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LOCAL AND GLOBAL EXISTENCE FOR MILD SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: In the present paper we shall investigate the local and global existence of mild solutions for a class of Ito type stochastic differential equations under the condition that the coefficients satisfy more general conditions than Lipschitz and linear growth.

1 – Introduction

In the present paper, we shall consider a stochastic differential equation of Ito type,

$$\begin{cases} dX(t) = \Big(AX(t) + F(t, X(t))\Big)\, dt + B(t, X(t))\, dW(t), \\ X(0) = \xi\ . \end{cases} \tag{1}$$

We will assume that a probability space $(\Omega, \mathcal{F}, P)$ together with a normal filtration $\mathcal{F}_t$, $t \geq 0$ are given. We denote by $\mathcal{P}$ and $\mathcal{P}_T$ the predictable σ-fields on $\Omega_\infty = [0, +\infty) \times \Omega$ and on $\Omega_T = [0, T) \times \Omega$ respectively.

We assume also that U and H are separable Hilbert spaces and that W is a Wiener process on U with covariance operator Q. We will assume that Q is a symmetric, positive, linear and bounded operator on U with $TrQ < \infty$. Let $U_0 = Q^{\frac{1}{2}}(U)$ with the induced norm $\|u\|_0 = \|Q^{-\frac{1}{2}}\, u\|$. The spaces U, H and $L_2^0 = L_2(U_0, H)$ (L_2^0 is the space of all Hilbert–Schmidt operators from U_0 into H) are equipped with Borel σ-fields $\mathcal{B}(U)$, $\mathcal{B}(H)$ and $\mathcal{B}(L_2^0)$. The space L_2^0 is also a separable Hilbert space equipped with the norm $\|\Psi\|_{L_2^0} = \|\Psi Q^{\frac{1}{2}}\|_{L_2(U,H)}$. Moreover, ξ is a H-valued random variable, $\mathcal{F}_0$-measurable.

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We fix $T > 0$ and impose first the following conditions on coefficients A, F and B of the equation (1):

i) A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geq 0$ in H.

ii) The mapping $F\colon\ [0,T]\times\Omega\times H \to H$, $(t,\omega,x) \to F(t,\omega,x)$ is measurable from $(\Omega_T \times H,\, \mathcal{P}_T \times \mathcal{B}(H))$ into $(H, \mathcal{B}(H))$.

iii) The mapping $B\colon\ [0,T]\times\Omega\times H \to L_2^0$, $(t,\omega,x) \to B(t,\omega,x)$ is measurable from $(\Omega_T \times H,\, \mathcal{P}_T \times \mathcal{B}(H))$ into $(L_2^0, \mathcal{B}(L_2^0))$.

A mapping $X : [0,T]\times\Omega \to H$ which is measurable from $(\Omega_T, \mathcal{P}_T)$ into $(H, \mathcal{B}(H))$, is said to be a *mild solution* of (1) if

$$P\Big(\int_0^T \Big(\Big\|S(t-s)\,F(s,X(s))\Big\| + \Big\|S(t-s)\,B(s,X(s))\Big\|_{L_2^0}\Big)\,ds < +\infty\Big) = 1$$

and, for arbitrary $t \in [0,T]$, we have

$$X(t) = S(t)\,\xi + \int_0^t S(t-s)\,F(s,X(s))\,ds + \int_0^t S(t-s)\,B(s,X(s))\,dW(s)\ P\ \text{ a.s..}$$

Existence and uniqueness theorem for solutions of the equation (1) under Lipschitz conditions on the coefficients are studied in [4], Th. 7.4.

Stochastic evolution equations in infinite dimensions are natural generalizations of stochastic ordinary differential equations and their theory has motivations coming both from mathematics and the natural sciences: physics, chemistry and biology, cf. [4].

In the present paper we shall present existence (local and global) and uniqueness results for solutions of the above mentioned equation under more general conditions. Similar results in finite dimensional case can be found in [1], [3], [6], [7].

A fundamental role in the proof of our theorems will play the following proposition ([4], P. 7.7.3).

Proposition 1.1. *Let $p > 2$, $T > 0$ and let Φ be a L_2^0-valued, predictable process, such that $E(\int_0^T \|\Phi(s)\|^p_{L_2^0}) < +\infty$. Then there exists a constant C_T such that*

$$E\Big(\sup_{t\in[0,T]}\Big\|\int_0^t S(t-s)\,\Phi(s)\,dW(s)\Big\|^p\Big) \leq C_T\,E\Big(\int_0^T \|\Phi(s)\|^p_{L_2^0}\,ds\Big)\ .$$

Moreover $W_A^\Phi(t) = \int_0^t S(t-s)\,\Phi(s)\,dW(s)$ has a continuous modification.

2 – The local existence of solutions

In the following we shall fix a real number p, $p > 2$. We shall denote by B_T the space of all H-valued predictable processes $X(t,\omega)$ defined on $[0,T]\times\Omega$ which are continuous in t for a.e. fixed $\omega \in \Omega$ and for which

$$\|X(\cdot,\cdot)\|_{B_T} \stackrel{\text{def}}{=} \left\{E\Big(\sup_{0\le t\le T} \|X(t,\omega)\|^p\Big)\right\}^{\frac{1}{p}} < \infty\ .$$

The next lemma is proved in [1]:

Lemma 2.1. *The space B_T is a Banach space with the norm $\|\cdot\|_{B_T}$.*

In the following we denote by $\Theta(X_0,r) \stackrel{\text{def}}{=} \{X \in B_T\colon\ \|X-X_0\|_{B_T} \le r\}$ the closed ball of center X_0 with radius r in B_T.

Theorem 2.1. *For the stochastic dzfferential equation (1), let the functions $F(t,\omega,x)$ and $B(t,\omega,x)$ be continuous in x for each fixed $(t,\omega)\in\Omega_T$, and let the following conditions be satisfied:*

(**1a**) *There exists a function $H\colon\ [0,\infty)\times[0,\infty)\to[0,\infty)$, $(t,u)\to H(t,u)$ such that*

$$E\Big(\|F(t,X)\|^p\Big) + E\Big(\|B(t,X)\|^p_{L_2^0}\Big) \le H\Big(t, E(\|X\|)^p\Big)$$

for all $t\in[0,T]$ and all $X \in L^p(\Omega,\mathcal{F},H)$.

(**1b**) *$H(t,u)$ is locally integrable in t for each fixed $u\in[0,\infty)$ and is continuous, monotone nondecreasing in u for each fixed $t\in[0,\infty)$.*

Then there exists $\tau\in[0,T]$ such that the operator $G\colon\ B_\tau\to B_\tau$

$$GX(t) = S(t)\,\xi + \int_0^t S(t-s)\,F(s,X(s))\,ds + \int_0^t S(t-s)\,B(s,X(s))\,dW(s),\quad t\in[0,\tau]$$

is well defined and has the property:

$$G\Big(\Theta(S(\cdot)\,\xi,\,r)\Big) \subset \Theta(S(\cdot)\,\xi,\,r)\ .$$

Proof. From Proposition 1.1 it follows that the operator G is well defined for all $\tau \in [0,T]$. Now we have:

$$
\begin{aligned}
E\Big(\sup_{0\le s\le\tau}\Big\|(GX)(s) - S(s)\,\xi\Big\|^p\Big) &\le \\
&\le 2^p E\bigg(\Big\|\int_0^\tau S(\tau-s)\,F(s,X(s))\,ds\Big\|^p\bigg) \\
&\quad + 2^p E\bigg(\sup_{0\le s\le\tau}\Big\|\int_0^s S(s-\theta)\,B(\theta,X(\theta))\,dW(\theta)\Big\|^p\bigg) \\
&\le 2^p\,M^p\,\tau^{p-1}\int_0^\tau E\Big(\|F(s,X(s))\|^p\Big)\,ds + 2^p\,C_T\int_0^\tau E\Big(\|B(s,X(s))\|^p_{L_2^0}\Big)\,ds \\
&\le C'_T\int_0^\tau H\Big(s, E\Big(\|X(s)\|^p\Big)\Big)\,ds\ .
\end{aligned}
$$

We have denoted $M = \sup_{t\in[0,T]}\|S(t)\|_{L(H)}$, $C'_T = 2^p\,M^p\,T^{p-1} + 2^p\,C_T$ and we applied the Hölder inequality for the first integral and used Proposition 1.1 for the second integral. If $X \in \Theta(S(\cdot)\,\xi,\, r) \subset B_\tau$ then $E(\|X(s) - S(s)\,\xi\|^p) \le r^p$ for every $s \in [0,\tau]$ and therefore

$$
\begin{aligned}
E\Big(\|X(s)\|^p\Big) &\le E\Big(\|X(s) - S(s)\,\xi\| + \|S(s)\,\xi\|\Big)^p \\
&\le 2^p\,r^p + 2^p\,E\Big(\|S(s)\,\xi\|^p\Big) \le C''_T\ ,
\end{aligned}
$$

where $C''_T = 2^p\,r^p + 2^p\,M^p\,E(\|\xi\|^p)$. The function $H(s,u)$ being monotone nondecreasing in u, we have

$$
E\Big(\sup_{0\le s\le\tau}\Big\|(GX)(s) - S(s)\,\xi\Big\|^p\Big) \le C'_T\int_0^\tau H(s, C''_T)\,ds
$$

for all $X \in \Theta(S(\cdot)\,\xi, r) \subset B_\tau$. But $H(\cdot, u_0)$ is locally integrable and therefore there exists τ' such that

$$
C'_T\int_0^{\tau'} H(s,u_0)\,ds \le r^p\ . \ \blacksquare
$$

In the following we consider the basic notions connected with measures of noncompactness and condensing operators (see [1]).

Definition 2.1. A function Ψ, defined on the family of all subsets of a Banach space E with values in some partially ordered set $(Q,\le)$, is called a measure of noncompactness (MNC for brevity) if $\Psi(\overline{co}\,O) = \Psi(O)$ for all $O \subset E$, where $\overline{co}\,O$ is the closure of the convex hull of O.

Definition 2.2. Let E_1 and E_2 be Banach spaces and let Φ and Ψ be MNC in E_1 and E_2, respectively, with values in some partially ordered set $(Q, \leq)$. A continuous operator $f\colon\ D(f) \subset E_1 \to E_2$ is said to be (Φ, Ψ)-condensing if $O \subset D(f)$, $\Psi[f(O)] \geq \Phi(O)$ implies O is relatively compact.

Definition 2.3. The Hausdorff measure of noncompactness $\chi(O)$ of the set O in a Banach space E is the infimum of the numbers $\varepsilon > 0$ such that O has a finite ε-net in E.

Recall that a set $C \subset E$ is called an ε-net of O if $O \subset C + \varepsilon\, \overline{B}(0,1) = \{s + \varepsilon\, b\colon s \in C,\ b \in \overline{B}(0,1)\}$ where $\overline{B}(0,1)$ is the closed ball of center 0 and radius 1 in E.

The MNC χ enjoy the following properties:

a) *regularity*: $\chi(O) = 0$ if and only if O is totally bounded;

b) *nonsingularity*: χ is equal to zero on every one-element set;

c) *monotonicity*: $O_1 \subset O_2$ implies $\chi(O_1) \leq \chi(O_2)$;

d) *semi-additivity*: $\chi(O_1 \cup O_2) = \max\{\chi(O_1), \chi(O_2)\}$;

e) *semi-homogeneity*: $\chi(t\, O) = |t|\, \chi(O)$ for any number t;

f) *algebraic semi-additivity*: $\chi(O_1 + O_2) \leq \chi(O_1) + \chi(O_2)$;

g) *invariance under translations*: $\chi(O + x_0) = \chi(O)$ for any $x_0 \in E$;

h) *invariance under passage to closure and to the convex hull*: $\chi(O) = \chi(\overline{O}) = \chi(co\, O)$.

The following result ([1], Th. 1.5.11 and generalisation 1.5.12) is fundamental for our considerations.

Theorem 2.2. *Let Ψ a MNC on a Banach space E which is additively-nonsingular (i.e. such that $\Psi(O \cup \{x\}) = \Psi(O)$ for all $O \subset E$ and $x \in E$) and a (Ψ, Ψ) condensing operator f which maps a nonempty, convex, closed subset M of the Banach space E into itself. Then f has at least one fixed point in M.*

Let $\mathcal{M}[0,T]$ denote the partially ordered linear space of all real monotone nondecreasing functions defined on $[0,T]$ and let us consider the following MNC on the space B_T defined above:

$$\Psi\colon\ B_T \to \mathcal{M}[0,T]\ ,$$

$$[\Psi(O)](t) = \chi_t[O_t]\ ,$$

where χ_t is the Hausdorff MNC on the space B_t and $O_t = \{x_{|[0,t]}\colon\ x \in O\} \subset B_t$.

Theorem 2.3. *For the stochastic differential equation (1), suppose that the following conditions are satisfied:*

(**3a**) *The functions $F(t,\omega,x)$ and $B(t,\omega,x)$ satisfy conditions (la), (lb) of Theorem 2.1 and are continuous in x for fixed $(t,\omega)\in\Omega_T$.*

(**3b**) *There exists a function $K:\ [0,\infty)\times[0,\infty)\to[0,\infty)$ that is locally integrable in t for each fixed $u\in[0,\infty)$ and is continuous, monotone nondecreasing in u for each fixed $t\in[0,\infty)$, $K(t,0)\equiv 0$ and*

$$E\Big(\Big\|F(t,X)-F(t,Y)\Big\|^p\Big)+E\Big(\Big\|B(t,X)-B(t,Y)\Big\|^p_{L_2^0}\Big)\le K\Big(t,E\Big(\|X-Y\|^p\Big)\Big)$$

for all $t\in[0,T]$ and $X,Y\in L^p(\Omega,\mathcal{F},H)$.

(**3c**) *If a nonnegative, continuous function $z(t)$ satisfies*

$$\begin{cases} z(t)\le\alpha\displaystyle\int_{t_0}^{t}K(s,z(s))\,ds\,, & t\in[0,T_1]\ ,\\ z(0)=0\ , \end{cases}$$

where $\alpha>0$, $T_1\in(0,T]$, then $z(t)=0$ for all $t\in[0,T_1]$.

Then the operator $G'\colon\ B_T\to B_T$,

$$(G'X)(t)=\int_0^t S(t-s)\,F(s,X(s))\,ds+\int_0^t S(t-s)\,B(s,X(s))\,dW(s),\quad t\in[0,T]\ ,$$

is condensing with respect to the MNC Ψ on any bounded subset of the space B_T.

Proof. We follow similar results for finite dimensional case ([1], Lemma 4.2.6). Suppose $\Psi(O)\le\Psi(G'O)$ for some bounded set $O\subset B_T$. We show that in this case $\Psi(O)=0$ from which results that O is relatively compact in B_T. In fact $\chi_T(O)=0$ and from this follow that O is totaly bounded in B_T, that is O is relatively compact. Let us notice that the function $t\to[\Psi(O)](t)$ is monotone nondecreasing and bounded and therefore for a fixed $\varepsilon>0$ there exists only a finite number of jumps of magnitude greather than ε. Remove the points corresponding to these jumps together with their disjoint δ_1-neighborhoods from the segment $[0,T]$, and using points β_j, $j=1,...,m$, divide the remaining part into intervals on which the oscillation of the function $\Psi(O)$ is smaller than ε. Now surround the points β_j by disjoint δ_2-neighborhoods and consider the family of all functions $Z=\{z_k\colon\ k=1,...,l\}$ continuous with probability one, constructed as follows: z_k coincides with an arbitrary element of a $[(\Psi(O))(\beta_j)+\varepsilon]$-net of the set O_{β_j} on the segment $\sigma_j = [\beta_{j-1} + \delta_2,\, \beta_j - \delta_2]$, $j = 1, ..., m$ and is linear on the complementar segments.

Let $u \in (G'O)$. Then $u = (G'z)$ for some $z \in O$ and

$$\|z - z_r^{\beta_j}\|_{B_{\beta_j}}^p \leq \Big[(\Psi(O))(\beta_j) + \varepsilon\Big]^p ,$$

where $z_r^{\beta_j}$ is some element of the $[(\Psi(O))(\beta_j) + \varepsilon]$-net of O_{β_j}. Since $z_r^{\beta_j}{}_{|\sigma_j} = z_{k|\sigma_j}$ for some element z_k of the set Z, it follows that for $s \in \sigma_j$ we have

$$\begin{aligned} E\Big(\|z(s) - z_k(s)\|^p\Big) &\leq E\Big(\sup_{\beta_{j-1}+\delta_2 \leq s \leq \beta_j - \delta_2} \|z(s) - z_k(s)\|^p\Big) \\ &\leq \|z - z_r^{\beta_j}\|_{B_{\beta_j}}^p \leq \Big[(\Psi(O))(\beta_j) + \varepsilon\Big]^p \leq \Big[(\Psi(O))(s) + 2\,\varepsilon\Big]^p . \end{aligned}$$

Then

$$\begin{aligned} E\Big(\sup_{0\leq s\leq t}\Big\|(G'z)(s) - (G'z_k)(s)\Big\|^p\Big) &\leq \\ &\leq 2^p\, M^p\, t^{p-1} \int_0^t E\Big(\Big\|F(s, z(s)) - F(s, z_k(s))\Big\|^p\Big)\, ds \\ &\quad + 2^p\, C_T \int_0^t E\Big(\Big\|B(s, z(s)) - B(s, z_k(s))\Big\|_{L_2^0}^p\Big)\, ds \\ &\leq C_T' \int_0^t K\Big(s, E\Big(\|z(s) - z_k(s)\|^p\Big)\Big)\, ds \\ &= C_T' \sum_{j=1}^m \int_{\sigma_j} K\Big(s, E\Big(\|z(s) - z_k(s)\|^p\Big)\Big)\, ds \\ &\quad + C_T' \int_{[0,t] - \bigcup_{j=1}^m \sigma_j} K\Big(s, E\Big(\|z(s) - z_k(s)\|^p\Big)\Big)\, ds , \end{aligned}$$

where $C_T' = 2^p\, M^p\, T^{p-1} + 2^p\, C_T$, $M = \sup_{t\in[0,T]} \|S(t)\|_{L(H)}$ and C_T is the constant from Proposition 1.1. The set O is bounded and Z is finite and therefore exists $u_0 > 0$ such that

$$E\Big(\|z(s) - z_k(s)\|^p\Big) < u_0 \quad \text{for all } \ z \in O, \quad z_k \in Z, \quad s \in [0, T] .$$

Using (2b) we can find $\delta_1 > 0$ and $\delta_2 > 0$ sufficiently small that can ensure that

$$\Big[(\Psi(O))(t)\Big]^p \leq \Big[(\Psi(G'O))(t)\Big]^p \leq \varepsilon + C_T' \int_0^t K\Big(s, \Big[(\Psi(O))(s) + 2\,\varepsilon\Big]^p\Big)\, ds .$$

From the arbitraryness of ε and the continuity of K in the second argument it follows that

$$\Big[(\Psi(O))(t)\Big]^p \leq C'_T \int_0^t K\Big(s, \Big[(\Psi(O))(s)\Big]^p\Big)\, ds\ .$$

By the last inequality, Lemma 2.2 and (2c) we deduce that $\Psi(O) = 0$.

The continuity of the operator G' follows easily. In fact, for $X, X_1, ...$ in B_T we have

$$\begin{aligned}
\|G'X - G'X_n\|^p_{B_T} &= E\Big(\sup_{t\in[0,T]} \|G'X(t) - G'X_n(t)\|^p\Big) \\
&\leq 2^p\, M^p\, T^{p-1} \int_0^T E\Big(\Big\|F(s,X(s)) - F(s,X_n(s))\Big\|^p\Big)\, ds \\
&\quad + 2^p\, C_T \int_0^T E\Big(\Big\|B(s,X(s)) - B(s,X_n(s))\Big\|^p_{L_2^0}\Big)\, ds \\
&\leq C'_T \int_0^T K\Big(s, E\Big(\Big\|X(s) - X_n(s)\Big\|^p\Big)\Big)\, ds \\
&\leq C'_T \int_0^T K\Big(s, \|X - X_n\|^p_{B_T}\Big)\, ds
\end{aligned}$$

from which we get $\|G'X - G'X_n\|^p_{B_T} \to 0$ as $\|X - X_n\|_{B_T} \to 0$. ∎

Remark 2.1.

i) Evidently, under the in conditions of Theorem 3.3 the operator $G\colon B_T \to B_T$ defined by

$$(GX)(t) = S(t)\,\xi + (G'X)(t)\,, \qquad t \in [0,T]\ ,$$

where $\xi \in L^p(\Omega, \mathcal{F}_0, H)$ is also Ψ-condensing.

ii) The inequality in (3b) of Theorem 2.3 is satisfied if the function K is concave with respect to u for each fixed $t \geq 0$ and

$$\Big\|F(t,x) - F(t,y)\Big\|^p + \Big\|B(t,x) - B(t,y)\Big\|^p_{L_2^0} \leq K\Big(t, \|x-y\|^p\Big)$$

for all $x, y \in H$ and $t \geq 0$. This follows immediately from Jensen's inequality.

iii) The function $K(t,u) = \lambda(t)\,\alpha(u)$, $t \geq 0$, $u \geq 0$, where $\lambda(t) \geq 0$ is locally integrable and $\alpha:\ R_+ \to R_+$ is a continuous, monotone nondecreasing function with $\alpha(0) = 0$, $\alpha(u) > 0$ for $u > 0$ and $\int_{0+} \frac{1}{\alpha(u)}\, du = \infty$ is an example for Theorem 2.3 (3c) (see [7]).

Lemma 2.2. *Let $K\colon [0,\infty)^2 \to [0,\infty)$, $(t,u) \to K(t,u)$ be a function which is locally integrable in t for each fixed $u \in [0,\infty)$ and continuous, monotone nondecreasing in u for each fixed $t \in [0,\infty)$, $K(t,0) \equiv 0$ and for which if there exists a continuous function $z\colon [0,T] \to [0,\infty)$, $z(0) = 0$ which satisfies*

$$z(t) \le \int_0^t K(s, z(s))\, ds\,, \qquad t \in [0,T]\ ,$$

then $z(t) = 0$ for all $t \in [0,T]$.

Then if a nonnegative monotone nondecreasing function $u\colon [0,T] \to [0,\infty)$, $u(0) = 0$, satisfies

$$u(t) \le \int_0^t K(s, u(s))\, ds\,, \qquad t \in [0,T]\ ,$$

it follows $u(t) = 0$ for all $t \in [0,T]$.

Proof. Let u as above and denote by $\mathcal{U}$ the class of functions $v\colon [0,T] \to [0,\infty)$ which satisfy $v(0) = 0$, $v(T) = u(T)$, $v(t) \ge u(t)$ for all $t \in [0,T]$, they are monotone nondecreasing and

$$v(t) \le \int_0^t K(s, v(s))\, ds\,, \qquad t \in [0,T]\ .$$

Evidently $u \in \mathcal{U}$ and $\mathcal{U}$ is partially ordered if we let $v_1 \le v_2$ if $v_1(t) \le v_2(t)$ for all $t \in [0,T]$.

We shall prove that $\mathcal{U}$ has maximal elements. For this it will be sufficient, in accordance with Zorn's Lemma to prove that a totally ordered subset of $\mathcal{U}$ has a majorant.

Let $\mathcal{U}' = \{v_i\}_{i\in I} \subset \mathcal{U}$ be a totally ordered subset of $\mathcal{U}$. We shall prove that $\sup_{i\in I} v_i \in \mathcal{U}$ and then $\sup_{i\in I} v_i$ will be a majorant for $\mathcal{U}'$. We have

$$\int_0^t K\Big(s, \sup_{i\in I} v_i(s)\Big)\, ds \ge \int_0^t K(s, v_i(s))\, ds \ge v_i(t) \quad \text{ for all }\ t \in [0,T],\ \ i \in I\ .$$

Therefore

$$\int_0^t K\Big(s, \sup_{i\in I} v_i(s)\Big)\, ds \ge \sup_{i\in I} v_i(t)\,, \qquad t \in [0,T]\ .$$

Obviously $\sup_{i\in I} v_i$ is monotone nondecreasing $(\sup_{i\in I} v_i)(0) = 0$ and $(\sup_{i\in I} v_i)(T) = u(T)$ that is $\sup_{i\in I} v_i \in \mathcal{U}$.

Let v be a maximal element of $\mathcal{U}$. We shall prove that v is continuous. Suppose v has a discontinuity point $t_0 \in (0,T]$ ($t=0$ is a continuity point) and $v(t_0+0) = v(t_0) > v(t_0-0)$. For other cases the proof will be the same. Let $\varepsilon = \frac{1}{2}(v(t_0+0) - v(t_0-0))$. We shall "raise up" v on the left (but close) of t_0. Let $\delta > 0$ such that $\int_J K(s,u(T))\,ds < \varepsilon$, for all $J \in \mathcal{B}([0,T])$, $m(J) < \delta$. We define $w\colon [0,T] \to [0,\infty)$

$$w(t) = \begin{cases} v(t), & t \in [0,T] - [t_0-\delta, t_0), \\ v(t_0-0)+\varepsilon, & t \in [t_0-\delta, t_0) \ . \end{cases}$$

Evidently $w > v$. We shall prove that $w \in \mathcal{U}$. For this it is sufficient to prove that

$$w(t) \le \int_0^t K(s,w(s))\,ds \ . \tag{2}$$

If $t < t_0 - \delta$, (2) is obviously satisfied. If $t \ge t_0$, then

$$\int_0^t K(s,w(s))\,ds \ge \int_0^t K(s,v(s))\,ds \ge v(t) = w(t) \ .$$

If $t \in [t_0-\delta, t_0)$, then

$$\begin{aligned} \int_0^t K(s,w(s))\,ds &\ge \int_0^{t_0} K(s,v(s))\,ds - \int_{t_0-\delta}^{t_0} K(s,v(s))\,ds \\ &\ge v(t_0) - \int_{t_0-\delta}^{t_0} K(s,u(T))\,ds \ge v(t_0) - \varepsilon = w(t) \ . \end{aligned}$$

We have proved that $w \in \mathcal{U}$. But $w \ge v$, $w \ne v$ which is a contradiction with the maximality of v. Therefore v is continuous on $[0,T]$ and from the hypothesis of lemma it follows $v(t) = 0$, for all $t \in [0,T]$. But $v(t) \ge u(t) \ge 0$, that is $u(t) = 0$ for all $t \in [0,T]$. ∎

Theorem 2.4. *Suppose the conditions of Theorem 2.3 are satisfied. Then there exists $T' \in (0,T]$ for which equation (1) has a unique solution in $B_{T'}$.*

Proof. In accordance with Theorem 2.1 there exists T' for which the operator G defined above has the property that

$$G\Big(\Theta(S(\cdot)\,\xi,\, r)\Big) \subset \Theta(S(\cdot)\,\xi,\, r) \subset B_{T'} \ .$$

But $\Theta(S(\cdot)\,\xi,\, r)$ is a nonempty, closed, convex subset of $B_{T'}$, G is a Ψ-condensing and then, from Theorem 2.2, it follows that G has at least one fixed point in

$\Theta(S(\cdot)\,\xi,\, r) \subset B_{T'}$. The fixed point is unique. Indeed, let $X, Y \in B_{T'}$ be two fixed points of G. Then we would have

$$\begin{aligned}
E\Big(\sup_{0\le s\le t} \|X(s)-Y(s)\|^p\Big) &\le 2^p\, M^p\, t^{p-1}\, E\bigg(\int_0^t \Big\|F(s,X(s))-F(s,Y(s))\Big\|^p\, ds\bigg) \\
&\quad + 2^p\, C_T\, E\bigg(\int_0^t \Big\|B(s,X(s)) - B(s,Y(s))\Big\|^p_{L_2^0}\, ds\bigg) \\
&\le (2^p\, M^p\, t^{p-1} + 2^p\, C_T) \int_0^t K\Big(s, E\Big(\|X(s)-Y(s)\|^p\Big)\Big)\, ds\,.
\end{aligned}$$

Therefore

$$\|X-Y\|^p_{B_t} \le (2^p\, M^p\, T^{p-1} + 2^p\, C_T) \int_0^t K\Big(s, \|X-Y\|^p_{B_t}\Big)\, ds\;.$$

From condition (2c) it follows that $\|X-Y\|^p_{B_t} \equiv 0$, that is $X \equiv Y$. ∎

3 – The global existence of solutions

In this section we shall discuss the existence of global solutions of equation (1). We suppose that the infinitesimal generator A generates a compact $\mathcal{C}_0$-semigroup (see [5]). Similar results in finite dimensional case can be found in [7].

Theorem 3.1. *For the stochastic differential equation (1), suppose that the following conditions are satisfied:*

(**5a**) *F and B satisfy conditions of Theorem 2.3 with $T=\infty$.*

(**5b**) *for all $T>0$, $\alpha>0$, equation*

$$\frac{du(t)}{dt} = \alpha\, H(t,u(t))$$

has a global solution on (t_0,∞) for any initial value (t_0,u_0), $t_0>0$, $u_0\ge 0$.

Then equation (1) with initial value $\xi \in L^p(\Omega,\mathcal{F}_0,H)$ has a global solution on $[0,\infty)$.

Proof. Let $\mathcal{U}$ the set of times s for which equation (1) has a mild solution on $[0,s]$ and let $s_1 = \sup_{s\in\mathcal{U}} s$. From Theorem 2.4 we have that $s_1>0$. Suppose $s_1<\infty$ and let T, $s_1<T<\infty$. We shall prove that the mild solution of equation (1) defined on $[0,s_1)$ has a continuous extension on $[0,s_1]$ and therefore, in accordance with Theorem 2.4, it has a "substantial" extension to the right of s_1 which is a contradiction with the definition of s_1.

Let $X(t)$, $t \in [0, s_1)$, be the mild solution of equation (1). Then for fixed $t \in [0, s_1)$ we have

$$E(\|X(t)\|^p) \le 3^p\, M^p\, E(\|\xi\|^p) + 3^p\, M^p\, t^{p-1} \int_0^t E\Big(\|F(s, X(s))\|^p\Big)\, ds \\ + 3^p\, C_T\, M^p \int_0^t E\Big(\|B(s, X(s))\|^p_{L_2^0}\Big)\, ds$$

that is

$$E(\|X(t)\|^p) \le 3^p\, M^p\, E(\|\xi\|^p) + (3^p\, M^p\, T^{p-1} + 3^p\, C_T\, M^p) \int_0^t H\Big(s, E(\|X(s)\|^p)\Big)\, ds\,.$$

Take $u_0 \in [0, \infty)$, $u_0 > 3^p\, M^p\, E(\|\xi\|^p)$, $\alpha = (3^p\, M^p\, T^{p-1} + 3^p\, C_T\, M^p)$ and let $u(t)$ be the global solution of equation

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha\, H(t, u(t))\ , \\ u(0) = u_0\ . \end{cases}$$

We have

$$E(\|X(t)\|^p) - \alpha \int_0^t H\Big(s, E(\|X(s)\|^p)\Big)\, ds < u_0 = u(t) - \alpha \int_0^t H(s, u(s))\, ds$$

for all $t \in [0, s_1)$. It follows, easily, (see [7], Lemma 4) that

$$E(\|X(t)\|^p) < u(t) \le u(T), \qquad \text{for all } \ t \in [0, s_1)\ .$$

Let $0 < \rho < s < t < s_1$. We have

$$E\Big(\|X(t) - X(s)\|^p\Big) =$$

$$= E\Bigg(\bigg\|\Big(S(t) - S(s)\Big)\,\xi + \int_0^s \Big(S(t-\theta) - S(s-\theta)\Big)\, F(\theta, X(\theta))\, d\theta \\ + \int_s^t S(t-\theta)\, F(\theta, X(\theta))\, d\theta + \int_0^s \Big(S(t-\theta) - S(s-\theta)\Big)\, B(\theta, X(\theta))\, dW(\theta) \\ + \int_s^t S(t-\theta)\, B(\theta, X(\theta))\, dW(\theta)\bigg\|^p\Bigg) \le$$

$$
\begin{aligned}
&\le 5^p \, \|S(t)-S(s)\|^p \, E(\|\xi\|^p) \\
&\quad + 5^p \, T^{p-1} \, E\Big(\int_0^s \Big\| \big[S(t-\theta)-S(s-\theta)\big] \Big\|^p \, \|F(\theta, X(\theta))\|^p \, d\theta\Big) \\
&\quad + 10^p \, M^p \, T^{p-1} \int_s^t E\Big(\|F(\theta, X(\theta))\|^p\Big) \, d\theta \\
&\quad + 5^p \, C_T \, E\Big(\int_0^s \Big\| \big[S(t-\theta)-S(s-\theta)\big] \Big\|^p \, \|B(\theta, X(\theta))\|^p \, d\theta\Big) \\
&\quad + 10^p \, M^p \, C_T \int_s^t E\Big(\|B(\theta, X(\theta))\|^p\Big) \, d\theta \\
&\le 5^p \, E(\|\xi\|^p) \, \|S(t)-S(s)\|^p \\
&\quad + (5^p \, T^{p-1} + 5^p \, C_T) \int_0^\rho \Big\|S(t-\theta)-S(s-\theta)\Big\|^p \, H(\theta, u(\theta)) \, d\theta \\
&\quad + (10^p \, M^p \, T^{p-1} + 10^p \, M^p \, C_T) \int_{s-\rho}^s H(\theta, u(\theta)) \, d\theta \\
&\quad + (10^p \, M^p \, T^{p-1} + 10^p \, M^p \, C_T) \int_s^t H(\theta, u(\theta)) \, d\theta \; .
\end{aligned}
$$

Using the continuity of the function $t \to S(t)$ in operator norm, for $t > 0$, the Lebesgue convergence theorem and the integrability of the function $\theta \to H(\theta, u(T))$ on $[0, T]$, we find

$$
\lim_{s,t \uparrow s_1} E\Big(\|X(t)-X(s)\|^p\Big) = 0 \; . \tag{3}
$$

From (3) it follows that there exists $\lim_{t \uparrow s_1} X(t) \overset{\text{def}}{=} X(s_1)$ and $E(\|X(s_1)\|^p) < \infty$. ∎

The following corollary is an immediat consequence of Theorem 3.1 and Remark 2.1.

Corollary 3.1. *For the stochastic differential equation (1), suppose that the following conditions are satisfied:*

(**a**) $\|F(t,x)-F(t,y)\|^p + \|B(t,x)-B(t,y)\|^p_{L_2^0} \le \lambda(t) \, \alpha(\|X-Y\|^p)$,

(**b**) $\|F(t,0)\|, \|B(t,0)\|_{L_2^0} \in F^p_{\text{loc}}([0,\infty), R^+)$,

for all $t \in [0,\infty)$ *and* $x, y \in H$, *where* $\lambda(t) \ge 0$ *is locally integrable and* $\alpha\colon R_+ \to R_+$ *is a continuous, monotone nondecreasing and concave function with* $\alpha(0) = 0$, $\alpha(u) > 0$ *for* $u > 0$ *and* $\int_{0+} \frac{1}{\alpha(u)} \, du = \infty$.

Let $E(\|\xi\|^p) < \infty$. Then, on any finite interval $[0,T]$, the equation (1) has a unique solution.

Remark 3.1.

i) If $\lambda(t) \equiv L$ $(L > 0)$ and $\alpha(u) = u$ $(u \geq 0)$ condition (a) implies a global Lipschitz condition.

ii) Another example is: $\alpha(u) = u \ln(\frac{1}{u})$ for $0 < u < u_0$ (u_0 sufficiently small), $\alpha(0) = 0$ and $\alpha(u) = (a\,u + b)$ for $u \geq u_0$, where $a\,u + b$ is the tangent line of the function $u \ln(\frac{1}{u})$ at point u_0.

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