PORTUGALIAE MATHEMATICA
Vol. 55 Fasc. 4 – 1998

ONE PROPERTY OF TRIANGULAR NUMBERS *

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Introduction

The following set of four numbers

$$\Big[1,\, 3,\, 8,\, 120\Big]$$

is well-known. This has a property that product of any two different numbers increased by the integer one is perfect square. Many attempts of generalising this result have been made. Hoggatt and Bergum [1] have shown that the *property* is satisfied by the following set of four numbers

$$\Big[F_{2n},\, F_{2n+2},\, F_{2n+4},\, F_{2n+1},\, F_{2n+2},\, F_{2n+3}\Big]$$

where $n \geq 1$ and F_n denotes the n^{th} Fibonacci members.

Horadam [2] has obtained a few more similar result and has given a good historical account.

In this note we explain a new procedure of obtaining similar sets of four numbers (possessing the *property*) from a triangular number. We also consider a further generalisation.

Received: November 25, 1996; *Revised*: February 14, 1997.
* The paper is prepared under N:B.H.M. grant 48/06/94-R&DV.

New procedure

Let T be any triangular number (It may be recalled that $T = m(m+1)/2$, m being a positive integer). Further more let a and b be two real numbers such that

1) $a < b$,

2) $a\,b = 2\,T$ and

3) $2a$ and $2b$ are positive integers.

By s we shall denote the positive square root of $8T + 1$. (It is obvious that s is an integer).

Theorem. *The set of the following four numbers satisfies the property mentioned in the Introduction*

(1) $$\Big[2a,\ 2b,\ 2a + 2b + 2s,\ 8s^3 + 8s^2\,(a+b) - 4s\Big]\ .$$

Proof. By using the fact that $4ab = 8T = s^2 - 1$ it is easy to show that

1) $$(2a)\,(2b) + 1 = s^2\ ,$$

2) $$2a\,(2a + 2b + 2s) + 1 = (2a + s)^2\ ,$$

3) $$2a\left(8s^3 + 8s^2\,(a+b) - 4s\right) + 1 = (2s^2 + 4as - 1)^2\ ,$$

4) $$2b\,(2a + 2b + 2s) + 1 = (2b + s^2)\ ,$$

5) $$2b\left(8s^3 + 8s^2\,(a+b) - 4s\right) + 1 = (2s^2 + 4bs - 1)^2\ ,$$

6) $$(2a + 2b + 2s)\left(8s^3 + 8s^2\,(a+b) - 4s\right) + 1 = \left(4s^2 + 4s\,(a+b) - 1\right)^2\ .\ ∎$$

Illustration. Let $T = 6$ then $s = 7$; we get 5 sets

a	b	set
0.5	24	(1, 48, 63, 12320)
1	12	(2, 24, 40, , 7812)
1.5	8	(3, 16, 33, 6440)
2	6	(4, 12, 30, 5852)
3	4	(6, 8, 28, 5460)

Further extension

Let n be a positive integer and a and b be real numbers such that

1) $a < b$

2) $2a$ and $2b$ are positive integers and

3) $4ab + n$ is a perfect square, say u^2.

We consider the following set of four positive integers

$$\Big(2a,\, 2b,\, 2a+2b+2u,\, 8u^3+8u^2\,(a+b)-4un\Big)\ . \tag{2}$$

This set has the *property* that the product of any two different numbers of the set increased by n or n^2 is a perfect square.

For notational convenience, let us denote these numbers by (a_1, a_2, a_3, A_1). Then we can easily check that the following six terms are perfect squares

$$\begin{array}{lll} a_1\,a_2+n, & a_1\,a_3+n, & a_2\,a_3+n \\ a_1\,A_1+n^2, & a_2\,A_1+n^2, & a_3\,A_1+n^2 \end{array}$$

Remarks.

1) The set (1) and the set (2) have many similar things.

2) Such properties were considered by Horadam [2].

3) For $n = 1$ this result coincides with the result obtained in the earlier section.

ACKNOWLEDGEMENT – The author is thankful to the referee.

REFERENCES

[1] Hoggatt, V.E. (Jr.) and Bergum, G.E. – A problem of Fermat and the Fibonacci sequence, *The Fibonacci Quarterly,* 15 (1977), 323–330.

[2] Horadam, A.F. – Generalisation of a result of Morgado, *Portugaliae Mathematica,* 44 (1987), 131–136.

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