(c) \\ &= \lambda(q_\kappa(c)) \ .\end{aligned}$$

We deduce that $(\varphi(\kappa\lambda)(e))(v_{\lambda,\kappa}(a)) \neq (\varphi(\kappa\lambda)(e))(v_{\lambda,\kappa}(b))$, whence we conclude that $v_{\lambda,\kappa}(a) \neq v_{\lambda,\kappa}(b)$. ∎

We could now state a brute force duality theorem for $\mathbb{HSP}(\underline{Q})$, based on the full set of algebras $\mathbb{H}(\underline{Q})$ and suitable carrier sets. This follows very directly from Theorem 3.6 below, using the maps $v_{\kappa,\lambda}$ as the endomorphisms to be included in our relational structure. The retention of *all* homomorphic images of $\underline{Q}$ ensures that we can satisfy the separation condition (S). In practice we usually simplify our dualities by omitting those algebras $\underline{Q}_\kappa$ which serve no necessary function. We therefore explore whether the separation condition can be met using a reduced set of algebras. Order the subalgebras in $\mathbb{H}(\underline{Q})$ by $\underline{Q}_\kappa \ll \underline{Q}_\lambda$ if and only if $\underline{Q}_\kappa \in \mathbb{S}(\underline{Q}_\lambda)$. Define

$$N[Q] := \Big\{ \kappa \in N \mid \ \underline{Q}_\kappa \text{ is } {\ll}\text{-maximal} \Big\} \ .$$

Given that $\underline{Q}$ is subdirectly irreducible,

$$\mathbb{HSP}(\underline{Q}) = \mathbb{ISP}\Big(\Big\{\, \underline{Q}_{\kappa} \mid \kappa \in N[Q] \,\Big\}\Big) .$$

Let $a \neq b$ in $\underline{Q}_{\kappa}$, where $\kappa \in N[Q]$. Since $\mathbf{N} \in \mathcal{N}^*$, $\underline{Q}_{\kappa}$ is itself subdirectly irreducible, and, applying Lemma 3.5 to it, we can find an endomorphism $f := v_{\kappa,\lambda} \colon \underline{Q}_{\kappa} \to \underline{Q}_{\lambda\kappa}$ such that $f(a) \neq f(b)$. We may not have $\lambda\kappa \in N[Q]$. However by composing f with an embedding of $\underline{Q}_{\lambda\kappa}$ into $\underline{Q}_{\mu}$, where μ does belong to $N[Q]$, we see that a and b can be separated by an endomorphism, of the same type, into $\underline{Q}_{\mu}$. We therefore have, for each pair $\kappa, \mu \in N[Q]$, an endomorphism $w_{\kappa,\mu} \colon \underline{Q}_{\kappa} \to \underline{Q}_{\mu}$ such that $a \neq b$ in $\underline{Q}_{\kappa}$ implies $w_{\kappa,\mu}(a) \neq w_{\kappa,\mu}(b)$. Here each $w_{\kappa,\mu}$ is a map of the form $v_{\kappa,\lambda}$, with (possibly) the codomain re-defined.

The following theorem is now a corollary of the Generalised Piggyback Duality Theorem and the proof of Lemma 3.5.

Theorem 3.6. *Let $\mathbf{N} \in \mathcal{N}^*$ and $\mathcal{A} := \mathcal{A}^{\mathbf{N}}$. Let $\underline{Q}$ be a subdirectly irreducible subalgebra of $\underline{P} := \underline{P}^{\mathbf{N}}$ and let*

$$\mathcal{B} = \mathbb{HSP}(\underline{Q}) = \mathbb{ISP}(\underline{\Pi}_{\mathcal{B}}) \quad \text{where } \underline{\Pi}_{\mathcal{B}} := \Big\{\, \underline{Q}_{\kappa} \mid \kappa \in N[Q] \,\Big\} .$$

For each $\kappa \in N[Q]$, let $\Omega_{\underline{Q}_{\kappa}} := \{\alpha_{\kappa}\}$ where $\alpha_{\kappa} := \varphi_{\kappa}(e)$. Take

$$\mathcal{E}[Q] := \Big\{\, w_{\kappa,\mu} \colon \underline{Q}_{\kappa} \to \underline{Q}_{\mu} \mid \kappa, \mu \in N[Q] \,\Big\}$$

and

$$\mathcal{S}[Q] := \Big\{\, (\alpha_{\kappa}, \alpha_{\mu})^{-1}(\leqslant)^{o} \leq \underline{Q}_{\kappa} \times \underline{Q}_{\mu} \mid \kappa, \mu \in N[Q] \,\Big\} .$$

Then $\mathcal{E}[Q] \cup \mathcal{S}[Q]$ yields a duality on $\mathbb{HSP}(\underline{Q})$.

More generally, let $\mathcal{C} := \mathbb{HSP}(\underline{Q}^1, \dots, \underline{Q}^k)$ (where $\underline{Q}^i \in \mathrm{Si}(\mathcal{A})$ for $i = 1, \dots, k$) be an arbitrary subvariety of $\mathcal{A}$. Then

$$R_{\mathcal{C}} := \bigcup \Big\{\, \mathcal{E}[Q^i] \cup \mathcal{S}[Q^i] \mid i = 1, \dots, k \,\Big\}$$

yields a duality on $\mathcal{C}$.

Note that in Theorem 3.6 the given set of relations $R_{\mathcal{C}}$ is the union of the sets of relations for dualities for the individual subvarieties $\mathbb{HSP}(\underline{Q}^i)$. This convenient localisation, observed previously for Ockham varieties, comes from the fact that the separation condition can be satisfied locally, as for the individual subvarieties $\mathbb{HSP}(\underline{Q}^i)$.

In order to appreciate how the duality in Theorem 3.6 works for an arbitrary subvariety $\mathcal{C}$, and to see its relation to the restricted Priestley duality, we need to describe the relational structure in terms of that of $\underset{\sim}{\mathbf{2}}^N$. Since the dualities for all subvarieties $\mathcal{C}$ of $\mathcal{A}$ work in exactly the same way, differing only in the subalgebras and carriers selected, we shall work below with the complete set

$$\underline{\Pi} := \left\{ \underline{Q}_\kappa \mid \underline{Q} \in \mathbb{S}(\underline{P}),\ \kappa \in N \right\}$$

of subalgebras (possibly with repetitions). Let $\underset{\sim}{\Pi}$ be the disjoint union of all the sets Q_κ equipped with all the relations $\mathcal{E}[Q_\kappa]$, $\mathcal{S}[Q_\kappa]$ ($\underline{Q} \in \mathbb{S}(\underline{P})$, $\kappa \in N$). The object $\underset{\sim}{\Pi}$ gives an excessively complicated piggyback duality for $\mathcal{A}$ itself — the antithesis of that in Theorem 3.1. Within this monster structure $\underset{\sim}{\Pi}$ sit all the objects $\underset{\sim}{\Pi}_{\mathcal{C}}$, and a uniform translation from natural duality to Priestley duality exists for them all.

The set P is a subset of 2^N, and every member of each $\underline{Q}_\kappa$ is a function from N to $\{0,1\}$. Then

$$Q_\kappa := \left\{ u_\kappa(a) \mid a = \langle a_\nu \rangle \in Q \right\} = \left\{ \langle a_{\nu\kappa} \rangle \mid \langle a_\nu \rangle \in Q \right\} = \gamma_\kappa(Q)\ .$$

Here the schizophrenic personality of Q_κ is clearly exposed. The set Q_κ wearing its algebraic hat is the image of Q under the endomorphism u_κ. The same set Q_κ, in terms of the dual structure, is characterised as the image of Q under the map γ_κ. The maps u_κ and γ_κ on 2^N are *the same map*.

We superscript elements of 2^N to denote elements of Q_κ regarded as belonging to $\underset{\sim}{\Pi}$. Since our piggybacking relations are subalgebras of $\underline{Q}_\kappa \times \underline{Q}_\mu$, with the same $\underline{Q}$ for each factor, we shall regard $\underline{Q}$ as fixed, and suppress it in our labelling. A typical element of Q_κ is therefore written $\langle a^\kappa_{\nu\kappa} \rangle$. We define $q_\kappa \colon Q_\kappa \to 2^N$ by $q_\kappa(\langle a^\kappa_{\nu\kappa} \rangle) := \langle a_{\nu\kappa} \rangle$. Thus q_κ is just the natural embedding of the free-standing set Q_κ into P.

The following lemma looks more ferocious than it is. It simply says that, when we regard the sets Q_κ as subsets in 2^N via the maps q_κ, the piggybacking maximal subalgebras are obtained simply by restricting the $\pm$-order, and that the piggybacking endomorphisms are given by the γ_μ-maps.

Lemma 3.7. *Under the assumptions of Theorem* 3.6*,*

$$(\langle a^\kappa_{\nu\kappa} \rangle, \langle b^\lambda_{\nu\lambda} \rangle) \in (\alpha_\kappa, \alpha_\lambda)^{-1}(\leqslant)^o \iff q_\kappa(\langle a^\kappa_{\nu\kappa} \rangle) \preccurlyeq q_\lambda(\langle b^\lambda_{\nu\lambda} \rangle)$$

and $q_\kappa \circ v_{\kappa,\lambda} \circ g_\kappa = \gamma_{\kappa\lambda} \circ q_\kappa$.

Proof: Similar proofs appear in [15], Lemma 3.6, and [25], Lemma 4.11. We therefore give here only an outline of the argument. Let $a = \langle a^{\kappa}_{\nu} \rangle \in \underline{Q}_{\kappa}$, $b = \langle b^{\lambda}_{\nu} \rangle \in \underline{Q}_{\lambda}$. Then

$$\begin{aligned}(a,b) \in (\alpha_\kappa, \alpha_\lambda)^{-1}(\leqslant)^o &\iff \alpha_\kappa(f(a)) \leqslant \alpha_\lambda(f(b)) \quad \text{for all unary operations } f \\ &\iff (\forall \nu \in N) \begin{cases} (g_\nu(\alpha_\kappa))(a) \leqslant (g_\nu(\alpha_\lambda))(b) & \text{if } \nu \in N^+, \\ (g_\nu(\alpha_\kappa))(a) \geqslant (g_\nu(\alpha_\lambda))(b) & \text{if } \nu \in N^- \end{cases} \\ &\iff (\forall \nu \in N) \begin{cases} (\nu\alpha_\kappa)(a) \leqslant (\nu\alpha_\lambda)(b) & \text{if } \nu \in N^+, \\ (\nu\alpha_\kappa)(a) \geqslant (\nu\alpha_\lambda)(b) & \text{if } \nu \in N^- . \end{cases}\end{aligned}$$

Also

$$q_{\kappa\lambda}(v_{\kappa,\lambda}(\langle a^{\kappa}_{\nu\kappa} \rangle)) = q_{\kappa\lambda}(\langle a^{\kappa\lambda}_{\nu\kappa\lambda} \rangle) = \langle a_{\nu\kappa\lambda} \rangle = \gamma_\lambda(\langle a_{\nu\kappa} \rangle) = \gamma_\lambda(q_\kappa(\langle a^{\kappa}_{\nu\kappa} \rangle)).$$

We conclude that $q_{\kappa\lambda} \circ v_{\kappa,\lambda} \circ g_\kappa = \gamma_\lambda \circ q_\kappa$. ∎

The natural dual of an algebra $A \in \mathcal{C}$, for $\mathcal{C}$ a subvariety of $\mathcal{A} := \mathcal{A}^{\mathbf{N}}$ is defined to be the disjoint union of hom-sets $\mathcal{A}(A, \underline{Q}_{\kappa})$ ($\underline{Q}_{\kappa} \in \underline{\Pi}_{\mathcal{C}}$), structured pointwise by the relations described in Theorem 3.6. When A is $F\mathcal{C}(1)$, the free algebra on 1 generator, there is a bijective correspondence between Q_κ and $\mathcal{A}(A, \underline{Q}_{\kappa})$. Thus $\underset{\sim}{\Pi}_{\mathcal{C}}$ belongs to the category $\mathcal{Z} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi}_{\mathcal{C}})$ dual to $\mathcal{C}$, and serves as the dual $D(F\mathcal{C}(1))$ of $F\mathcal{C}(1)$ (see [15], Lemma 1.2). We shall show that a natural simultaneous embedding of the sets in $\Pi_{\mathcal{C}}$ into 2^N gives $H(F\mathcal{C}(1))$, and that this embedding lifts pointwise to give the Priestley duals of arbitrary algebras in $\mathcal{C}$. We shall state without proof this translation between the natural dual and the Priestley dual of an algebra $A \in \mathcal{C}$. We denote the dual category **N-P** by $\mathcal{Y}$. To avoid complicated notation we present it just for the case of a variety generated by a single subdirectly irreducible algebra. The general case works in the same manner. The specialisation to the Ockham case appears as Theorem 3.8 of [15].

Theorem 3.8. *Let $\mathcal{B} := \mathbb{HSP}(\underline{Q})$ ($\underline{Q}$ a subdirectly irreducible subalgebra of $\underline{P}$), let $\underline{\Pi}_{\mathcal{B}} := \{\underline{Q}_{\kappa} \mid \kappa \in N[Q]\}$ and let $\underset{\sim}{\Pi}_{\mathcal{B}} := (\Pi_{\mathcal{B}}; \tau, \mathcal{E}[Q] \cup \mathcal{S}[Q])$, where $\mathcal{E}[Q]$ and $\mathcal{S}[Q]$ are as in Theorem 3.6 and τ is the discrete topology. Fix $A \in \mathcal{B}$ and let $X^\kappa := \mathcal{A}(A, \underline{Q}_{\kappa})$, for $\kappa \in N[Q]$, so that $D(A) = \dot{\bigcup}\{\, X^\kappa \mid \kappa \in N[Q]\,\}$. Let $Y = H(A)$ and and let*

$$Y^\kappa := \Big\{\, y \in H(A) \mid \ y = \alpha_\kappa \circ \varphi \ \text{ for some } \varphi \in D(A) \Big\} .$$

From $D(A) \in \mathcal{X} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi}_{\mathcal{B}})$, construct $(Z; \mathcal{T}, \leqslant, \{g_\mu\}_{\mu \in N}) \in \mathcal{Y}$ as follows. Define an equivalence relation ρ on X by

$$\rho := \bigcup \Big\{ \, s \cap s^{\smile} \mid \; s \in \mathcal{S}[Q] \, \Big\}$$

(where $(x, y) \in s^{\smile}$ if and only if $(y, x) \in s$). More explicitly,

$$\text{for } x \in X^\kappa,\ y \in X^\lambda, \quad x \ \rho\ y \iff (x, y) \in \ker(\alpha_\kappa, \alpha_\lambda)^o$$
$$(:= (\alpha_\kappa, \alpha_\lambda)^{-1}(\leqslant)^o \cap (\alpha_\lambda, \alpha_\kappa)^{-1}(\leqslant)^o) \ .$$

Let $\pi \colon X \to X/\rho$ be the canonical projection. Then define

- **(i)** *$Z := X/\rho$, with the quotient topology;*
- **(ii)** *$\pi(x) \leqslant \pi(y)$ in Z if and only if $(x, y) \in s$ for some $s \in \mathcal{S}[Q]$;*
- **(iii)** *for $\mu \in N$, and $x \in X^\kappa$, $g_\mu(\pi(x)) = \pi(w_{\kappa,\lambda}(x))$, where $\lambda \in N[Q]$ is such that $\underline{Q}_\mu \subseteq \underline{Q}_\lambda$.*

Then the order and operations on Y are well defined, and Y is isomorphic in $\mathcal{Y}$ to $H(A)$.

From $Y \in \mathcal{Y}$ construct a space $\mathbf{X} = (X; \mathcal{T}, S \cup E) \in \mathcal{X}$ as follows.

- **(i)′** *$X = \dot{\bigcup}\{ \, Y^\kappa \mid \kappa \in N[Q] \, \}$;*
- **(ii)′** *X carries the sum topology induced by subspace topologies on the components Y^κ;*
- **(iii)′** *$S := \{ \leqslant \cap (Y^\kappa \times Y^\lambda) \mid \; \kappa, \lambda \in N[Q] \, \}$;*
- **(iv)′** *$E = \{ \, g_\lambda{\restriction}Y^\kappa \mid \; \kappa, \lambda \in N[Q] \, \}$.*

Then $D(A)$ is isomorphic to $\mathbf{X}$.

The fact that the quotient map π maps $D(A)$ *onto* $H(A)$ relies on the fact that the maps $\Phi_{\alpha_\kappa} := \alpha_\kappa \circ -$ associated with our chosen carrier maps α_κ are jointly onto (so that $Y = \bigcup\{Y^\kappa \mid \kappa \in N[Q]\}$, for $Y = H(A)$). This does not need direct proof: it is forced by the separation condition (S); see Section 2 of [15]. Nevertheless, it is worth recording the version of the Joint-ontoness Lemma which applies to dualities as formulated here. We do not give the proof, since this follows very closely that of Lemma 3.3 of [15]. To minimise double subscripting we have changed the notation a little.

Lemma 3.9. *Assume the same definitions and the same conditions as in Theorem* 3.6. *For each* κ, *define maps*

$$\Psi_\kappa \ (= \Phi_z) := \alpha_\kappa \circ - \colon \mathcal{A}(A, \underset{\sim}{Q}_\kappa) \to \operatorname{im}\Phi_z \ ,$$
$$\Theta_\kappa := k_A(-) \circ h_\kappa \colon \operatorname{im}\Phi_z \to \mathcal{A}(A, \underset{\sim}{Q}_\kappa) \ ,$$

where $h_\kappa(y)(g_\nu(z)) := g_\nu(y)$. *Then* Ψ_κ *and* Θ_κ *are mutually inverse bijections.*

Free algebras are now extremely easy to describe. Since we wish to use our descriptions in Section 5 to obtain equational bases, we need to consider general subvarieties of $\mathbb{ISP}(\underset{\sim}{P})$. Let $\underset{\sim}{Q}^1, \dots, \underset{\sim}{Q}^k$ belong to $\mathbb{S}(\underset{\sim}{P})$. We adapt the notation employed earlier for a single subalgebra $\underset{\sim}{Q}$ to the present setting by appending superscript is to indicate the subalgebra under consideration. Thus, for example, we denote the $\ll$-maximal homomorphic images of $\underset{\sim}{Q}^i$ by $\underset{\sim}{Q}^i_\kappa$ ($\kappa \in N[Q^i]$). Once again we are extending results from [15] (Theorem 3.15) and [25] (Theorem 4.12), and the proof strategy is the same. We therefore omit the proof, noting merely that we must carry out the translation process on $\underset{\sim}{\Pi}_{\mathcal{C}} = D(F\mathcal{C}(1))$.

Theorem 3.10. *Let* $\mathbf{N} \in \mathcal{N}^*$ *and let* $\mathcal{C}$ *be the variety generated by the subalgebras* $\underset{\sim}{Q}^1, \dots, \underset{\sim}{Q}^k$ *of* $\underset{\sim}{P}^{\mathbf{N}}$. *Then*

$$\begin{aligned} H(F\mathcal{C}(1)) &= \Big\{\, g_\mu(y) \mid \ y \in Q^i_\kappa, \ \kappa, \mu \in N[Q^i], \ i = 1, \dots, k \,\Big\} \\ &\qquad \text{(as a subspace of } \underset{\sim}{\mathbf{2}}^N) \\ &= \Big\{\, g_\mu(y) \ (\mu \in N) \mid \ \underset{\sim}{\mathbf{2}}^N \models \sigma_e(y) \,\Big\} \ . \end{aligned}$$

More generally, the Priestley dual of $F\mathcal{B}(s)$ *is obtained by*

(**a**) *taking the* s*-fold power of* $\underset{\sim}{\Pi}_{\mathcal{B}}$ *in* $\mathcal{X}$, *that is,*

$$\underset{\sim}{\Pi}^s_{\mathcal{B}} := \bigcup \Big\{\, (Q^i_\kappa)^s \mid \ \kappa \in N[Q^i], \ i = 1, \dots, k \,\Big\} \ ,$$

with relational structure lifted pointwise from $\underset{\sim}{\Pi}_{\mathcal{B}}$,

(**b**) *applying the translation process described in Theorem 3.8.*

4 – Double MS-algebras

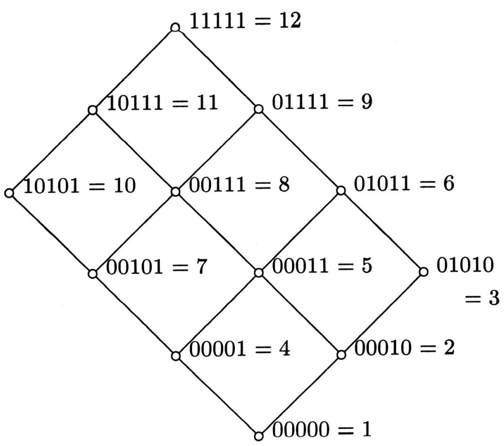
Here we have P as the set of quintuples $\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle$ in which $a_{h^2} \leqslant a_1 \leqslant a_{g^2}$ and $a_h \leqslant a_g$. We shall represent these 12 elements as binary strings of length 5. The algebra $\underline{P}$ is as shown in Figure 2. The same set under its $\pm$-order is shown as $\underset{\sim}{P}$ in Figure 2. The operations are given by

$$\begin{aligned}
\varphi_g(\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle) &= \langle \overline{a_g}, \overline{a_{g^2}}, \overline{a_g}, \overline{a_{g^2}}, \overline{a_g}\rangle,\\
\varphi_h(\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle) &= \langle \overline{a_h}, \overline{a_{h^2}}, \overline{a_h}, \overline{a_{h^2}}, \overline{a_h}\rangle,\\
\varphi_{g^2}(\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle) &= \langle a_{g^2}, a_g, a_{g^2}, a_g, a_{g^2}\rangle,\\
\varphi_{h^2}(\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle) &= \langle a_{h^2}, a_h, a_{h^2}, a_h, a_{h^2}\rangle\ .
\end{aligned}$$

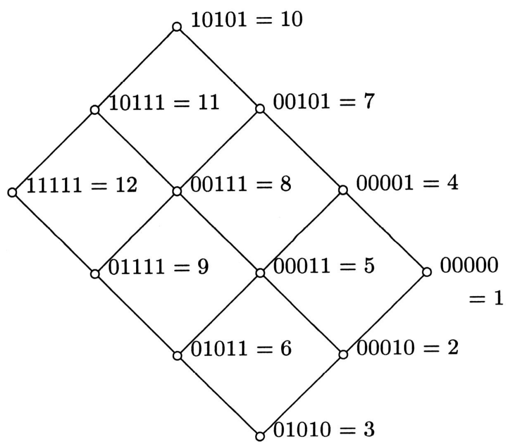
Likewise,

$$\begin{aligned}
\gamma_g(\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle) &= \langle a_h, a_{h^2}, a_g, a_{g^2}, a_g\rangle,\\
\gamma_h(\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle) &= \langle a_h, a_{h^2}, a_h, a_{g^2}, a_g\rangle,\\
\gamma_{g^2}(\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle) &= \langle a_{h^2}, a_h, a_{g^2}, a_g, a_{g^2}\rangle,\\
\gamma_{h^2}(\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle) &= \langle a_{h^2}, a_h, a_{h^2}, a_g, a_{g^2}\rangle\ .
\end{aligned}$$

Also $\varphi_1 = \mathrm{id}$ and $\gamma_1 = \mathrm{id}$.



$\underline{P}$



$\underset{\sim}{P}$

Fig. 2

Observe that the transformation from $\leqslant$ on $\underline{P}$ to $\preccurlyeq$ on $\underset{\sim}{P}$ is effected simply by reflecting in the central NW–SE diagonal. The maps g and h have geometric interpretations too, though these are less transparent. For g, consider $\underset{\sim}{P}$ with the points $4, 5, 6$ removed, and re-drawn as $\underline{\mathbf{3}} \times \underline{\mathbf{3}}$ with horizontal and vertical symmetry. The map g on this subset is just reflection in the horizontal axis joining 1 and 12. Finally, $g(4+i) = g(7+i)$ for $i = 0, 1, 2$. The map h arises in the dual manner.

The methods for treating the subvariety lattice of a finitely generated congruence-distributive variety stem from Jónsson's Lemma, and have been applied many times to varieties of distributive-lattice-ordered algebras; see for example [13], [2] and [22]. The lattice $\Lambda(\mathbf{DMS})$ of subvarieties of **DMS** has been partially analysed by T.S. Blyth, A.S.A. Noor and J.C. Varlet in [5], [3], and a summary of these results appears in [6]. The general theory tells us that the lattice $\Lambda(\mathbf{DMS})$ is isomorphic to the lattice of order ideals of the poset of join-irreducible members of $\Lambda(\mathbf{DMS})$. This poset, V, may be identified with $\mathrm{Si}(\mathbf{DMS})$, the (isomorphism classes of) non-trivial subdirectly irreducible algebras in **DMS**, ordered by $\underline{Q} \leq \underline{R}$ if and only if $\underline{Q} \in \mathbb{HS}(\underline{R})$. In fact $\mathrm{Si}(\mathbf{DMS})$ can be identified with the isomorphism classes of subalgebras of $\underline{P}$ (see [5] and [25], Theorem 3.5). There are 21 elements in V, ordered as shown in Figure 3. The numbering follows that used in [6], p. 198; our $\mathbf{k}$ is denoted SID_k in [6], the trivial algebra being omitted from the listing there.

Our representation of the order of V looks different from that given in [5] and [6], p. 198. We have taken advantage of knowledge of $\Lambda(\mathbf{DMS})$ gained through work on identities in [27] to give a picture which better reflects the equational relationships involved. In V we have a central 'spine', consisting of

5,6, 8,10, 11,12, 15,16, 17,18, 1, 3, 9, 13, 20,

of which the first 10 elements occur in mutually dual pairs $\mathbf{k}$, $\mathbf{k}^d$ (that is, the lattices are order duals and $^+$ and ^{o} are interchanged) and the remainder are self-dual. Added to this spine are two 'flaps': **2, 4** and **7, 14, 19**, all of which are self dual. The former flap supplies the 'Stone-like' subvarieties: double Stone algebras ($\mathbb{HSP}(\mathbf{4})$) and doubled Stone algebras ($\mathbb{HSP}(\mathbf{2})$).

Let $\underline{Q}$ be a subdirectly irreducible algebra in **DMS**. Since every homomorphic image of $\underline{Q}$ is isomorphic to a subalgebra of $\underline{Q}$, we have $\mathbb{HSP}(\underline{Q}) = \mathbb{ISP}(\underline{Q})$ and can base a natural duality for this variety on Theorem 3.4. Proceeding this way we can obtain natural dualities for each of the join-irreducible subvarieties in $\Lambda(\mathbf{DMS})$. These dualities were first derived in [28], by direct application of the Generalised Piggyback Duality Theorem to each variety individually. Whether the determination of the endomorphisms and the selection of carriers is treated algebraically as in [28] or is based on Lemma 3.3 a substantial amount of work is involved in describing each duality completely.

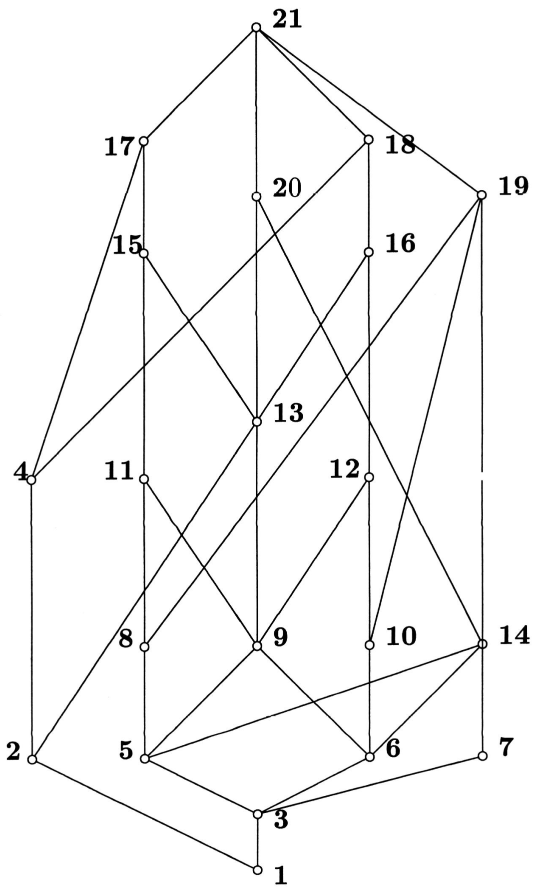


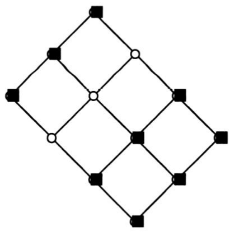
Fig. 3: $V = \mathrm{Si}(\mathbf{DMS})$.

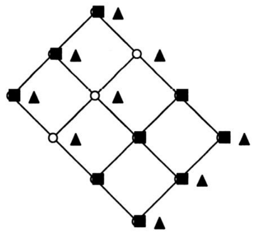
We contend here that, rather than invoking Theorem 3.4, it is preferable to appeal to Theorem 3.6 and the interpretation of its results in terms of $\underset{\sim}{\mathbf{2}}^N$. The advantages of this approach is that a minimum of calculation is required. For a given variety $\mathcal{B} := \mathbb{ISP}(\underline{Q})$ ($\underline{Q}$ subdirectly irreducible, Q regarded as a subset of the 12 quintuples in $\underline{P}$) we proceed as indicated below. We shall henceforth write γ_μ simply as μ for $\mu \in \{g, g^2, h, g^2\}$ and suppress parentheses to make our notation more compact in the table which follows. Remember that γ_μ is the same map as u_μ.

20

S 2 3 4 5 6 10 11

gS 2 3 7 8 9 10 11

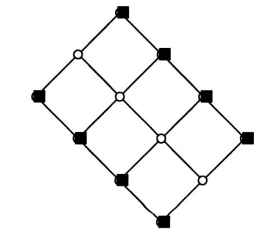


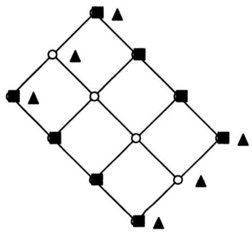


19

S 3 4 6 7 9 10

gS 2 3 10 11





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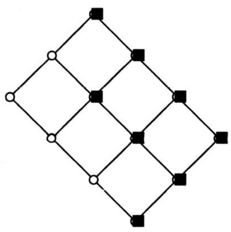
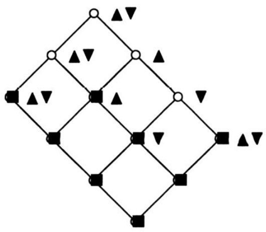
17 / 18^d

S 2 3 5 6 8 9

gS 7 8 10 11

hS 4 5 10 11





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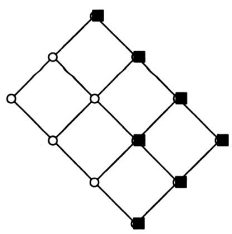
15 / 16^d

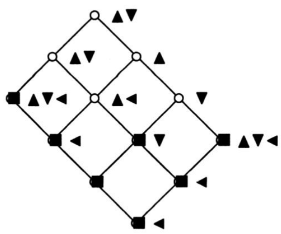
S 2 3 5 6 9

gS 7 8 10 11

hS 4 5 10 11

g^2S 2 3 8 9



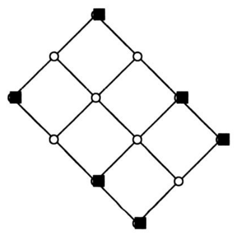


14

S 3 4 6 10

gS 2 3 10 11

g^2S 3 7 9 10



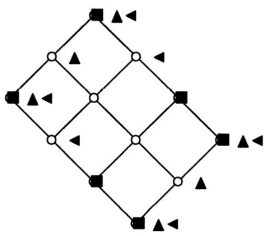


Table 1

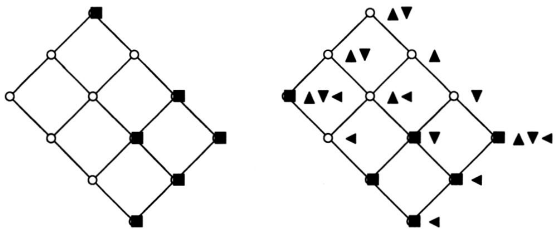
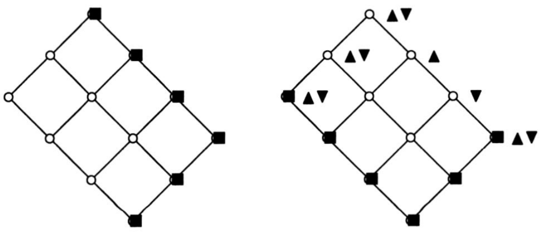
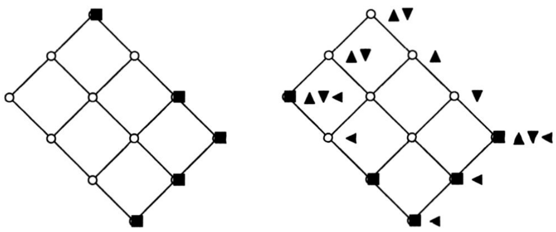
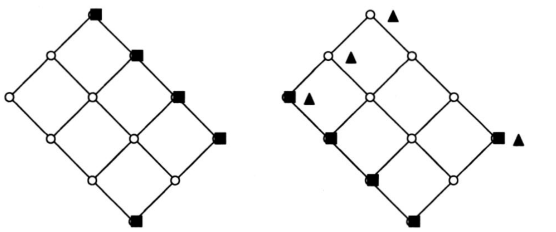
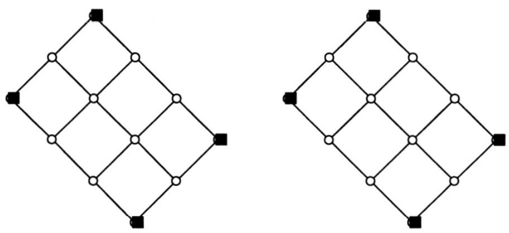
13 S 2 3 5 6 gS 7 8 10 11 hS 4 5 10 11 g^2S 2 3 8 9	
11 / 12^d S 2 3 6 9 gS 7 10 11 hS 4 10 11	
9 S 2 3 6 gS 7 10 11 hS 4 10 11 g^2S 2 3 9	
8 / 10^d S 3 6 9 gS 10 11	
7 S 3 10	

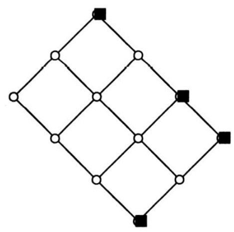
Table 1 (cont.)

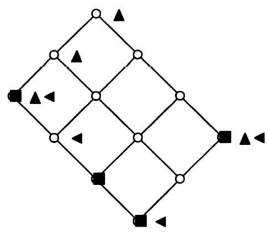
5 / 6^d

S 3 6

gS 10 11

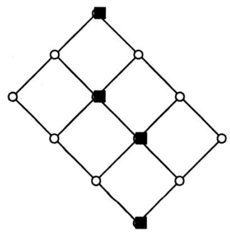
g^2S 3 9

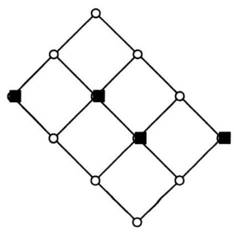




4

S 5 8

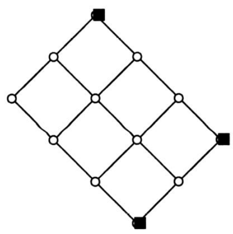


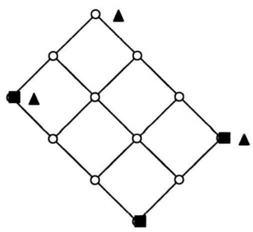


3

S 3

gS 10

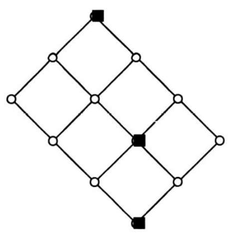


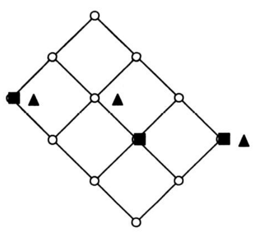


2

S 5

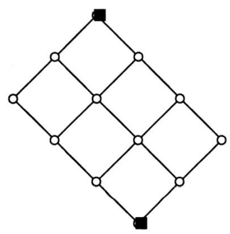
gS 8





1

S $\varnothing$



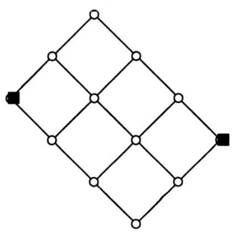


Table 1 (cont.)

(**1**) Compute the images gQ, hQ, g^2Q, h^2Q; these are the homomorphic images of $\underline{Q}$, regarded as subsets of P.

(**2**) Select the elements of $\{\, Q, gQ, hQ, g^2Q, h^2Q \,\}$ which are maximal with respect to inclusion; these elements, regarded as subalgebras of $\underline{P}$, comprise the set $\underline{\Pi}_{\mathcal{B}}$ on which the duality is based.

(**3**) Regard the sets R ($\underline{R} \in \underline{\Pi}_{\mathcal{B}}$) as subsets of $\underset{\sim}{P}$; the disjoint union of the sets R acquire a relational structure by restricting the $\pm$-order and the maps g, h, g^2, h^2.

The resulting structure serves as the *alter ego* $\underset{\sim}{\Pi}_{\mathcal{B}}$ of $\underline{\Pi}_{\mathcal{B}}$ we require for our duality for $\mathbb{HSP}(\underline{Q})$. Since the relational structure is determined by that of $\underset{\sim}{P}$ we can record the duality simply by specifying the subsets of $\underline{P}$ obtained in (3).

Table 1 gives natural dualities for the join-irreducible subvarieties of **DMS**, in a diagrammatic form. For each listed subvariety, $\mathbb{HSP}(\underline{Q})$, we give the following information.

(**a**) The code number of the subalgebra generating the variety.

(**b**) The set $S := Q \backslash \{00000, 11111\}$, labelled as in Table 1. The omitted points occur in every subset μQ and so do not need to be recorded explicitly.

(**c**) Those sets μS for which the corresponding subalgebras $\gamma_\mu(\underline{Q})$ are $\ll$-maximal. The sets μQ give the components of $\underset{\sim}{\Pi}_{\mathcal{B}}$.

(**d**) The algebra $\underline{Q}$ as a subalgebra of $\underline{P}$, the points being indicated by $\blacksquare$.

(**e**) The sets μQ, for μ as in (c), as subsets of $\underset{\sim}{P}$. To save space, we have marked all these sets on a single copy of $\underset{\sim}{P}$. The symbols have the following meanings.

$$Q \quad \blacksquare, \qquad gQ \quad \blacktriangle, \qquad hQ \quad \blacktriangledown, \qquad g^2Q \quad \blacktriangleleft, \qquad h^2Q \quad \blacktriangleright$$

We have included in the table the diagram for only one of each pair of mutually dual subdirectly irreducible algebras. Consider the map on the Boolean algebra $\underline{\mathbf{2}}^5$ sending $\langle a_{h^2}, a_h, a_1, a_g, a_{g^2}\rangle$ to $\langle \overline{a_{g^2}}, \overline{a_g}, \overline{a_1}, \overline{a_h}, \overline{a_{h^2}}\rangle$. Restricted to P, this map is simply the rotation of P in its own plane through the angle π, and converts any subalgebra of $\underline{P}$ to a dually isomorphic subalgebra. Given the object $\underset{\sim}{\Pi}$ giving a duality for $\mathbb{HSP}(\underline{Q})$ we obtain an object $\underset{\sim}{\Pi}^d$ giving a duality for $\mathbb{HSP}(\underline{Q}^d)$ just by turning the picture upside down. More precisely, consider P with its $\pm$-order as a fixed template, and consider the labelling obtained for $\mathbb{HSP}(\underline{Q})$ written on a transparent sheet as an overlay. Then we rotate the transparent sheet through π, keeping the template fixed. Our notation ensures that the resulting configuration gives a correct labelling (▲ and ▼ swop over, as do ◄ and ►). In certain cases our inversion process converts the algebra $\underline{Q}$ not into $\underline{Q}^d$ as defined in our list of subalgebras but into an isomorphic copy of it. Thus inversion applied to the duality for $\mathbb{HSP}(\underline{Q})$ will not always give the same duality as we get if we work with $\mathbb{HSP}(\underline{Q}^d)$ directly. This minor awkwardness could be avoided by suitably picking the generating algebras for the join-irreducible subvarieties. However, for maximum compatibility with the existing literature we have elected to work with Blyth and Varlet's original choice. For an example, consider **5** and **6**: inversion of **5** gives the subalgebra with elements $1, 7, 10, 12$, and this is isomorphic as a **DMS**-algebra to the subalgebra with elements $1, 2, 3, 12$, that is, **6**.

The image under the natural embedding map into 2^5 of the structure $\underset{\sim}{P}$ is a **DMS**-space, dual to $F\mathbf{DMS}(1)$. The set of all tagged points, as a subspace of $\underset{\sim}{P}$, gives $H(F\mathcal{B}(1))$. For the free algebras on s generators, $s > 1$, we proceed by the following steps.

(**1**) Separate the components, labelling the points to retain the knowledge of which point of P each point corresponds to.

(**2**) Take the s-fold power, component by component, to obtain $D(F\mathcal{B}(s))$.

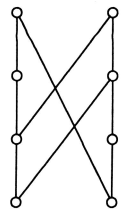
(**3**) Go from $D(F\mathcal{B}(s))$ to $H(F\mathcal{B}(s))$ by identifying points which carry the same label. The order and the maps g, h, g^2 and h^2 are inherited from $\underset{\sim}{P}^s$.

From Table 1 we may immediately read off descriptions of $F\mathcal{B}(s)$ for every join-irreducible subvariety $\mathcal{B}=\mathbb{HSP}(\underline{Q})$. For **DMS** itself we have $H(F\mathbf{DMS}(s))=\underset{\sim}{P}^s$. Its underlying lattice is the s-fold **D**-copower of the **D**-coproduct $\mathbf{4}\coprod\mathbf{5}$.

For $\mathcal{B} = \mathbb{HSP}(\underline{Q})$, for $\underline{Q}$ in the following sets, $F\mathcal{B}(1)$ is the same for all, and has the indicated distributive lattice reduct, M being the lattice whose dual is as shown in Figure 4:

$\underline{Q}$	**D**-reduct of $F\mathcal{B}(1)$
13, 15, 16, 17, 18, 20, 21	$\mathbf{4}\coprod\mathbf{5}$
9, 11, 12, 14, 19	$\mathbf{1} \oplus \mathbf{M} \oplus \mathbf{1}$
5, 6, 8, 10	$\mathbf{1} \oplus (\mathbf{2} \times \mathbf{5}) \oplus \mathbf{1}$
3, 7	$\mathbf{1} \oplus \mathbf{2}^2 \oplus \mathbf{1}$
2, 4	$\mathbf{2}^2 \times \mathbf{3}$
1	$\mathbf{2}^2$

Table 2



$H(M)$

Fig. 4

Note that it is immediate from Table 1 that the indicated free algebras are the same as algebras, rather than merely as lattices — no calculation of the operations is needed.

5 – Equational bases

We now fix a monoid $\mathbf{N} \in \mathcal{N}^*$ such that $N^+ \cap N^- = \emptyset$. That is, we require $\mathbf{N}$ to be an ordered $\pm$-monoid in the sense of [25]. Given a natural duality for $\mathcal{A}^{\mathbf{N}}$ we can write down in an algorithmic way identities of a canonical type defining each meet-irreducible subvariety in the lattice of subvarieties $\Lambda(\mathcal{A}^{\mathbf{N}})$, and thence identities for all subvarieties. This programme was carried through for finitely generated Ockham varieties, with examples, in [22], [1] and [24]. The general case introduces no essentially new ideas, and we shall simply state the results without proof and record the identities thereby produced for the variety **DMS**.

As usual we let $\underline{P} := \underline{P}^{\mathbf{N}}$ and $\mathcal{A} := \mathcal{A}^{\mathbf{N}}$, and denote the identity of N by e. Fix $\mathcal{C} \in \Lambda(\mathcal{A})$. We first need to identify free generators in $F\mathcal{C}(s)$, which we represent concretely as $KH(F\mathcal{C}(s))$. The following lemma generalises the second part of Theorem 4.6 of [1]. It relies on the fact that, under the identification of $D(F\mathcal{C}(s))$ with $\mathcal{X}(\underset{\sim}{\Pi}{}^s_{\mathcal{C}}, \underset{\sim}{\Pi}{}_{\mathcal{C}})$, the free generators in $F\mathcal{C}(s)$, regarded as $ED(F\mathcal{C}(s))$, are the co-ordinate projections. The lemma is then just the interpretation, via the translation process, of this statement in terms of the Priestley duality. We shall henceforth denote a typical element of $(2^N)^s$ in the form $\mathbf{c} = \langle \mathbf{c}_t \rangle$, where $\mathbf{c}_t = \langle c^t_\nu \rangle$, and let π_t be the natural projection of $(\underset{\sim}{\mathbf{2}}^N)^s$ onto the t^{th} component of $(\underset{\sim}{\mathbf{2}}^N)^s$, so that $\pi_t(\mathbf{c}) = \langle c^t_\nu \rangle$ $(t = 1, \ldots, s)$.

Lemma 5.1. *Let $1 \leqslant s < \omega$. Identify $H(F\mathcal{C}(s))$ with a subset of $(\underset{\sim}{\mathbf{2}}^N)^s$, as in* 3.10. *Then the algebra $F\mathcal{C}(s)$ is freely generated by $\{a_1, \ldots, a_s\}$, where*

$$(\forall\, y \in H(F\mathcal{C}(s))) \;\; a_t(y) = y^t_e \;.$$

We now introduce our canonical identities, in the form of inequalities. Let $L_1, \ldots, L_s$ be subsets of N and let $R_t = N \setminus L_t$ $(t = 1, \ldots, s)$. For any subset J of N let $J^+ := J \cap N^+$ and $J^- := J \cap N^-$. Associate with $\mathbf{L} := (L_1, \ldots, L_s)$ the inequality

$$\bigwedge_{r=1}^{s} \Big(\bigwedge_{\lambda \in L_r^+ \cup R_r^-} \varphi_\lambda(x_r) \Big) \leqslant \bigvee_{t=1}^{s} \Big(\bigvee_{\mu \in L_t^- \cup R_t^+} \varphi_\mu(x_t) \Big) ,$$

with the usual convention that an empty meet is 1 and an empty join is 0. Such an identity is taken relative to $\mathbf{D}\text{-}\mathbf{N}_=$, rather than with respect to $\mathcal{A}$ $(= \mathbf{D}\text{-}\mathbf{N})$. It can often be simplified by taking into account identities or inequalities defining $\mathcal{A}$ within $\mathbf{D}\text{-}\mathbf{N}_=$. To illustrate, we give in Table 3 the 1-variable identities associated with those quintuples we shall require when considering **DMS**, the monoid here being $\mathbf{N}_=^{\mathbf{DMS}}$ (that is, we work in the variety known as $\mathbf{DK}_{1,1}$, in which $a^\circ = a^{\circ\circ}$ and $a^+ = a^{++}$ but the other **DMS** identities do not hold). Within **DMS**, the identity derived from quintuple 2 reduces to $a^+ \leqslant a^\circ \vee a^{\circ\circ}$, that derived from 6 to $a^{\circ\circ} \leqslant a^+ \vee a^{++}$, and so on. Note that the quintuples occur in mutually dual pairs, giving mutually dual identities: the pairs are

$$2, 11, \quad 3, 10, \quad 4, 9, \quad 5, 8, \quad 6, 7 .$$

	$\mathbf{c}$	$I_{\mathbf{L}(\mathbf{c})}$	**DMS** identity
2	00010	$a^+ \leqslant a \vee a^\circ \vee a^{\circ\circ} \vee a^{++}$	$a^+ \leqslant a^\circ \vee a^{\circ\circ}$
3	01010	$1 = a \vee a^\circ \vee a^{\circ\circ} \vee a^+ \vee a^{++}$	$1 = a^{\circ\circ} \vee a^+$
4	00001	$a^\circ \wedge a^{\circ\circ} \wedge a^+ \leqslant a \vee a^{++}$	$a^\circ \wedge a^{\circ\circ} \leqslant a$
5	00011	$a^{\circ\circ} \wedge a^+ \leqslant a \vee a^\circ \vee a^{++}$	$a^{\circ\circ} \wedge a^+ \leqslant a \vee a^\circ$
6	01011	$a^{\circ\circ} \leqslant a \vee a^\circ \vee a^+ \vee a^{++}$	$a^{\circ\circ} \leqslant a \vee a^+$
7	00101	$a \wedge a^\circ \wedge a^+ \wedge a^{\circ\circ} \leqslant a^{++}$	$a \wedge a^\circ \leqslant a^{++}$
8	00111	$a \wedge a^{\circ\circ} \wedge a^+ \leqslant a^\circ \vee a^{++}$	$a \wedge a^+ \leqslant a^\circ \vee a^{++}$
9	01111	$a \wedge a^{\circ\circ} \leqslant a^\circ \vee a^+ \vee a^{++}$	$a \leqslant a^+ \vee a^{++}$
10	10101	$a \wedge a^\circ \wedge a^{\circ\circ} \wedge a^+ \wedge a^{++} = 0$	$a^\circ \wedge a^{++} = 0$
11	10111	$a \wedge a^{\circ\circ} \wedge a^+ \wedge a^{++} \leqslant a^\circ$	$a^+ \wedge a^{++} \leqslant a^\circ$

Table 3: Identities associated with **DMS** quintuples.

The next lemma follows from Lemma 5.1. For a subvariety $\mathcal{C}$ of $\mathcal{A}$ we shall write $F_{\mathcal{C}}^s$ for $H(F\mathcal{C}(s))$ regarded as a subset of $(\underset{\sim}{\mathbf{2}}^N)^s$.

Lemma 5.2. *Let $\mathcal{B}$ and $\mathcal{C}$ be subvarieties of $\mathcal{A}$ with $\mathcal{B} \subsetneqq \mathcal{C}$. Suppose that $F^i_{\mathcal{B}} = F^i_{\mathcal{C}}$ for $i < s$ and that $\langle c^t_\nu \rangle \in F^s_{\mathcal{C}} \setminus F^s_{\mathcal{B}}$. Let*

$$L_t := \Big\{ \nu \in N^+ \mid \; c^t_\nu = 1 \Big\} \cup \Big\{ \nu \in N^- \mid \; c^t_\nu = 0 \Big\} \quad (t = 1, \dots, s)$$

and let $\mathbf{L}(\mathbf{c}) := (L_1, \dots, L_s)$. Then $I_{\mathbf{L}(\mathbf{c})}$ holds in $\mathcal{B}$ and fails in $\mathcal{C}$, and no identity in fewer than s variables distinguishes $\mathcal{B}$ and $\mathcal{C}$.

The final step in determining defining identities for the subvarieties of $\mathcal{A}$ is to give identities for the meet-irreducible subvarieties. For this we appeal to [24], where some elementary theory of varieties (Lemma 5.3) is combined with some well-known duality theory (Lemma 5.4).

Lemma 5.3. *Let $\mathcal{K}$ be any finitely generated congruence-distributive variety. Then any meet-irreducible subvariety $\mathcal{B}$ is determined within $\Lambda(\mathcal{K})$ by an identity holding in $\mathcal{B}$ and failing in its upper cover.*

Lemma 5.4. *Let V be a finite poset and $\mathcal{O}(V)$ its lattice of order-ideals. Then*

(**i**) *there is a bijection between the join-irreducible elements, $\mathcal{J}(\mathcal{O}(V))$, and the meet-irreducible elements, $\mathcal{M}(\mathcal{O}(V))$, given by*

$$\zeta \colon {\downarrow}p \mapsto \bigvee \Big\{ y \mid \; y \text{ is maximal in } (V \setminus {\uparrow}p) \Big\} ,$$
$$\zeta^{-1} \colon V \setminus {\uparrow}p \mapsto \bigwedge \Big\{ y \mid \; y \text{ is minimal in } (V \setminus {\downarrow}p) \Big\} ;$$

(**ii**) *for each meet-irreducible element $V \setminus {\uparrow}p$ the unique upper cover is $(V \setminus {\uparrow}p) \cup {\downarrow}p$.*

Putting together the preceding results we have the following algorithmic procedure. We take $V := \mathrm{Si}(\mathcal{A})$ ordered as usual by $\underline{Q} \leqslant \underline{R}$ if and only if $\underline{Q} \in \mathbb{HS}(\underline{R})$. The join-irreducible elements of $\Lambda(\mathcal{A})$ are then labelled by the elements of V. Let $\underline{Q}_0 \in V$.

(**1**) Express via the map ζ defined in Lemma 5.4 the meet-irreducible element associated with $\underline{Q}_0$ as the join of join-irreducible elements $\underline{Q}_1, \dots, \underline{Q}_k$. Let $\mathcal{B} := \mathbb{HSP}(\underline{Q}_1, \dots, \underline{Q}_k)$ and let $\mathcal{C} := \mathbb{HSP}(\underline{Q}_0, \underline{Q}_1, \dots, \underline{Q}_k)$ be its unique upper cover.

(**2**) Identify $Q_0, Q_1, \dots, Q_k$ with (suitably tagged) subsets of 2^N, as in Section 4.

(**3**) Form the sets

$$F^s_{\mathcal{B}} := \dot{\bigcup}\Big\{ (\gamma_\nu(Q_i))^s \mid \nu \in N, i = 1, \dots, k \Big\},$$

$$F^s_{\mathcal{C}} := \dot{\bigcup}\Big\{ (\gamma_\nu(Q_i))^s \mid \nu \in N, i = 0, \dots, k \Big\},$$

where s is chosen as the smallest value $\geqslant 1$ for which these sets differ.

(**4**) Pick $\mathbf{c} \in F^s_{\mathcal{C}} \setminus F^s_{\mathcal{B}}$ and define a corresponding identity $I_{\mathbf{L}(\mathbf{c})}$ in the manner described in Lemma 5.2.

Then $\mathcal{B}$ is determined within $\Lambda(\mathcal{A})$ by $I_{\mathbf{L}(\mathbf{c})}$, and by no identity in fewer variables.

We now present the results of applying this algorithm to the **DMS** subvarieties. Table 4 gives the correspondences between join-irreducible elements and meet-irreducible elements in $\Lambda(\mathbf{DMS})$, identified with $\mathcal{O}(V)$, where V is as shown in Figure 3. The first column gives the code numbers of the elements of V, the join-irreducibles of $\mathcal{O}(V)$. The second column indicates the associated meet-irreducible elements: for example, the meet-irreducible associated with **5** is $\mathbf{4} \vee \mathbf{7} \vee \mathbf{10}$. For each meet-irreducible we give a point $\mathbf{c}$ as in Step 4 of our algorithm, by listing the code numbers of its components $\mathbf{c}_t$ $(t = 1, \dots, s)$ (column 3). The final column gives $I_{\mathbf{L}(\mathbf{c})}$. This list of identities, together with the identity $a = b$ for the trivial variety, is sufficient to define via conjuncts every **DMS** subvariety. In certain cases the identities can be simplified somewhat; see the comments below.

As noted in [24] we can say more. Since $\mathcal{M}(\mathcal{O}(V)) \cong \mathcal{J}(\mathcal{O}(V)) \cong V$, these identities are ordered by implication according to the ordering of V (so that, for example, the identity $a^+ \wedge b^\circ \wedge b^{++} \leqslant a^\circ \vee a^{\circ\circ}$ implies the identity $a^\circ \wedge a^{\circ\circ} \wedge b \leqslant a \vee b^+ \vee b^{++}$ because $\mathbf{14} < \mathbf{19}$ in V). In other words, we are asserting that the lattice of equational theories of **DMS** is given by $\mathcal{O}(V)$), where the points of V are labelled with corresponding identities as given in Table 4.

The number of subvarieties of **DMS**, or equivalently the number of equational theories, was first found to be 381 by Blyth, Noor and Varlet ([4], p. 53). It is possible to write down equations for any given subvariety expressed as a join of join-irreducible subvarieties simply by re-expressing this join as a meet of meet-irreducible subvarieties; see [22]. We do not pursue this since in [27] we are able to present a much more direct approach based just on Priestley duality. In [27] we undertake a systematic study of re-write rules for **DMS** identities, or more generally for identities in any variety $\mathcal{A}^{\mathbf{N}}$. However a few brief remarks should be made about variants on our listed identities.

ji	mi	c	**DMS** identity
1			a=b
2	11 12 19	5	$a^{\circ\circ} \wedge a^{+} \leqslant a \vee a^{\circ}$
3	4	3	$1 = a^{\circ\circ} \vee a^{+}$
4	15 16 19 20	5 8	$a^{\circ\circ} \wedge a^{+} \wedge b \wedge b^{+} \leqslant a \vee a^{\circ} \vee b^{\circ} \vee b^{++}$
5	4 7 10	11	$a^{+} \wedge a^{++} \leqslant a^{\circ}$
6	4 7 8	2	$a^{+} \leqslant a^{\circ} \vee a^{\circ\circ}$
7	17 18	3 10	$b^{\circ} \wedge b^{++} \leqslant a^{\circ\circ} \vee a^{+}$
8	18 20	6 9	$a^{\circ\circ} \wedge b \leqslant a \vee a^{+} \vee b^{+} \vee b^{++}$
9	4 19	2 9	$a^{+} \wedge b \leqslant a^{\circ} \vee a^{\circ\circ} \vee b^{+} \vee b^{++}$
10	17 20	7 4	$a \wedge a^{\circ} \wedge b^{\circ} \wedge b^{\circ\circ} \leqslant a^{++} \vee b$
11	18 19 20	2 6 9	$a^{+} \wedge b^{\circ\circ} \wedge c \leqslant a^{\circ} \vee a^{\circ\circ} \vee b \vee b^{+} \vee c^{+} \vee c^{++}$
12	17 19 20	11 7 4	$a^{+} \wedge a^{++} \wedge b \wedge b^{\circ} \wedge c^{\circ} \wedge c^{\circ\circ} \leqslant a^{\circ} \vee b^{++} \vee c$
13	4 11 12 19	2 5	$a^{+} \wedge b^{\circ\circ} \wedge b^{+} \leqslant a^{\circ} \vee a^{\circ\circ} \vee b \vee b^{\circ}$
14	7 17 18	2 10	$a^{+} \wedge b^{\circ} \wedge b^{++} \leqslant a^{\circ} \vee a^{\circ\circ}$
15	11 18 19 20	9 5	$a \wedge b^{\circ\circ} \wedge b^{+} \leqslant a^{+} \vee a^{++} \vee b \vee b^{\circ}$
16	12 17 19 20	4 8	$a^{\circ} \wedge a^{\circ\circ} \wedge b \wedge b^{+} \leqslant a \vee b^{\circ} \vee b^{++}$
17	15 18 19 20	6 8	$a^{\circ\circ} \wedge b \wedge b^{+} \leqslant a \vee a^{+} \vee b^{\circ} \vee b^{++}$
18	16 17 19 20	7 5	$a \wedge a^{\circ} \wedge b^{\circ\circ} \wedge b^{+} \leqslant a^{++} \vee b \vee b^{\circ}$
19	17 18 20	4 9	$a^{\circ} \wedge a^{\circ\circ} \wedge b \leqslant a \vee b^{+} \vee b^{++}$
20	17 18 19	2 4	$a^{+} \wedge b^{\circ} \wedge b^{\circ\circ} \leqslant a^{\circ} \vee a^{\circ\circ} \vee b$
21	17 18 19 20	2 4 8	$a^{+} \wedge b^{\circ} \wedge b^{\circ\circ} \wedge c \wedge c^{+} \leqslant a^{\circ} \vee a^{\circ\circ} \vee b \vee c^{\circ} \vee c^{++}$

Table 4: DMS identities.

First of all, certain of the given identities can easily be seen to equivalent to simpler ones. For example, in line 8 of Table 4, b^{+} can be removed from the right-hand side by replacing b by $a \vee b$. Since such manipulations involve human input, and our aim in this paper is to show that identities can be generated automatically, we have not presented substitute identities derived in this manner. Further, every allowable choice of **c** in Step 4 of our algorithm gives an equivalent identity, always in a minimum number of variables. For the following cases, there is only one way to choose **c**: $4, 7$, and $8, 11, 16, 17$ and their duals. Where we had a choice we picked a point **c** so that $I_{\mathbf{L}(\mathbf{c})}$ contained a minimum number of total occurrences of the variables, and thereafter preferred identities in which the number of iterations of the operations $^{+}$ and $^{\circ}$ was minimal. Thus we only used either of the 5-tuples 5 and 8 as a component of **c** when we had no alternative, and used 3 or 10 whenever this was possible. Each of 5 and 8 leads to 4 occurrences of a variable, 2 on each side of the inequality, whereas 3 and 10 lead to 2 occurrences only, both on the same side. The remaining 5-tuples each lead to a variable appearing 3 times. Assuming this has been done, we select the component quintuples of **c** as far as possible so that, in $I_{\mathbf{L}(\mathbf{c})}$, a will override $a^{\circ\circ}$ or a^{++} rather than the other way round when the **DMS** inequalities $a^{++} \leqslant a \leqslant a^{\circ\circ}$ are invoked. So 6 would be preferred to 2, for example. While it is possible in a specific variety such as **DMS** to give empirical rules for selecting 'optimal' identities generated by the algorithm above, it does not seem profitable to try to write down rules for doing this in an arbitrary variety $\mathcal{A}^{\mathbf{N}}$. In general the relative merits of equivalent canonical inequalities depend on the interaction between the $\pm$-order of **N**, which dictates the way variables are distributed to left and right, and the order of **N**, which determines which points in 2^N are in play and the $\underline{\mathbf{2}}^N$-ordering between these points.

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