PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 4 – 1999

COMMUTING EXPANDING DYNAMICS

M. Carvalho

Abstract: There are several indications that the commutativity between two dynamics induces restrictions at very important levels. The event of two commuting distinct dynamics is rather rare, at least among special families like Anosov diffeomorphisms, as explained in [PY1] and [PY2]. Since it is unusual, if commutativity is present a price is expected, as a relative scarcity of common invariants. The best sources to evince rigidity instances among commuting expanding dynamics are [F], [L] and [R]. We intend to add here another example.

1 – Introduction

The subject we address here had its origin in [F]. It was proved there that every closed invariant set for a non-lacunary semigroup of endomorphisms of the circle is finite or the whole circle. And it was conjectured that the set of ergodic probabilities invariant by both the expanding maps $z \in S^1 \mapsto z^p$ and $z \in S^1 \mapsto z^q$, where p and q are positive relatively prime integers greater than 2, is reduced to the Lebesgue measure and probabilities supported on common periodic orbits. This is so far still unproved. But in [L] a partial incentive was brought to light when it was concluded that a borelean non-atomic measure invariant for both z^p and z^q and mixing for one of the dynamics is the Lebesgue measure. In [R] it was achieved a wider characterization of the common invariant measures in this context, namely that a probability invariant by both z^p and z^q and ergodic for the semigroup they generate must either be the Lebesgue measure (the measure of maximal entropy of z^p and z^q) or have zero metric entropy with respect to both z^p and z^q. These results are extensible to commuting differentiable expanding maps of the circle through their conjugacies with the models z^k.

Received: January 12, 1998; *Revised*: March 9, 1998.
Mathematics Classification Subject: Primary 58F11, 28D05.

The natural environments to generalize this discussion are the tori $\mathbb{T}^d$, $d \geq 2$, similarly considered as compact additive groups. For Anosov systems, the dynamics, like the ones of z^p or z^q or any other differentiable map (see [Sh1]), may be codified by shifts, also have a unique fixed point (due to the hyperbolicity) and always induce an expanding invariant direction. However, to understand the array of rigidity present in this context we have to begin establishing the corresponding vocabulary. For example, a suitable analogue to a non-lacunary semigroup generated by multiplications of integers may be a semigroup generated by the composition of linear Anosov diffeomorphisms provided their centralizers are not trivial; this is a convenient way to translate the relative primeness demanded above between the degrees of the expanding maps, and it is achieved if we reduce the analysis to diffeomorphisms f and g verifying the independence condition

$$\left[f^n \circ g^m = \mathrm{Id},\ (n,m) \in \mathbb{Z}^2\right] \;\Rightarrow\; \left[n = m = 0\right] .$$

The most comprising works on conjectures similar to the ones mentioned above, within semigroups containing a codimension one Anosov element, belong to [Fk2], [H], [B] and [KS]. In the first two it is settled that there are topological constraints to commutativity of hyperbolic automorphisms acting on the bidimensional torus. Namely that a compact common invariant set which contains a C^2 arc must be the whole torus and, moreover, the only common invariant compact connected submanifold of class C^r, $r \geq 2$, is the torus. The result in [B] is the version of [F] on the tori: it is established that the minimal subsets for linear hyperbolic multi-parameter actions of homomorphisms on $\mathbb{T}^d$ are finite. In particular, common orbits of commuting automorphisms (diffeomorphisms which are projections on $\mathbb{T}^d$ of hyperbolic matrices of $\mathcal{G}l_d(\mathbb{Z})$ acting on $\mathbb{R}^d$ through the universal covering $\pi\colon \mathbb{R}^d \to \mathbb{T}^d$) are dense, apart from the finite (periodic) ones. In [KS] it was concluded the analogue of [R], that the set of common invariant probabilities, ergodic by the action of commuting codimension one Anosov diffeomorphisms, reduces to the Haar measure (this is the Lebesgue measure on $\mathbb{T}^d$) and probabilities with zero metric entropy. So far it remains unknown if, in this latter case, the support of the measure is a periodic orbit.

Codimension one Anosov diffeomorphisms are always transitive and, in fact, they may only live on tori where all Anosov diffeomorphisms are conjugate to linear models. See [Fk1] and [Mn] for details. This simplification of the environments and the dynamics justifies our choice of commuting codimension one Anosov diffeomorphisms as the starting point to add to the list above one more example of rigidity among expanding maps. There is indeed more to trace in the general context but we will stay within the bidimensional torus, although some conclusions hold in higher dimensions.

Let $\mathcal{A}$ and $\mathcal{B}$ denote two commuting codimension one Anosov automorphisms on $\mathbb{T}^2$ such that, if n and m are in $\mathbb{Z}$, then

$$\Big[\mathcal{A}^n \circ \mathcal{B}^m = \text{Identity}\Big] \;\Rightarrow\; \Big[m = n = 0\Big] \,.$$

Denote by A and B the corresponding matrices acting in $\mathbb{R}^2$. The hyperbolicity of A and B is in general not inherited by their composition, so we assume in the sequel that

$$\Big[\text{spectrum of}\, \mathcal{A} \circ \mathcal{B}\Big] \cap S^1 = \emptyset \,.$$

This assumption is essential to entangle the dynamics of $\mathcal{A}$ and $\mathcal{B}$ through $\mathcal{A} \circ \mathcal{B}$. We will verify in what follows that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \circ \mathcal{B}$ share the same unique fixed point, say P, and that we may assume without loss of generality that they also have in common the one-dimensional unstable foliation. Denote by $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}}$ the restrictions of $\mathcal{A}$ and $\mathcal{B}$ to the unstable manifolds $\mathcal{W}^u(P, \mathcal{A})$ and $\mathcal{W}^u(P, \mathcal{B})$. As assumed, $\mathcal{W}^u(P, \mathcal{A}) = \mathcal{W}^u(P, \mathcal{B})$, so these maps have the same domain, we will abbreviate into $\mathcal{W}(P)$. Due to the hyperbolicity of $\mathcal{A}$ and $\mathcal{B}$, the maps $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}}$ are differentiable and expanding: there is a constant ρ bigger than 1 such that, in the adapted Riemannian metric, for each x belonging to $\mathcal{W}(P)$, each nonzero vector v of $T_x \mathcal{W}^u(P)$ and positive iterate n, we have

$$\|Df^n_x(v)\| = \|D\mathcal{F}^n_x(v)\| \geq \rho^n \, \|v\| \,.$$

As $\mathcal{A}$ and $\mathcal{B}$ are linear projections, the liftings to $\mathbb{R}^2$ of $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}}$, within the subspace $E^u(\mathcal{A}) = E^u(\mathcal{B})$, are multiplications by numbers $\beta_{\mathcal{A}}$ and $\beta_{\mathcal{B}}$ greater than 1. These liftings are essentially commuting expanding maps defined on $\mathbb{R}$ and, since the quotient $\frac{\operatorname{Log} \beta_{\mathcal{A}}}{\operatorname{Log} \beta_{\mathcal{B}}}$ is irrational, otherwise $\mathcal{A}$ and $\mathcal{B}$ would not be independent, the expanding maps $x \in \mathbb{R} \mapsto \beta_{\mathcal{A}}\, x$ and $x \in \mathbb{R} \mapsto \beta_{\mathcal{B}}\, x$ form a non-lacunary semigroup acting on $\mathbb{R}$. But to apply [F] we should envisage $\mathcal{F}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{B}}$ as liftings to $\mathbb{R}$ of commuting maps of the circle and this is not possible because, as $\mathcal{A}$ and $\mathcal{B}$ are hyperbolic, $\beta_{\mathcal{A}}$ and $\beta_{\mathcal{B}}$ are never integers. So this does not seem to be the right approach.

We are then led to another way to lift the Anosov diffeomorphisms to a lesser dimensional dynamics. Denote the stable foliations assigned to $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{W}^s(\mathcal{A})$ and $\mathcal{W}^s(\mathcal{B})$. Given two close points R and S in $\mathbb{T}^2$ and local unstable manifolds $\mathcal{W}^u_{\text{loc}}(R)$ and $\mathcal{W}^u_{\text{loc}}(S)$, the holonomy between these two sections maps each point x in $\mathcal{W}^u_{\text{loc}}(R)$ to the intersection of the local stable manifold of x with $\mathcal{W}^u_{\text{loc}}(S)$. In general, the holonomy associated to a foliation is not even absolutely continuous (see [RY]). Anyway, among Anosov diffeomorphisms, we know that

Theorem ([HP], [PS]).

(**a**) *If a hyperbolic diffeomorphism is of class* $C^{1+\epsilon}$, $\epsilon > 0$, *then the holonomy is Hölder continuous.*

(**b**) *If a* C^2 *hyperbolic diffeomorphism has codimension one, then the holonomy is* C^1.

(**c**) *The holonomy of a* C^2 *hyperbolic diffeomorphism is absolutely continuous.*

(**d**) *For a toral Anosov automorphism the holonomy is* C^∞. ∎

So the holonomies of the foliations $\mathcal{W}^s(\mathcal{A})$ and $\mathcal{W}^s(\mathcal{B})$ are C^∞ and, in particular, they are absolutely continuous. That is, their holonomies take zero Lebesgue measure subsets of $\mathcal{W}^s(\mathcal{A})$ into zero Lebesgue measure sets. More precisely, they have a positive Jacobian with respect to the probability induced on the foliations by the Lebesgue measure inside $\mathbb{T}^2$. Using the holonomy of the stable leaves and a convenient Markov partition, we will construct two maps $F_\mathcal{A}$ and $F_\mathcal{B}$ that are essentially one-dimensional, since they are defined on a non-trivial subset of the unstable foliation, and are expanding maps according to a more general definition we will present on a comming section. Their domain, say $\mathbb{X}$, is made of a finite union of local stable manifolds intersected with the subset of the torus, which we denote by $\mathbb{M}_0$, whose points have itineraries, along a chosen Markov partition, that never reach its boundary; and is endowed with a Lebesgue measure $m_\mathbb{X}$ on its boreleans.

The most interesting dynamical attributes of the Anosov diffeomorphisms are evinced through these expanding maps. For instance, $F_\mathcal{A}$ shares with $\mathcal{A}$ the same topological entropy and preserves a unique exact probability, say $\lambda_\mathcal{A}$, that is absolutely continuous with respect to the Lebesgue measure of its domain, verifies

$$\lim_{n\to+\infty} m_\mathbb{X}\left(F_\mathcal{A}^{-n}(L)\right) = \lambda_\mathcal{A}(L) \qquad \forall \text{ borelean } L$$

and whose projection on the torus is the Bowen–Ruelle–Sinai measure of $\mathcal{A}$. It is expected that the rigidity found in [B] or [KS] manifests through geometric restrictions on the common $F_\mathcal{A}, F_\mathcal{B}$-invariants. We thereby conclude that

Theorem.

(**a**) *If* K *is a* $F_\mathcal{A}$, $F_\mathcal{B}$ *invariant closed set, then* $K = \mathbb{X}$ *or* K *is finite.*

(**b**) *If* σ *is a* $F_\mathcal{A}$, $F_\mathcal{B}$ *invariant exact probability, then* $\sigma = m_\mathbb{X}$ *or its entropy is zero.*

We will start with some definitions and notation, pursue with straight consequences of the commutativity of the dynamics and resume the proof of the Theorem in the last section.

2 – Definitions

We now recall the basic tenets of dynamical systems we will use. We follow Mañé's presentation of the theory of expanding maps, to be found in his book [M].

(1) Given a family of diffeomorphisms $\mathbb{F}$, a set L is $\mathbb{F}$-*invariant* if $f(L) = L$ for all $f \in \mathbb{F}$.

(2) A subset L is *minimal* if it is non-empty, closed, $\mathbb{F}$-invariant and it contains no closed $\mathbb{F}$-invariant proper subset. That is, $L \neq \emptyset$, L is closed, $f(L) = L$ for all f in $\mathbb{F}$, and if N is a closed non-empty subset of L such that $f(N) = N$ for all f in $\mathbb{F}$, then $N = L$.

(3) Let $f\colon M \to M$ be a diffeomorphism and $\mathcal{T} = \bigcup_{x\in M} T_xM$ the tangent bundle of M. Denote the derivative of f by $Df\colon \mathcal{T} \to \mathcal{T}$, where $D_xf\colon T_xM \to T_{f(x)}M$ is the corresponding linear map. A closed subset $\Lambda \subseteq M$ is *hyperbolic* if $f(\Lambda) = \Lambda$ and each tangent space T_xM, with x in Λ, can be written as a direct sum $T_xM = E^s_x(f) \oplus E^u_x(f)$ of subspaces such that

(**i**) $D_xf(E^s(f)) = E^s_{f(x)}(f)$;

(**2i**) $D_xf(E^u(f)) = E^u_{f(x)}(f)$;

(**3i**) $\exists\, C > 0\ \exists\, \lambda \in\,]0,1[\colon \forall\, x \in \Lambda\ \forall\, v \in E^s_x(f)\ \forall\, w \in E^u_x(f)\ \forall\, n \in \mathbb{N}$ we have

$$\|D_xf^n(v)\| \le C\,\lambda^n\,\|v\| \quad \text{and} \quad \|D_xf^{-n}(w)\| \le C\,\lambda^n\,\|w\| \ ;$$

(**4i**) the maps $x \mapsto E^s_x(f)$ and $x \mapsto E^u_x(f)$ are continuous.

This hyperbolic splitting of the tangent bundle at each x is integrable and the corresponding foliations define two families of submanifolds immersed in M, the stable and unstable manifolds, which are transversal at x and invariant by f.

(4) Given a diffeomorphism f, the *spectrum* of D_xf is the set of eigenvalues of D_xf. A family $\mathbb{F}$ of diffeomorphisms acting on M is *hyperbolic* if $[\mathrm{spectrum}\, D_xf] \cap S^1 = \emptyset$ for all f in $\mathbb{F}$ and x in M. Denote by $\mathrm{spc}(f)$ the union $\bigcup_{x\in M} \mathrm{spectrum}(D_xf)$.

(5) If M is a hyperbolic set of f, we say that f is an *Anosov diffeomorphism*. This restricts the possible eigenvalues of the linear maps $D_x f$ and means that $\mathrm{spc}(f) \cap S^1 = \emptyset$. By the continuity demanded in (3) and the connectedness of M, the dimension of each of the subspaces forming the splitting of (3) is constant on M. We say that f is of *codimension one* if its hyperbolic splitting $T_x M = E^s_x(f) \oplus E^u_x(f)$ verifies one of the equalities

$$\text{dimension}\, E^u(f) = 1 \quad \text{or} \quad \text{dimension}\, E^s(f) = 1 \ .$$

(6) Two elements f and g of $\mathbb{F}$ are *rationally independent* if for all $n, m \in \mathbb{Z}$, we have

$$f^n \circ g^m = \text{Identity} \ \Leftrightarrow \ n = m = 0 \ .$$

(7) Let $(X, \mathbb{D}, \nu)$ be a compact metric space endowed with a distance $\mathbb{D}$ and a borelean probability ν and G an endomorphism of $(X, \mathbb{D}, \nu)$. G is *expanding* if there is a sequence of partitions $\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1, ...$ such that

(**i**) the closure of $\bigcup_{\mathcal{O} \in \mathcal{P}} \mathcal{O}$ is X and $\inf_{\mathcal{O} \in \mathcal{P}} \nu[G(\mathcal{O})] > 0$;

(**2i**) $\forall\, n \geq 0\ \forall\, \mathcal{O} \in \mathcal{P}_{n+1}$, $G(\mathcal{O})$ is covered, modulo a set with zero ν-measure, by atoms of $\mathcal{P}_n$ and G is injective in $\mathcal{O}$;

(**3i**) the local inverses of G are contractions, that is, there are $0 < \omega < 1$ and $c > 0$ such that, in each atom $\mathcal{P}_n(x)$ that contains x, we have

$$\mathbb{D}(x, y) \leq c\, \omega^n\, \mathbb{D}[G^n(x), G^n(y)] \quad \forall\, n \geq 0\ \ \forall\, x \in X\ \ \forall\, y \in \mathcal{P}_n(x) \ ;$$

(**4i**) $\forall \mathcal{U}, \mathcal{V} \in \mathcal{P}\ \exists\, n > 0$ such that $\nu[G^{-n}(\mathcal{U}) \cap \mathcal{V}] \neq 0$;

(**5i**) $\exists\, J \colon\ X \to \mathbb{R}^+\ \exists\, 0 < \theta < 1\ \exists\, C > 0$ such that

1. $\forall$ borelean L inside an atom of $\mathcal{P}$, $\nu(G(L)) = \int_L J\, d\nu$;

2. $\forall n \geq 0\ \forall\, x, y$ in the same atom of $\mathcal{P}_n$ we have

$$\left| \frac{J(x)}{J(y)} - 1 \right| \leq C \left\{ \mathbb{D}[G(x), G(y)] \right\}^\theta \ .$$

This definition, although cumbersome, extends the notion of differentiable expanding map. It includes Markov maps and the topogically mixing subshifts.

(8) Given an Anosov diffeomorphism f on a Riemannian manifold $(M, \mathcal{D})$, there is a constant (*of expansivity*) $\epsilon_f > 0$ such that

$$\forall\, x, y \in M,\ x \neq y,\ \ \exists\, n \in \mathbb{Z} \ \text{ such that } \mathcal{D}[f^n(x), f^n(y)] > \epsilon_f \ .$$

It is known that M has a *local product structure*, that is, there is a positive constant ρ_f less than ϵ_f such that the *crochet*

$$[\![x,y]\!]\colon\ (x,y)\mapsto \mathcal{W}^s_{\rho_f}(x)\cap\mathcal{W}^u_{\rho_f}(y)$$

is a well defined continuous map. ρ_f may be chosen so that the crochet determines a homeomorphism between the product $\overset{\circ}{\mathcal{W}}{}^u_{\rho_f}(x)\times\overset{\circ}{\mathcal{W}}{}^s_{\rho_f}(x)$ and a neighbourhood of x in M. By $\mathcal{W}^s_{f,\rho_f}(x)$ we mean the set

$$\left\{z\in\mathcal{W}^s_f(x)\colon\ \mathcal{D}(x,y)\le\rho_f\right\}\ ,$$

which by hyperbolicity coincides with

$$\left\{z\in M\colon\ \mathcal{D}[f^k(x),f^k(z)]\le\rho_f\ \ \text{for all } k\in\mathbb{N}_0\right\}\ ;$$

$\overset{\circ}{\mathcal{W}}{}^s_{\rho_f}(x)$ is the subset of $\mathcal{W}^s_{\rho_f}(x)$ given by the points z such that $\mathcal{D}(x,y)<\rho_f$.

A subset R of M is called a *rectangle* for f if its diameter is less than ρ_f and it is stable for the crochet that is, for all x and y in R, $[\![x,y]\!]$ belongs to R. We say that R is *proper* if it is the closure of its interior (notice that, in this case, R is closed).

A *Markov partition* of M for f is a finite collection $\mathcal{R}=\{R_1,...,R_m\}$ by proper rectangles verifying

(**a**) $M=\bigcup_{i=1}^{m}R_i$;

(**b**) the interiors of distinct rectangles are disjoint and the diameter of each rectangle is less than ρ_f;

(**c**) when x is in the interior of R_i and $f(x)$ is in the interior of R_j, then

$$f[\mathcal{W}^s_{f,\rho_f}(x)]\cap R_i\ \subseteq\ \mathcal{W}^s_{f,\rho_f}(f(x))\cap R_j\ ;$$

(**d**) when x is in the interior of R_i and $f^{-1}(x)$ is in the interior of R_j, then

$$f[\mathcal{W}^u_{f,\rho_f}(x)]\cap R_i\ \supseteq\ \mathcal{W}^u_{f,\rho_f}(f(x))\cap R_j\ .$$

A general construction of Markov partitions for Axiom A diffeomorphisms may be found in [Sh2].

3 – Preliminaries

Consider two Anosov automorphisms $\mathcal{A}$ and $\mathcal{B}$ which commute and the corresponding matrices A and B acting in $\mathbb{R}^2$. Obviously all the iterates of $\mathcal{A}$ and $\mathcal{B}$ also commute. Let us start with an account of the role played by the facts that $\mathcal{A}$ and $\mathcal{B}$ commute and the hyperbolicity of $\mathcal{A} \circ \mathcal{B}$.

Let $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ be the fixed points of $\mathcal{A}$ and $\mathcal{B}$ respectively. They are unique due to the hyperbolicity and

Claim 1.

(**a**) $P_{\mathcal{A}} \equiv P_{\mathcal{B}}$.

(**b**) *If* $\wp(\mathcal{A})$ *denotes the* $\mathcal{A}$*-invariant probabilities, then* $\wp(\mathcal{A}) \cap \wp(\mathcal{B})$ *is a non-empty convex set.*

(**c**) *If* μ *is in* $\wp(\mathcal{A}) \cap \wp(\mathcal{B})$*, then* μ *is also* $\mathcal{A} \circ \mathcal{B}$*-invariant.*

Proof:

(**a**) Since $\mathcal{A}(\mathcal{B}(P_{\mathcal{B}})) = \mathcal{A}(P_{\mathcal{B}}) = \mathcal{B}(\mathcal{A}(P_{\mathcal{B}}))$ and the fixed point for an Anosov diffeomorphism is unique, we conclude that $P_{\mathcal{B}} = \mathcal{A}(P_{\mathcal{B}})$ and thus $P_{\mathcal{B}} = P_{\mathcal{A}}$. In the sequel we will denote by P this common fixed point.

(**b**) By continuity, $\wp(\mathcal{A})$ and $\wp(\mathcal{B})$ are non-empty. Besides if ν belongs to $\wp(\mathcal{A})$ and the probability $\mathcal{B}_* \nu$ is given by

$$(\mathcal{B}_* \nu)(K) = \nu[\mathcal{B}^{-1}(K)] \quad \forall \text{measurable set } K ,$$

then the sequence

$$\left(\frac{1}{n}\left[\nu + \mathcal{B}_* \nu + \cdots + (\mathcal{B}^{n-1})_* \nu\right]\right)_{n\in\mathbb{N}}$$

has an accumulation point which is in $\wp(\mathcal{A}) \cap \wp(\mathcal{B})$ because $\mathcal{A}$ and $\mathcal{B}$ commute.

(**c**) For all borelean K, we have

$$\begin{aligned}\mu\Big[(\mathcal{A}\circ\mathcal{B})^{-1}(K)\Big] = \mu\Big[(\mathcal{B}^{-1}\circ\mathcal{A}^{-1})(K)\Big] &= \mu(\mathcal{A}^{-1}(K)) \\ &\qquad \text{because } \mu \text{ is } \mathcal{B}\text{-invariant} \\ &= \mu(K) \\ &\qquad \text{because } \mu \text{ is } \mathcal{A}\text{-invariant.}\end{aligned}$$ ∎

Claim 2. *If* [spectrum $\mathcal{A} \circ \mathcal{B}$] $\cap S^1 = \emptyset$, *then the stable / unstable foliations of* $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{A} \circ \mathcal{B}$ *coincide.*

Proof: Consider the hyperbolic splitting of $\mathcal{A}$ and $\mathcal{B}$ in stable / unstable vector spaces $E^s(\mathcal{A}) \oplus E^u(\mathcal{A})$ and $E^s(\mathcal{B}) \oplus E^u(\mathcal{B})$. If $v \neq 0$ is an eigenvector for $\mathcal{A}$ with eigenvalue $\alpha_{\mathcal{A}}$ with $|\alpha_{\mathcal{A}}| < 1$, then

$$A(B(v)) = B(A(v)) = B(\alpha_{\mathcal{A}}\, v) = \alpha_{\mathcal{A}}\, B(v)$$

and therefore the non-zero vector $B(v)$ is also an eigenvector for $\mathcal{A}$ with eigenvalue $\alpha_{\mathcal{A}}$. But the eigenspace $E(\alpha_{\mathcal{A}})$ is $E^s(\mathcal{A})$ and its dimension is equal to 1. Thus $B(v)$ is a generator of $E^s(\mathcal{A})$ and so there exists ρ in $\mathbb{R}$ such that $B(v) = \rho\, v$. However B has only two proper invariant directions, the ones corresponding to $E^s(\mathcal{B})$ and $E^u(\mathcal{B})$, so we must have either

$$|\rho| > 1 \quad \text{and} \quad E(\rho) = E^s(\mathcal{A}) = E^u(\mathcal{B})$$

or

$$|\rho| < 1 \quad \text{and} \quad E(\rho) = E^s(\mathcal{A}) = E^s(\mathcal{B}) \ .$$

An analogous argument leads to the conclusion that $E^u(\mathcal{A})$ is invariant by B and therefore $E^{s,\,u}(\mathcal{A}) = E^{s,\,u}(\mathcal{B})$. That is, the subspaces generated by $E^s(\mathcal{A})$ and $E^u(\mathcal{A})$ coincide with the ones generated by $E^s(\mathcal{B})$ and $E^u(\mathcal{B})$, although they may have complementary dynamics and certainly have different rates of expansion or contraction. In particular, since we are dealing with codimension one systems, A and B always exhibit one real eigenvalue which we may assume (considering $\mathcal{A}^{-1}$ or $\mathcal{B}^{-1}$ if necessary) having absolute value greater than one, whose eigenspace generates the unstable direction of both $\mathcal{A}$ and $\mathcal{B}$. This reduces the dimension of the subspace where we have to pursue the analysis of the dynamics of $\mathcal{A} \circ \mathcal{B}$. In the complement of this unstable direction the dynamics of both $\mathcal{A}$ and $\mathcal{B}$ is contracting (stable) and so is the one of $\mathcal{A} \circ \mathcal{B}$.

Therefore, in dimension two, if in the basis given by the direct sum $E^s(\mathcal{A}) \oplus E^u(\mathcal{A})$ and $E^s(\mathcal{B}) \oplus E^u(\mathcal{B})$, the matrices A and B are written as

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad B = \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \overline{\beta} \end{pmatrix} ,$$

where $|\alpha| < 1$, $|\beta| > 1$, $|\overline{\alpha}| < 1$, $|\overline{\beta}| > 1$, $|\det A| = 1 = |\alpha\, \beta|$ and $|\det B| = 1 = |\overline{\alpha}\, \overline{\beta}|$, then:

(1) If $E^s(\mathcal{A}) = E^s(\mathcal{B})$, $\mathcal{A} \circ \mathcal{B}$ is the Anosov diffeomorphism whose linear lifting in the basis $E^s(\mathcal{A}) \oplus E^u(\mathcal{A}) = E^s(\mathcal{B}) \oplus E^u(\mathcal{B})$ is written as

$$AB = \begin{pmatrix} \alpha\, \overline{\alpha} & 0 \\ 0 & \beta\, \overline{\beta} \end{pmatrix}$$

with $|\alpha\, \overline{\alpha}| < 1$ and $|\beta\, \overline{\beta}| > 1$.

(2) If $E^s(\mathcal{A}) = E^u(\mathcal{B})$, $\mathcal{A} \circ \mathcal{B}$ is the Anosov diffeomorphism whose linear lifting in the basis $E^s(\mathcal{A}) \oplus E^u(\mathcal{A}) = E^u(\mathcal{B}) \oplus E^s(\mathcal{B})$ is expressed as

$$AB = \begin{pmatrix} \alpha\,\overline{\beta} & 0 \\ 0 & \overline{\alpha}\,\beta \end{pmatrix}$$

with $|\alpha\,\overline{\beta}| \neq 1$, because $|\det AB| = |\alpha\,\beta\,\overline{\alpha}\,\overline{\beta}| = 1$ yields $|\alpha\,\overline{\beta}| = 1$ and $|\overline{\alpha}\,\beta| = 1$, contradicting the initial hypothesis that does not allow an eigenvalue of AB in S^1; and so $|\alpha\,\overline{\beta}| < 1$ or $|\alpha\,\overline{\beta}| > 1$, in which case $|\overline{\alpha}\,\beta| \neq 1$.

These equalities extend to the invariant manifolds (stable, unstable) tangent to these subspaces. In fact, if $\pi\colon\ \mathbb{R}^2 \to \mathbb{T}^2$ denotes the universal covering of $\mathbb{T}^2$ and v, w are eigenvectors of A which determine the directions of $E^s(\mathcal{A})$ and $E^u(\mathcal{A})$, respectively, then, for each point $Q = \pi(x_0, y_0)$ of the torus, the stable manifold of f at Q is the projection by π of the line

$$(x,y) - (x_0,y_0) = t\,v \;\;:\;\; t \in \mathbb{R}$$

and the unstable manifold at Q is the projection of

$$(x,y) - (x_0,y_0) = t\,w \;\;:\;\; t \in \mathbb{R}\ .$$

These are orthogonal lines for the metric in $\mathbb{R}^2$. Since v and w also determine the stable and unstable directions for $\mathcal{B}$, the corresponding foliations are the ones of $\mathcal{A}$ for all point Q. In particular, if ρ is the least number among $\rho_{\mathcal{A}}$, $\rho_{\mathcal{B}}$ and $\rho_{\mathcal{A}\circ\mathcal{B}}$ (see definition (8)), then the corresponding local stable manifolds for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \circ \mathcal{B}$ coincide:

$$\begin{aligned} \mathcal{W}^s_{\mathcal{A},\rho}(x) &= \Big\{ z \in \mathcal{W}^s_{\mathcal{A}}(x)\colon\ \mathcal{D}(x,z) \leq \rho \Big\} = \\ = \mathcal{W}^s_{\mathcal{B},\rho}(x) &= \Big\{ z \in \mathcal{W}^s_{\mathcal{B}}(x)\colon\ \mathcal{D}(x,z) \leq \rho \Big\} = \\ = \mathcal{W}^s_{\mathcal{A}\circ\mathcal{B},\rho}(x) &= \Big\{ z \in \mathcal{W}^s_{\mathcal{A}\circ\mathcal{B}}(x)\colon\ \mathcal{D}(x,z) \leq \rho \Big\}\ . \end{aligned}$$

Notice that, for each Q in $\mathbb{T}^2$, $\mathcal{W}^s(Q)$ and $\mathcal{W}^u(Q)$ are C^∞ submanifolds, invariant by both $\mathcal{A}$ and $\mathcal{B}$, but they are not closed. Their closures are the whole torus, which serves as a fair example of what was settled in [Fk] and [H]. ∎

Corollary. *$\beta_{\mathcal{A}}$ and $\beta_{\mathcal{B}}$ cannot be rational numbers.*

Proof: Since A belongs to $\mathcal{G}l_d(\mathbb{Z})$, we have $\det A = 1$ and $\operatorname{tr} A \in \mathbb{Z}$, so $\det(\rho\,\mathrm{Id} - A)$ is a polinomial in ρ with integer coefficients, independent term equal to 1 and coefficient of highest degree equal to 1. Therefore its rational zeros can only be 1 or -1, which are forbidden by hyperbolicity. ∎

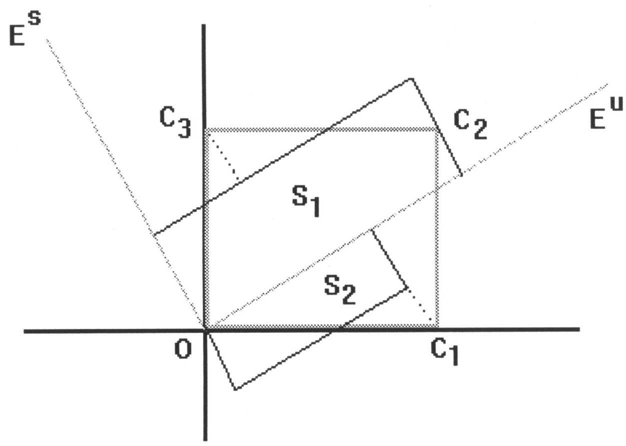
Corollary. *If $d = 2$, there is a Markov partition, with arbitrarily small diameter, for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \circ \mathcal{B}$.*

Proof: Let us start sketching the construction of a Markov partition for a linear Anosov diffeomorphism f on $\mathbb{T}^2$. The torus may be decomposed in two closed sets, we denote by S_1 and S_2, whose interiors have no elements in common and which are the projections by π of the two rectangles in $\mathbb{R}^2$ with sides parallel to the directions of the eigenspaces of f (the stable and unstable directions, as we know). We get a Markov partition for f considering as rectangles the sets

$$\bigcap_{k=-Z}^{+Z} f^k(S_{\alpha_k})$$

where α_k belongs to $\{1,2\}$ and $Z \in \mathbb{N}$ is big enough in order to guarantee that these sets have a requested small diameter.

As mentioned in the preceding Claim, the stable and unstable foliations associated with $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \circ \mathcal{B}$ coincide. Therefore, the starting sets S_1 and S_2 serve as well in the general procedure described above when applied to $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{A} \circ \mathcal{B}$. Notice that the sides of the two rectangles S_1 and S_2 in $\mathbb{R}^2$, when projected onto $\mathbb{T}^2$ by π, become part of the stable manifold of the fixed point P since $\pi(O) = \pi(C_1) = \pi(C_2) = \pi(C_3)$.



Thus the collections of sets

$$\bigcap_{k=-J}^{+J} \mathcal{A}^k(S\alpha_k)\,, \qquad \bigcap_{k=-L}^{+L} \mathcal{B}^k(S_{\alpha_k})\,, \qquad \bigcap_{k=-N}^{+N} (\mathcal{A} \circ \mathcal{B})^k\,(S_{\alpha_k})\ ,$$

where J, L and N are positive integers big enough to guarantee that each of these sets has diameter less than $\rho = \min\{\rho_{\mathcal{A}}, \rho_{\mathcal{B}}, \rho_{\mathcal{A}\circ\mathcal{B}}\}$, are Markov partitions for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}\circ\mathcal{B}$. Moreover, if we let $\mathcal{B}$ act on the Markov partition of $\mathcal{A}$, that is, if we consider instead the family of all sets of the form

$$\bigcap_{k=-L}^{+L} \mathcal{B}^k \left(\bigcap_{j=-J}^{+J} \mathcal{A}^j(S_{\alpha_j}) \right)$$

then we obtain a Markov partition for $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}\circ\mathcal{B}$. This is so on account of the hypothesis that $\mathcal{A}$ and $\mathcal{B}$ commute and the connection between these sets and the already defined rectangles of the Markov partitions for $\mathcal{B}$ and $\mathcal{A}\circ\mathcal{B}$. ∎

4 – Proof of the Theorem

Let $\mathcal{R} = (R_i)_{i=1,\cdots,k}$ be a Markov partition with small diameter common to $\mathcal{A}$ and $\mathcal{B}$ and $(p_i)_{i=1,\cdots,k}$ be chosen points inside each rectangle R_i. We may assume that these points $(p_i)_i$ belong to the same connected component of an unstable manifold $\mathcal{W}^u(Q)$, because this foliation has dense leaves on the torus. By $\mathcal{W}^u(Q)$ we mean the unstable manifold of Q for the dynamics of $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{A}\circ\mathcal{B}$ since they coincide. Consider then

(1) $\epsilon > 0$ smaller than $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{B}}$.

(2) $M_0 = \bigcap_{m\in\mathbb{Z}} \mathcal{A}^m \left(\mathbb{T}^2 - \bigcup_{i=1}^{k} \partial R_i\right) = \mathbb{T}^2 - \bigcup_{m\in\mathbb{Z}} \mathcal{A}^m \left(\bigcup_{i=1}^{k} \partial R_i\right).$

M_0 is also given by the intersection $\bigcap_{m\in\mathbb{Z}} \mathcal{B}^m(\mathbb{T}^2 - \bigcup_{i=1}^k \partial R_i)$, is invariant by $\mathcal{A}$ and $\mathcal{B}$, not closed and the Lebesgue measure in $\mathbb{T}^2$ of its complement is zero as a consequence of the absolute continuity of the holonomy (which implies that the Lebesgue measure of the boundary of $\mathcal{R}$ is zero) and the fact that $\mathcal{A}$ and $\mathcal{B}$ are area preserving. We recall that

$$M_0 = \mathbb{T}^2 - \left(\mathcal{W}^s(P) \cup \mathcal{W}^u(P)\right)$$

since $\mathcal{R}$ was constructed upon these two manifolds because the boundary of S_1 and S_2 was made of the stable/unstable foliation of the fixed point P.

(3) If $\mathcal{W}^u_\epsilon(p_i)$ is the local unstable manifold of p_i of size ϵ, consider the set

$$\mathbb{X} = \bigcup_{i=1}^{k} \left[\mathcal{W}^u_\epsilon(p_i) \cap R_i\right] \cap M_0 \ .$$

The subset $\bigcup_{i=1}^{k}[\mathcal{W}_\epsilon^u(p_i) \cap R_i]$ is a compact non-connected manifold with boundary and dimension one, but the intersection with M_0 turns $\mathbb{X}$ into a domain with a very intrincate structure. In any case, $\mathbb{X}$ has a well defined Lebesgue measure $m_{\mathbb{X}}$ on its boreleans.

(4) $h\colon M_0 \to \mathbb{X}$ such that $h(x) = \mathcal{W}_\epsilon^s(x) \cap \mathbb{X}$.

The map h is well defined since each point of M_0 has an unambiguous itinerary through the Markov partition and the choice of ϵ implies that each local stable manifold has only one point of intersection with $\mathbb{X}$. Recall that $\mathcal{A}$ and $\mathcal{B}$ have the same stable foliations, so h is the same for both dynamics.

Lemma 1. *If m and $m_{\mathbb{X}}$ denote the Lebesgue measure on $\mathbb{T}^2$ and on $\mathbb{X}$ respectively, then*

$$m_{\mathbb{X}}\,(A) = 0 \;\Rightarrow\; m\,(h^{-1}(A)) = 0 \;.$$

Proof: This is a direct consequence of the fact that the holonomy of the stable foliation of $\mathcal{A}$ or $\mathcal{B}$ is absolutely continuous. ∎

(5) Define in $\mathbb{X}$ the following metric $\mathbb{D}$

$\mathbb{D}(x,y) = 1$ if x and y are in different rectangles of the partition $\mathcal{R}$;

$$\mathbb{D}(x,y) = \sup_{z \in R_j} \mathcal{D}_u \Big(h^{-1}(x) \cap \mathcal{W}_\epsilon^u(z) \cap R_j,\; h^{-1}(y) \cap \mathcal{W}_\epsilon^u(z) \cap R_j\Big)$$

if x and y are in $\mathcal{W}_\epsilon^u(p_j) \cap R_j$.

Here $\mathcal{D}_u$ denotes the Riemannian distance in $\mathcal{W}^u(z)$ induced by the Riemannian metric $\mathcal{D}$ in $\mathbb{T}^2$. Notice that, with respect to this metric, we have

$$\mathcal{D}_u\Big[g^{-1}(x), g^{-1}(y)\Big] \leq \gamma\, \mathcal{D}_u(x,y) \qquad \forall\, x,y \in \mathbb{T}^2 \;,$$

where y is in $\mathcal{W}_\epsilon^u(x)$, $g = \mathcal{A}$ or $\mathcal{B}$ and γ is equal to $\gamma_{\mathcal{A}}$ or $\gamma_{\mathcal{B}}$ given respectively by

$$\gamma_{\mathcal{A}} = \sup\Big\{\|D_x\, \mathcal{A}_{|\,E^s(x)}\|,\; \|D_x\, \mathcal{A}^{-1}_{|\,E^u(x)}\|\Big\}_{x \in \mathbb{T}^2} \;,$$

$$\gamma_{\mathcal{B}} = \sup\Big\{\|D_x\, \mathcal{B}_{|\,E^s(x)}\|,\; \|D_x\, \mathcal{B}^{-1}_{|\,E^u(x)}\|\Big\}_{x \in \mathbb{T}^2} \;.$$

(6) Consider $F_{\mathcal{A}}\colon \mathbb{X} \to \mathbb{X}$, $F_{\mathcal{B}}\colon \mathbb{X} \to \mathbb{X}$ given by $F_{\mathcal{A}} = h \circ \mathcal{A}$, $F_{\mathcal{B}} = h \circ \mathcal{B}$.

Lemma 2. *$F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ are commuting expanding maps with respect to the metric $\mathbb{D}$.*

Proof: By definition, $F_{\mathcal{A}}(h(x)) = h(\mathcal{A}(x))$ and $F_{\mathcal{B}}(h(x)) = h(\mathcal{B}(x))$ for all x in M_0, and the same holds for $(F_{\mathcal{A}})^n$ and $\mathcal{A}^n$, $n \geq 0$. Therefore

$$F_{\mathcal{A}} \circ F_{\mathcal{B}} = F_{\mathcal{A}} \circ (h \circ \mathcal{B}) = (h \circ \mathcal{A}) \circ \mathcal{B} = h \circ (\mathcal{B} \circ \mathcal{A}) = \\ = (F_{\mathcal{B}} \circ h) \circ \mathcal{A} = F_{\mathcal{B}} \circ (h \circ \mathcal{A}) = F_{\mathcal{B}} \circ F_{\mathcal{A}} \ .$$

Take now the following ingredients for $F_{\mathcal{A}}$. They are enough to check definition (7) on $\mathrm{F}_{\mathcal{A}}$. Similar argument for $\mathrm{F}_{\mathcal{B}}$.

(**a**) $m_{\mathbb{X}} =$ Lebesgue measure on the boreleans of $\mathbb{X}$.

(**b**) $\mathcal{P}$ given by $P_j = \mathcal{W}^u_\epsilon(p_j) \cap R_j \cap M_0$.

(**c**) $\mathcal{P}_n = \bigvee_0^n (F_{\mathcal{A}})^{-j}(\mathcal{P})$.

(**d**) $c_{\mathcal{A}} = 1$.

(**e**) $\omega = \gamma_{\mathcal{A}}$.

(**f**) $\theta_{\mathcal{A}}$, $C_{\mathcal{A}}$ the constants associated to the Hölder continuity of the map

$$z \to \left|\det D_z \, \mathcal{A}_{|\, E^u(z)}\right| \ .$$

(**g**) In each atom $\mathcal{P}_n$,

$$J_{\mathcal{A}}(x) = \mathcal{H}_n \, * \, \left|\det D_x \, \mathcal{A}_{|\, E^u(x)}\right| = \mathcal{H}_n \, * \, \beta_{\mathcal{A}} \ ,$$

where $\mathcal{H}_n$ is the Jacobean of the holonomy of the stable foliation in this atom.

Since $\mathcal{A}$ and $\mathcal{B}$ are linear Anosov diffeomorphisms, the maps $\mathcal{H}_n$ are constant and equal to 1. See for instance their construction in [M]. Therefore, in this context, $J_{\mathcal{A}} \equiv \beta_{\mathcal{A}}$ and $J_{\mathcal{B}} \equiv \beta_{\mathcal{B}}$. Thus the property (5i) of definition (7) means that, for all borelean L inside an atom of $\mathcal{P}$,

$$m_{\mathbb{X}}\,(F_{\mathcal{A}}(L)) = \int_L J_{\mathcal{A}}\, dm_{\mathbb{X}} = \int_L \beta_{\mathcal{A}}\, dm_{\mathbb{X}} = \beta_{\mathcal{A}} \, * \, m_{\mathbb{X}}(L) \ . \ \blacksquare$$

Corollary. *The probability $m_{\mathbb{X}}$ is ergodic for both $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$.*

Proof: From [M] we know that the expanding maps $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ leave invariant (unique) probabilities $\lambda_{\mathcal{A}}$ and $\lambda_{\mathcal{B}}$ which are exact (in particular ergodic) and equivalent to $m_{\mathbb{X}}$, with an Hölder continuous Radon-Nykodim derivative and metric entropy given by

$$h_{\lambda_{\mathcal{A}}}(F_{\mathcal{A}}) = \int_{\mathbb{X}} \log(J_{\mathcal{A}})\, d\lambda_{\mathcal{A}} = \int_{\mathbb{X}} \log \beta_{\mathcal{A}}\, d\lambda_{\mathcal{A}} = \log(\beta_{\mathcal{A}}) = h_{\text{top}}(\mathcal{A}) \ ,$$
$$h_{\lambda_{\mathcal{B}}}(F_{\mathcal{B}}) = \int_{\mathbb{X}} \log(J_{\mathcal{B}})\, d\lambda_{\mathcal{B}} = \int_{\mathbb{X}} \log \beta_{\mathcal{B}}\, d\lambda_{\mathcal{B}} = \log(\beta_{\mathcal{B}}) = h_{\text{top}}(\mathcal{B}) \ .$$

This implies that $m_{\mathbb{X}}$ is ergodic even though we do not know yet if it is invariant. ∎

Our goal now is to complement this transition between $(\mathbb{X}, F_{\mathcal{A}}, F_{\mathcal{B}}, m_{\mathbb{X}}, \lambda_{\mathcal{A}}, \lambda_{\mathcal{B}})$ and $(\mathbb{T}^2, \mathcal{A}, \mathcal{B}, m)$ regarding invariant probabilities and closed invariant sets with the purpose of detecting rigidity among $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$. We will analyse first the invariant sets for both dynamics and resume the proof studying the common invariant probabilities.

(I) Closed invariant sets

Let $\emptyset \neq L \subseteq \mathbb{T}^2$ be a $\mathcal{A}$, $\mathcal{B}$-minimal set. Then, by [B], L is finite. Moreover, if $L \cap \mathcal{W}^u(P) \neq \emptyset$ (or $L \cap \mathcal{W}^s(P) \neq \emptyset$), then P is in L because, for all x in $\mathcal{W}^u(P)$, we know that

$$\lim_{n \to +\infty} \mathcal{A}^{-m}(x) = P$$

and L is closed. Therefore L must be $\{P\}$ in account of its minimality. Similar argument with the stable manifold. Besides, if $L \cap \mathcal{W}^s(P) = \emptyset$, then $L \cap (\mathbb{T}^2 - M_0) = L \cap [\mathcal{W}^u(P) \cup \mathcal{W}^s(P)] = \emptyset$. And so we may lift L to $\mathbb{X}$, that is, $K^* = h(L)$ is well defined. And K^* is $F_{\mathcal{A}}, F_{\mathcal{B}}$-minimal. In fact

(**i**) K^* is non-empty;

(**ii**) K^* is $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ invariant since $F_{\mathcal{A}}(K^*) = F_{\mathcal{A}}(h(L)) = h\,\mathcal{A}(L) = h(L) = K^*$;

(**iv**) K^* is closed in X because L is compact and h is continuous;

(**iv**) K^* is minimal: if $\emptyset \neq N^* \subseteq K^*$ is closed in $\mathbb{X}$ and $F_{\mathcal{A}}, F_{\mathcal{B}}$-invariant, then $L_1 = h^{-1}(N^*) \cap L$ is a closed subset of L (since N^* and L are closed and h is continuous), and L_1 is invariant by both $\mathcal{A}$ and $\mathcal{B}$ because

$$\begin{aligned}\mathcal{A}(L_1) = \mathcal{A}(h^{-1}(N^*) \cap L) &= \mathcal{A}(h^{-1}(N^*)) \cap \mathcal{A}(L) \\ &= h^{-1}F_{\mathcal{A}}(N^*) \cap L = h^{-1}(N^*) \cap L = L_1 \ ;\end{aligned}$$

thus $L_1 = L$ and therefore, by definition of h, $N^* = K^*$.

And K^* is finite because L is finite and h is a function. This is a way to produce $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ minimal sets and they suit the part (a) of the Theorem. But there may be others not yet obtained.

Given a $F_{\mathcal{A}}, F_{\mathcal{B}}$-invariant closed minimal set K in $\mathbb{X}$, its pre-image by h, $h^{-1}(K)$, is a subset of M_0 which is $\mathcal{A},\mathcal{B}$-invariant since M_0 is $\mathcal{A},\mathcal{B}$-invariant and

$$h^{-1}(K) = \Big\{x\colon\ h(x) = \mathcal{W}^s(x,\epsilon)\cap\mathbb{X}\in K\Big\} = \bigcup_{x\in K}\mathcal{W}^s(x,\epsilon)\cap M_0\ .$$

But in general it is not closed in $\mathbb{T}^2$. So we have to consider its closure, say L, which is still $\mathcal{A},\mathcal{B}$-invariant, since $\mathcal{A}$ and $\mathcal{B}$ are homeomorphisms. Therefore, by [B], either L is the whole torus $\mathbb{T}^2$ or it contains a minimal, which is finite.

If $L=\mathbb{T}^2$, then

$$\begin{aligned}\mathbb{X} = h\Big[\overline{h^{-1}(K)}\cap M_0\Big] &\subseteq h\Big[\overline{h^{-1}(K)}\Big]\cap h(M_0)\\ &= \overline{h[h^{-1}(K)]}\cap h(M_0)\\ &= \overline{h[h^{-1}(K)]}\cap X\\ &= \overline{h[h^{-1}(K)]}\ .\end{aligned}$$

But $h[h^{-1}(K)]=K$ because h is surjective. Hence we conclude that, in this case, $\mathbb{X}=K$ as claimed.

If L has a finite minimal Λ, consider $\Lambda\cap M_0$ and $\Gamma = h(\Lambda\cap M_0)$. Γ is a closed subset of K which is $F_{\mathcal{A}}, F_{\mathcal{B}}$ invariant. If Γ is non-empty then, by the minimality of K, $\Gamma=K$. As Γ is finite, K is finite. We are then left to know if there is always a minimal inside L which intersects M_0. Suppose, on the contrary, that there is a set L for which all minimals are disjoint from M_0. If Λ is one of these minimals, then $\Lambda\cap M_0=\emptyset$, that is, Λ is contained in $\mathcal{W}^u(P)\cup\mathcal{W}^s(P)$; but inside these two manifolds, only $\{P\}$ is closed and invariant. Therefore $\Lambda=\{P\}$. Meanwhile, the set L is given by

$$L = \overline{h^{-1}(K)} = \overline{\Big(\bigcup_{x\in K}\mathcal{W}^s(x,\epsilon)\cap M_0\Big)}$$

which is contained in $\bigcup_{x\in K}\overline{\mathcal{W}^s(x,\epsilon)}$. Thus P belongs to one of the sets in these union, that is, there exists x_0 in K such that P is in the closure of $\mathcal{W}^s(x_0,\epsilon)$. Moreover as $\mathcal{W}^s(x_0,\epsilon)$ is an embbeded disk, if P is in its closure, then x_0 in $\mathcal{W}^s(P)$. But this contradicts the fact that K is a subset of $\mathbb{X}$ and $\mathbb{X}$ is contained in M_0. ∎

(II) Invariant probabilities

Let us now analyse part (b) of the Theorem. Take an ergodic probability μ on $\mathbb{T}^2$, invariant by $\mathcal{A}$ and $\mathcal{B}$. By [KS], μ is the Haar measure or has zero entropy. Its support, say $\mathcal{S}_\mu$, is closed and $\mathcal{A},\mathcal{B}$-invariant so we have

Claim.

(**a**) $\mathcal{S}_\mu \cap \mathcal{W}^u(P) \neq \emptyset \Leftrightarrow \{P\} \subseteq \mathcal{S}_\mu$.

(**b**) $\mathcal{S}_\mu \cap \mathcal{W}^s(P) = \emptyset \Rightarrow \mathcal{S}_\mu \subseteq M_0 \Rightarrow \mu(M_0) = 1$ *and* $\mathcal{S}_\mu = \mathbb{T}^2$.

(**c**) *If* $\mathcal{S}_\mu \subseteq \mathcal{W}^u(P) \cup \mathcal{W}^s(P)$, *then* $\mu(M_0) = 0$ *and* $\mu = \delta_P$.

Proof:

(**a**) If x belongs to $\mathcal{S}_\mu \cap \mathcal{W}^u(P)$, then the sequence $((\mathcal{A})^{-n}(x))_{n\in\mathbb{N}}$ approaches P as n goes to $+\infty$; as $\mathcal{S}_\mu$ is $\mathcal{A}$-invariant and closed, $\{P\} \subseteq \mathcal{S}_\mu$.

(**b**) As M_0 is invariant and μ is ergodic, we have $\mu(M_0) = 0$ or $\mu(M_0) = 1$. Besides if $\mathcal{S}_\mu \cap \mathcal{W}^s(P)$ is empty, then P is not in $\mathcal{S}_\mu$ and therefore $\mathcal{S}_\mu \cap \mathcal{W}^u(P) = \emptyset$. This implies, by definition of M_0, that $\mathcal{S}_\mu$ is contained in M_0 and so $\mu(M_0) \geq \mu(\mathcal{S}_\mu)= 1$. Moreover since $\mathcal{S}_\mu$ is closed and M_0 is dense on the torus, we conclude that $\mathcal{S}_\mu = \mathbb{T}^2$.

(**c**) If $\mathcal{S}_\mu \subseteq \mathcal{W}^u(P) \cup \mathcal{W}^s(P)$, then M_0 is contained in the complement of $\mathcal{S}_\mu$; this implies that $\mu(M_0) = 0$. Besides, in $\mathcal{W}^u(P) \cup \mathcal{W}^s(P)$, only $\{P\}$ is closed in $\mathbb{T}^2$ and left invariant by $\mathcal{A}$ (or $\mathcal{B}$); so $\mathcal{S}_\mu = \{P\}$ and $\mu = \delta_P$. ∎

For our purposes, we reduce further analysis to ergodic probabilities μ such that $\mu(M_0) = 1$, although it is not known if there are exceptions to this equality besides δ_P or other Dirac probabilities supported on periodic orbits. Given one of such a probability μ, define ν at each borelean K of $\mathbb{X}$ by

$$\nu(K) = \frac{\mu\,[h^{-1}(K)]}{\mu\,(M_0)} = \mu[h^{-1}(K)] \ .$$

Lemma 3. *ν is a $F_\mathcal{A}, F_\mathcal{B}$-invariant ergodic probability on $\mathbb{X}$. And either $\nu = \lambda_\mathcal{A} = \lambda_\mathcal{B}$ or it has zero entropy.*

Proof: Given a borelean K in $\mathbb{X}$, we have

$$\nu\,[F_\mathcal{A}^{-1}(K)] = \mu\,[h^{-1}(F_\mathcal{A}^{-1}(K))] = \mu\,[\mathcal{A}^{-1}h^{-1}(K))] = \mu\,[h^{-1}(K)] = \nu(K) \ .$$

Let now K verify $F_{\mathcal{A}}(K) = K$. Clearly $\mathcal{A}\, h^{-1}(K) = h^{-1} F_{\mathcal{A}}(K) = h^{-1}(K)$, so $\mu[\mathrm{h}^{-1}(K)]$ is equal to 0 or 1, which yields $\nu(K) = 0$ or 1. Analogous argument for $\mathcal{B}$ and $\mathrm{F}_{\mathcal{B}}$. Thus ν is ergodic with respect to both dynamics.

If μ is the Lebesgue measure on $\mathbb{T}^2$, then ν is absolutely continuous with respect to $m_{\mathbb{X}}$ according to Lemma 1. By uniqueness, $\nu = \lambda_{\mathcal{A}} = \lambda_{\mathcal{B}}$.

If μ has zero entropy, then ν has zero entropy since the entropy of μ is mainly aware of the expanding nature of its component along $\mathbb{X}$. In fact, the map $h\colon M_0 \to \mathbb{X}$ is surjective and continuous, that is, a semiconjugacy between $\mathcal{A}_{|\, M_0}$ and $F_{\mathcal{A}}$; so $\mathbb{X}$ may be seen as a quotient of M_0 under an equivalence relation $\sim$ given by

$$x \sim y \;\Leftrightarrow\; h(x) = h(y)\ .$$

Moreover $\mu[h^{-1}(K)] = \nu(K)$ for all K, therefore h is an isomorphism in measure. Then

$$0 = h_\mu(\mathcal{A}) \geq h_\nu(F_{\mathcal{A}})$$

and the claim follows. Similar comments hold for $\mathcal{B}$ and $\mathrm{F}_{\mathcal{B}}$. ∎

This way we produce probabilities, ergodic and invariant by $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$, directly from the ones invariant by $\mathcal{A}$ and $\mathcal{B}$ and they fit the contents of the Theorem. We need now to check how are the others.

Given a $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ invariant probability σ, there is a natural way to lift it into a measure on $\mathbb{T}^2$. Associated to σ and the algebra of the boreleans of $\mathbb{X}$, say $\mathfrak{B}_{\mathbb{X}}$, we may consider, for each $m \geq 0$ and $j \geq 0$, the sequence of probabilities

(1) $\mu_0\colon h^{-1}(\mathfrak{B}_{\mathbb{X}} \to \mathbb{R}^+$ such that $\mu_0[h^{-1}(Y)] = \sigma(Y)\ \forall\, Y \in \mathfrak{B}_{\mathbb{X}}$.

Since $F_{\mathcal{A}} \circ h = h \circ \mathcal{A}$, we have, for each element Y of $\mathfrak{B}_{\mathbb{X}}$, $\mathcal{A}^{-1} h^{-1}(Y) = (h \circ \mathcal{A})^{-1}(Y) = (F_{\mathcal{A}} \circ h)^{-1}(Y) = h^{-1}(F_{\mathcal{A}}(Y))$ and this set belongs to $h^{-1}(\mathfrak{B}_{\mathbb{X}})$. That is, $\mathcal{A}^{-1} h^{-1}(\mathfrak{B}_{\mathbb{X}})$ is contained in $h^{-1}(\mathfrak{B}_{\mathbb{X}})$. From here it follows that, for all $m \in \mathbb{N}_0$, $\mathcal{A}^{m+1} h^{-1}(\mathfrak{B}_{\mathbb{X}})$ contains $\mathcal{A}^m h^{-1}(\mathfrak{B}_{\mathbb{X}})$.

(2) $\mu_{m,j}\colon \mathcal{A}^m \mathcal{B}^j\, h^{-1}(\mathfrak{B}_{\mathbb{X}}) \to \mathbb{R}^+$ given, for each Y in $\mathfrak{B}_{\mathbb{X}}$, by

$$\mu_{m,j}\left[\mathcal{A}^m \mathcal{B}^j\, h^{-1}(Y)\right] = \mu_0\left[\mathcal{A}^{-m} \mathcal{B}^{-j}\, \mathcal{A}^m \mathcal{B}^j\, h^{-1}(Y)\right] = \mu_0\Big[h^{-1}(Y)\Big] = \sigma(Y)\ .$$

Notice that, for each such Y, the following equalities hold,

$$\begin{aligned}\mu_{m+1,j}\left[\mathcal{A}^m \mathcal{B}^j\, h^{-1}(Y)\right] &= \mu_0\Big[\mathcal{A}^{-m-1} \mathcal{B}^{-j} \mathcal{A}^m \mathcal{B}^j\, h^{-1}(Y)\Big] = \\ &= \mu_0\Big[\mathcal{A}^{-1}\, h^{-1}(Y)\Big] = \mu_0\Big[(h \circ \mathcal{A})^{-1}(Y)\Big] = \mu_0\Big[(F_{\mathcal{A}} \circ h)^{-1}(Y)\Big] = \\ &= \mu_0\Big[h^{-1}(F_{\mathcal{A}})^{-1}(Y)\Big] = \sigma\Big[(F_{\mathcal{A}})^{-1}(Y)\Big] = \sigma(Y)\end{aligned}$$

and the same is valid for the iterates by $\mathcal{B}$ since it commutes with $\mathcal{A}$; so we may proceed defining

(3) $\mu_\infty : \bigcup_{m\geq 0} \bigcup_{j\geq 0} \mathcal{A}^m \mathcal{B}^j \, h^{-1}(\mathfrak{B}_{\mathbb{X}}) \to \mathbb{R}^+$ such that, if Y is in $\mathfrak{B}_{\mathbb{X}}$ and $Y = \mathcal{A}^m\mathcal{B}^j(h^{-1}(Y))$, then $\mu_\infty(Y) = \mu_{m,j}(Y)$.

μ_∞ is σ-aditive, $\mathcal{A}, \mathcal{B}$-invariant, satisfies $\mu_\infty(M_0) = 1$ and is a probability acting on the σ-algebra $\bigcup_{m\geq 0} \bigcup_{j\geq 0} \mathcal{A}^m\mathcal{B}^j \, h^{-1}(\mathfrak{B}_{\mathbb{X}})$. It may be extended to the σ-algebra of the boreleans of M_0, which is given by

$$\bigvee_{m\geq 0} \bigvee_{j\geq 0} \mathcal{A}^m\mathcal{B}^j \, h^{-1}(\mathfrak{B}_{\mathbb{X}}) \ ,$$

yielding an $\mathcal{A}, \mathcal{B}$-invariant probability $\mu^\vee$. Notice that the σ-algebra of M_0 is the σ-algebra of $\mathbb{T}^2$ modulus a set of measure zero, since $\mu_\infty(M_0) = 1$; to reach all boreleans of $\mathbb{T}^2$ we only have to settle that, for each borelean $\mathcal{M}$,

$$\mu(\mathcal{M}) = \mu^\vee(\mathcal{M} \cap M_0)$$

and the result is a probability μ invariant by $\mathcal{A}$ and $\mathcal{B}$ such that $\mu(M_0) = 1$. Besides, by definition, $\mu[h^{-1}(Y)] = \sigma(Y)$ for all Y in $\mathfrak{B}_{\mathbb{X}}$. Moreover

Lemma 4. *If σ is exact, then so is μ with respect to $\mathcal{A}$ and $\mathcal{B}$.*

Proof: We will use the dynamics of $\mathcal{A}$, the argument is easily repeated for $\mathcal{B}$. As mentioned already, the σ-algebra $h^{-1}(\mathfrak{B}_{\mathbb{X}})$ verifies $\mathcal{A}^{-1} \, h^{-1}(\mathfrak{B}_{\mathbb{X}}) \subseteq h^{-1}(\mathfrak{B}_{\mathbb{X}})$ and the union $\bigcup_{m\geq 0} \mathcal{A}^m h^{-1}(\mathfrak{B}_{\mathbb{X}})$ is the σ-algebra of boreleans of M_0, so it is, modulus zero, the σ-algebra of boreleans of $\mathbb{T}^2$. Finally, given Z in $\bigcap_{m\geq 0} \mathcal{A}^m h^{-1}(\mathfrak{B}_{\mathbb{X}})$, then $\mathcal{C}$ belongs to $\bigcap_{m\geq 0} h^{-1}(F_{\mathcal{A}})^m \, (\mathfrak{B}_{\mathbb{X}}) = h^{-1}(\bigcap_{m\geq 0} (F_{\mathcal{A}})^m (\mathfrak{B}_{\mathbb{X}}))$ and so there is C in $\bigcap_{m\geq 0} (F_{\mathcal{A}})^m \, (\mathfrak{B}_{\mathbb{X}})$ such that $\mathcal{C} = h^{-1}(\mathrm{C})$. As σ is exact, we have $\sigma(\mathrm{C}) = 0$ or $\sigma(\mathrm{C}) = 1$, thus $\mu(\mathcal{C}) = \mu[h^{-1}(\mathrm{C})] = \sigma(\mathrm{C})$ is equal to 0 or 1. ∎

Corollary. *If σ is an exact probability for $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$, then $\sigma = m_{\mathbb{X}}$ or has zero entropy. In particular, $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}} = m_{\mathbb{X}}$.*

Proof: By previous Lemma, the probability μ is invariant and ergodic by both $\mathcal{A}$ and $\mathcal{B}$, and so by [KS], μ is the Lebesgue measure or has zero entropy. Use now the proof of the Lemma to conclude that, in the former case, σ is $\lambda_{\mathcal{A}}$ (that is, the unique probability invariant by $F_{\mathcal{A}}$ and absolutely continuous with respect to $m_{\mathbb{X}}$ is exactly the lifting of the Bowen–Ruelle–Sinai measure of $\mathcal{A}$ which is the Lebesgue measure on the torus) and, in the latter, σ has zero entropy. Analogous calculations hold for $\mathcal{B}$. Thus we have $\lambda_{\mathcal{A}} = \lambda_{\mathcal{B}} = \sigma$.

Besides since $\mathcal{A}$ and $\mathcal{B}$ are linear Anosov diffeomorphisms, by previous Lemma we also conclude that, for the Lebesgue measure m and a borelean L of $\mathbb{X}$, we have

$$\lambda_{\mathcal{A}}(L) = m(h^{-1}(L)) \ ;$$

and moreover this is $m_{\mathbb{X}}(L)$ because, as mentioned before, the Jacobean $\mathcal{H}_n$ of the holonomy is identically one and so the density of $m_{\mathbb{X}}$ with respect to m (see Lemma 1) is identically one. Therefore we ultimately settle that there does exist an exact invariant probability for both $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ and is precisely $m_{\mathbb{X}}$. ∎

ACKNOWLEDGEMENT – The author acknowledges support from JNICT through the project PRAXIS/2/2.1/MAT/19/94.

REFERENCES

[B] Berend – Minimal sets on tori, *Ergod. Th. & Dynam. Sys.,* 4 (1984), 499–507.

[F] Furstenberg – Disjointness in ergodic theory, minimal sets and a problem in diophantine approximation, *Math. Syst. Theory,* 1(1) (1967), 1–49.

[Fk1] Franks – Anosov diffeomorphisms on tori, *Trans. Amer. Math. Soc.,* 145 (1969), 117–124.

[Fk2] Franks – Invariant sets of hyperbolic toral automorphisms, *Amer. Jour. of Math.,* 99(5) (1977), 1089–1095.

[H] Hirsch – *On invariant sets of hyperbolic automorphisms*, In "Essays on Topology and Related Topics" (Haefliger and Narasimhan, Eds.), Springer, 1970.

[HP] Hirsch and Pugh – Stable manifolds and hyperbolic sets, *Proc. Symposia in Pure Math.,* 14 (1970), 133–163.

[KS] Katok and Spatzier – Invariant measures for higher rank hyperbolic abelian actions, *Ergodic Th. & Dynam. Sys,* 16(4) (1996), 751–778.

[L] Lyons – On measures simultaneously 2 and 3 invariant *Israel J. of Math.,* 61(2) (1988), 219–224.

[M] Mañé – *Differentiable Ergodic Theory*, Springer-Verlag.

[Mn] Manning – There are no new Anosov diffeomorphisms on tori, *Amer. Jour. of Math.,* 96 (1974), 422–429.

[N] Newhouse – On codimension one Anosov diffeomorphisms, *Amer. Jour. of Math.,* 92 (1970), 761–770.

[PS] Pugh and Shub – Ergodicity of Anosov actions, *Invent. Math.,* 15 (1972), 1–23.

[PY1] Palis and Yoccoz – Rigidity of centralizers of diffeomorphisms, *Ann. Sci. E. N. S.,* 22 (1989), 81–98.

[PY2] Palis and Yoccoz – Centralizers of Anosov diffeomorphisms on tori, *Ann. Sci. E. N. S.,* 22 (1989), 99–108.

[R] Rudolph – $\times 2$ and $\times 3$ invariant measures and entropy, *Ergodic Th. & Dynam. Sys.,* 10 (1990), 395–406.

[RY] Robinson and Young, L.-S. – Nonabsolutely continuous foliations for an Anosov diffeomorphism, *Invent. Math.,* 61 (1980), 159–176.

[Sh1] Shub – Endomorphisms of compact differentiable manifolds, *Amer. Jour. of Math.,* 91 (1969), 175–199.

[Sh2] Shub – Stabilité globale des systemes dynamiques, *Astérisque,* 56 (1978).

Maria Carvalho,
Centro de Matemática, Faculdade de Ciências, Universidade do Porto,
Praça Gomes Teixeira, 4050 Porto – PORTUGAL
E-mail: mpcarval@fc.up.pt