PORTUGALIAE MATHEMATICA
Vol. 57 Fasc. 1 – 2000

SIGNAL EXTRACTION FOR A CLASS OF NONSTATIONARY PROCESSES

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Abstract: In 1952, Cramér introduced a class of nonstationary processes. This broad class of processes contains the important harmonizable and stationary classes of processes. The Cramér class can have additional structure imposed upon it through Cesàro summability considerations. These refined Cramér classes, termed (c,p)-summable Cramér, have recently been considered by Swift 1997. This paper considers the problem of signal extraction for a process of the form $X(t) = Y(t) + N(t)$ where $Y(\cdot)$ and $N(\cdot)$ are (c,p)-summable Cramér.

1 – Preliminaries

In the following work, let (Ω, Σ, P) be the underlying probability space.

Definition 1.1. For $p \geq 1$, define $L_0^p(P)$ to be the set of all centered complex valued $f \in L^p(\Omega, \Sigma, P)$, that is $E(f) = 0$, where $E(f) = \int_\Omega f(\omega)\,dP(\omega)$ is the expectation. □

In the following work, we will consider second order stochastic processes, that is, mappings $X\colon \mathbb{R} \to L_0^2(P)$. The classical results for $X(\cdot)$ are based upon the following assumption on the covariance $r(\cdot,\cdot)$:

Definition 1.2. A stochastic process $X\colon \mathbb{R} \to L_0^2(P)$ is *stationary* (stationary in the wide or Khintchine sense) if its covariance $r(s,t) = E(X(s)\,\overline{X(t)})$ is continuous and is a function of the difference of its arguments, so that

$$r(s,t) = \tilde{r}(s-t)\ . \ □$$

Received: June 19, 1998; *Revised*: November 16, 1998.
AMS Subject Classification: 60G12, 60G35.
Keywords: Harmonizable processes; Cramér processes; Signal detection.

An equivalent definition of a stationary process is one whose covariance function can be represented as

$$\tilde{r}(\tau) = \int_{\mathbb{R}} e^{i\lambda\tau}\, dF(\lambda)\ , \tag{1}$$

for a unique non-negative bounded Borel measure $F(\cdot)$. This alternate definition is a consequence of a classical theorem of Bochner (cf. Gihman and Skorohod [3]). This representation allows the powerful Fourier analytic methods in the analysis of stationary processes. In many applications, the assumption of stationarity is not always valid, this provides the motivation for the following.

Definition 1.3. A stochastic process $X\colon \mathbb{R} \to L_0^2(P)$ is *weakly harmonizable* if its covariance $r(\cdot,\cdot)$ is expressible as

$$r(s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda s - i\lambda' t}\, dF(\lambda,\lambda') \tag{2}$$

where $F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is a positive semi-definite bimeasure. □

A stochastic process, $X(\cdot)$, is *strongly harmonizable* if the bimeasure $F(\cdot,\cdot)$ in (2) extends to a complex measure and hence is of bounded Vitali variation.

In either case, $F(\cdot,\cdot)$ is termed the *spectral bi-measure* (or *spectral measure*) of the harmonizable process.

Comparison of equation (2) with equation (1) shows that when $F(\cdot,\cdot)$ concentrates on the diagonal $\lambda=\lambda'$, both the weak and strong harmonizability concepts reduce to the stationary concept. Harmonizable processes retain the powerful Fourier analytic methods inherent with stationary processes, as seen in Bochner's theorem, (1); but they relax the requirement of stationarity.

Recently, the structure and properties of harmonizable processes has been investigated and developed extensively by M.M. Rao and others.

2 – A class of nonstationary processes

A general class of nonstationary processes which extends the ideas of the harmonizable class was first considered by Cramér in 1952 [2]. Rao has refined and studied [7] these processes and gave the following definition.

Definition 2.1. A second-order process $X\colon T \to L^2(P)$ is of Cramér class (or class (C)) if its covariance function $r(\cdot,\cdot)$ is representable as

$$r(t_1,t_2) = \int_S \int_S g(t_1,\lambda)\, \overline{g(t_2,\lambda')}\, dF(\lambda,\lambda') \tag{3}$$

relative to a family $\{g(t,\cdot),\ t \in T\}$ of Borel functions and a positive definite function $F(\cdot,\cdot)$ of locally bounded variation on $S \times S$, [S will be in the classical case $\hat{T}$ the dual of an LCA group T, and generally $(S, \boldsymbol{B})$ is a measurable space] with each g satisfying the (Lebesgue) integrability condition:

$$0 \le \int_S \int_S g(t_1,\lambda)\,\overline{g(t_2,\lambda')}\,dF(\lambda,\lambda') < \infty, \qquad t \in T\ . \ \square$$

If $F(\cdot,\cdot)$ has a locally finite Fréchet variation, then the integrals in equation (3) are in the sense of (strict) Morse–Transue and the corresponding concept is termed *weak class (C)*.

An integral representation of weak class (C) processes was obtained by Chang and Rao [1] and is given by

Theorem 2.1. *If* $X\colon T \to L_0^2(P)$ *is of weak Cramér class relative to a family* $\{g(t,\cdot),\ t \in T\}$ *of Borel functions and a positive definite bimeasure* $F(\cdot,\cdot)$ *of locally bounded Fréchet variation on* $S \times S$*, then there exists a stochastic measure* $Z\colon \boldsymbol{B} \to L_0^2(P)$*,* $\boldsymbol{B}$ *a* σ*-algebra of* S*, such that*

$$X(t) = \int_S g(t,\lambda)\,dZ(\lambda) \tag{4}$$

where

$$E(Z(A)\,\overline{Z(B)}) = F(A,B) \quad \text{for } (A,B) \in \boldsymbol{B} \times \boldsymbol{B}\ .$$

Conversely, if $X(\cdot)$ *is a second-order process defined by (4) then it is a process of weak class (C).* ∎

The weak class (C) processes include the weakly harmonizable class and so (4) is the form of the integral representation for the harmonizable class as well.

A finer analysis of class (C) processes is obtained by requiring some additional structure. The desired concept was introduced by Swift [8] and is obtained by considering $g(\cdot,\lambda)$ satisfying a Cesàro summability condition.

Definition 2.2. A second-order process $X\colon \mathbb{R} \to L_0^2(P)$ is (c,p)-summable weak Cramér class, $p \ge 1$ (or (c,p)-summable weak class (C)) if its covariance $r(\cdot,\cdot)$ has representation

$$r(t_1,t_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(t_1,\lambda)\,\overline{g(t_2,\lambda')}\,dF(\lambda,\lambda') \tag{5}$$

relative to a family $\{g(\cdot,\lambda),\ \lambda \in \mathbb{R}\}$ of Borel functions and a positive definite function $F(\cdot,\cdot)$ of locally bounded Fréchet variation on $\mathbb{R} \times \mathbb{R}$, with each g satisfying the condition that

$$\lim_{T\to\infty} a_T^{(p)}(|h|,\lambda,\lambda') \tag{6}$$

exists uniformly in h relative to a set $D \subset \mathbb{R}$ and is bounded for all $h,\ p \geq 1$, where

$$a_T^{(p)}(|h|,\lambda,\lambda') = \begin{cases} \dfrac{1}{T}\displaystyle\int_0^T a_\alpha^{(p-1)}(|h|,\lambda,\lambda')\,d\alpha & \text{for } p>1\ , \\ \dfrac{1}{T}\displaystyle\int_0^{T-|h|} g(s,\lambda)\,\overline{g(s+|h|,\,\lambda')}\,ds & \text{for } p=1\ . \ \square \end{cases} \tag{7}$$

If $F(\cdot,\cdot)$ has a locally finite Vitali variation, then $F(\cdot,\cdot)$ determines a (Lebesgue–Stieltjes) measure in the plane. The corresponding processes will be termed class (c,p)*-summable Cramér*.

The concept of the limit in (6) existing uniformly relative to a set (not necessarily restricted to a point) was introduced by T. Yoshizawa [9] and is a weaker concept than the limit existing uniformly. *This concept is used here as well as in [8].*

The classes of (c,p)-summable weak Cramér processes are wide. Classical summability results (cf. Hardy [4]) imply that if

$$\lim_{T\to\infty} a_T(|h|,\lambda,\lambda')$$

exists uniformly in h relative to a set $D \subset \mathbb{R}$ then

$$\lim_{T\to\infty} a_T^{(p)}(|h|,\lambda,\lambda')$$

exists uniformly in h relative to a set $D \subset \mathbb{R}$ for each integer $p \geq 1$. The converse implication is false. Hence,

(c,p)-summable weak Cramér processes $\subset$
$\subset$ $(c,p{+}1)$-summable weak Cramér processes

for $p \geq 1$. The inclusions are proper.

It should be noted here that a further extension of the preceding class is obtainable by considering the still weaker concept of Abel summability. The consequences of such an extension are not yet known and await a further investigation.

A subclass of $(c,1)$-summable Cramér processes was introduced by Rao in 1978 [6]. This subclass requires the theory of uniformly almost periodic functions depending upon a parameter, (cf. Yoshizawa [9]). Swift [8] gave the following extension.

Definition 2.3. A second-order process $X\colon \mathbb{R} \to L^2(P)$ whose covariance is of weak class (C) is termed almost weakly harmonizable if $g(\cdot,\lambda)$ is a uniformly almost periodic function relative to $(\lambda \in)\, D = \mathbb{R}$. □

If the spectral bimeasure $F(\cdot,\cdot)$ admits a finite Vitali variation in the plane, the corresponding concept will be termed *almost strongly harmonizable*. The class of almost weakly harmonizable processes contains the class of weakly harmonizable processes. This can be immediately seen by setting $g(t,\lambda) = e^{i\lambda t}$. Further, if the spectral bimeasure $F(\cdot,\cdot)$ concentrates on the diagonal $\lambda = \lambda'$ the representation of the covariance would become

$$r(t_1,t_2) = \int_{\mathbb{R}} g(t_1,\lambda)\, \overline{g(t_2,\lambda)}\; d\tilde{F}(\lambda) \;, \tag{8}$$

where $g(\cdot,\lambda)$ is a uniformly almost periodic function relative to $(\lambda \in)\, D = \mathbb{R}$. Processes with a covariance representable by (8) will be termed *almost stationary*.

Swift showed that the class of almost weakly harmonizable processes is contained in the $(c,1)$-summable weak Cramér class. In addition, a sample path analysis for the almost harmonizable class is considered in [8].

3 – Signal extraction

Detailed analysis of the problem of filtering a process of the form

$$X(t) = Y(t) + N(t)$$

where $Y(\cdot)$ and $N(\cdot)$ are weakly harmonizable and correlated is given in Chang and Rao [1]. The following theorem extends their result to (c,p)-summable Cramér processes.

Theorem 3.1. *Let $X\colon \mathbb{R} \to L^2(P)$ be a stochastic processes given by*

$$X(t) = Y(t) + N(t) \qquad \textit{for} \quad -\infty < t < \infty \tag{9}$$

where $Y(t)$ and $N(t)$ are uncorrelated and are (c,p)-summable Cramér processes so that their respective covariances have representation

$$r_Y(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_Y(s,\lambda)\, \overline{g_Y(t,\lambda')}\; dF_1(\lambda,\lambda') \;, \tag{10}$$

(11) $$r_N(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_N(s,\lambda)\,\overline{g_N(t,\lambda')}\; dF_2(\lambda,\lambda')\ ,$$

where g_Y and g_N satisfy (7) and the $F_i(\cdot,\cdot)$ are of bounded Frechét variation. Then the optimal filter $\hat{Y}(a)$, i.e. the linear estimator of $Y(a)$ that minimizes the mean square error, is given by

(12) $$\hat{Y}(a) = \int_{-\infty}^{\infty} k_a(\lambda)\; dZ_X(\lambda)$$

where $k_a(\cdot)$ is a solution of the integral equation

(13) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(\lambda)\, d\Big[F_1(\lambda,\lambda') + F_2(\lambda,\lambda')\Big] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_Y(\lambda)\, dF_1(\lambda,\lambda')\ ,$$

and $Z_X(\cdot)$ is the stochastic measure associated with the process X (cf. (4)). Moreover, there is only one solution of (13) that gives the optimal filter in (12).

Proof: Since $Y(t)$ and $N(t)$ are (c,p)-summable Cramér processes it follows that $X(t)$ is also (c,p)-summable Cramér with representation

$$X(t) = \int_{-\infty}^{\infty} g(t,\lambda)\, dZ_X(\lambda)$$

for some $g(t,\cdot)$ satisfying (7), and where $Z_X(\cdot)$ is the stochastic measure associated with the process X, (cf. (4)).

Let G be the closed linear span determined by $X(\cdot)$, that is

$$G = \overline{\mathrm{sp}}\Big\{X(t)\colon\ -\infty < t < \infty\Big\}\ .$$

Now consider for the Borel functions $g_i\colon \mathbb{R} \to \mathbb{C}$, $i=1,2$,

$$(g_1, g_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(\lambda)\,\overline{g_2(\lambda')}\; dF(\lambda,\lambda)\ ,$$

with the integrals being in the MT-sense. Cramér has shown for class (C) ([2], p. 336) that the space of all such g defines a semi-inner product space. (The argument extends to the MT-integrals as well.) Considering the equivalence classes, this space can be completed, which we shall call H, a Hilbert space. The completion is a function space, not with some ideal elements. A proof (even for vector valued functions) is in Y. Kakihara (cf. [5], pp. 163–164.) This fact will be used here.

Now let $L^2(Z_X)$ be the space of all random variables generated by the linear combinations of the form

$$\sum_{i=1}^{n} a_i\, Z_X(S_i)$$

for constants a_i and bounded Borel sets S_i of the line. It follows that $L^2(Z_X)$ is also a Hilbert space [if z_1, z_2 in $L^2(Z_X)$, $(z_1, z_2) = E(z_1 \cdot \overline{z_2})$ is the inner product] and G is a subspace of $L^2(Z_X)$.

Now, if $z \in L^2(Z_X)$ then by a theorem of Cramér ([2], p. 336)

$$z = \int_{-\infty}^{\infty} g(\lambda)\, dZ_X(\lambda)$$

for some $g \in H$, and $L^2(Z_X)$ and H are isometrically isomorphic with

$$||z||_{L^2(Z_X)} = ||g||_H \ .$$

Let z_n in G be such that

$$z_n = \sum_{i=1}^{n} a_i^n\, X(t_i) \ .$$

Now to complete this argument, a z_n in G (or a limit of such elements), must be found such that

$$||z_n - Y(a)||$$

is a minimum. Note that $Y(a) \notin G$. It should also be noted that the minimizing z in G is a linear function of the $X(t_i)'s$.

Considering Y as an element of $L^2(P)$, the space of all L^2 random variables, so that

$$G \subset L^2(Z_X) \subset L^2(P) \ ,$$

and noting that the L^2 is a uniformly convex space, with G a complete subspace of $L^2(P)$, it follows from the projection theorem in Hilbert space, that there exists a unique element z in G such that $||z_n - Y(a)||$ is a minimum. Further, there exists a sequence of $\{a_i^n\}$ such that

$$z_n = \sum_{i=1}^{n} a_i^n X(t_i) \to z$$

in G. The z_n can thus be expressed as

$$z_n = \sum_{i=1}^{n} a_i^n \int_{-\infty}^{\infty} g(t_i, \lambda)\, dZ_X(\lambda)$$

where g satisfies the condition (7). Letting

$$k_n(\lambda) = \sum_{i=1}^{n} a_i^n\, g(t_i, \lambda)$$

then

$$z_n = \int_{-\infty}^{\infty} k_n(\lambda)\, dZ_X(\lambda) \ .$$

If $\tilde{L}^2(F)$ is the closed subspace of H corresponding to G, then $k_n(\cdot) \in \tilde{L}^2(F)$ and $k_n(\cdot) \to k(\cdot)$ in this subspace. Further, $k(\cdot) \in \tilde{L}^2(F)$ by closure, so that

$$z = \int_{-\infty}^{\infty} k(\lambda)\, dZ_X(\lambda) \ .$$

Now it follows from the integral representations (cf. (4)) of $X(\cdot), Y(\cdot)$ and $N(\cdot)$ that

$$Z_X = Z_Y + Z_N \ ,$$

so that the required element z can be written as,

$$\begin{aligned} z &= \int_{-\infty}^{\infty} k(\lambda)\, dZ_X(\lambda) \\ &= \int_{-\infty}^{\infty} k(\lambda)\, dZ_Y(\lambda) + \int_{-\infty}^{\infty} k(\lambda)\, dZ_N(\lambda) \\ &= Y + N \ , \quad \text{(say)} \end{aligned}$$

where Y and N are uncorrelated. If the minimal mean square error is denoted by e_a, then

$$\begin{aligned} e_a &= ||Y(a) - z||^2 = ||Y(a) - Y||^2 + ||N||^2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(f_a(\lambda) - k(\lambda)\Big) \overline{\Big(f_a(\lambda') - k(\lambda')\Big)}\, dF_1(\lambda, \lambda') \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(\lambda)\, \overline{k(\lambda')}\, dF_2(\lambda, \lambda') \ , \end{aligned} \tag{14}$$

where $f_Y(a, \lambda) = f_a(\lambda)$. Thus the desired $k(\cdot)$ in $\tilde{L}^2(F)$ is the unique function that gives the minimal value for e_a.

Now to find the minimizing k $(= k_a$, say), one applies the well-known Hilbert space variational type argument. Thus letting

$$k = k_a + \varepsilon\, h$$

where $h \in \tilde{L}^2(F)$ and denoting the mean square error of (14) by $e_a(\varepsilon, h)$, then

$$e_a(\varepsilon, h) \geq e_a$$

for all ε and h. Thus (14) becomes

$$
\begin{aligned}
e_a(\varepsilon,h) \;&= \\
&= e_a+\varepsilon\biggl(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\lambda)\,\overline{\Bigl(k_a(\lambda')-f_a(\lambda')\Bigr)}+\overline{h(\lambda')}\,\Bigl(k_a(\lambda)-f_a(\lambda)\Bigr)\,dF_1(\lambda,\lambda') \\
&\quad+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_a(\lambda)+\overline{k_a(\lambda')}\,dF_2(\lambda,\lambda')\biggr) \\
&\quad+\varepsilon^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\lambda)\,\overline{h(\lambda')}\,\Bigl(dF_1(\lambda,\lambda')+dF_2(\lambda,\lambda')\Bigr)\;.
\end{aligned} \tag{15}
$$

Since $e_a(\varepsilon,h)\ge e_a$, the sum of the other two terms must be non-negative for all ε, and h in $\tilde L^2(F)$. Thus, taking $\varepsilon>0$ and letting $\varepsilon\to 0$, it follows that the coefficient of ε in (15) must vanish. From this one finds that k_a must satisfy

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(\lambda)\,d\Bigl(F_1(\lambda,\lambda')+F_2(\lambda,\lambda')\Bigr) \;=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_a(\lambda)\,dF_1(\lambda,\lambda')\;. \tag{16}
$$

Now, there is a unique k in $\tilde L^2(F)$ that makes $e_a(\varepsilon,h)=e_a$ and such a k satisfies (16). However from (15), since the last term does not involve k and the minimum occurs for only one k in $\tilde L^2(F)$, it follows that the middle term of (15) vanishes for only one element k in $\tilde L^2(F)$, and hence there is only one solution of the integral equation (16) that gives an optimal filter. ∎

ACKNOWLEDGEMENTS – The author expresses his thanks to Professor M. M. Rao for his advice and constant encouragement, and a referee for a substantial clarification to an earlier draft of this paper. The author also expresses his gratitude to the Mathematics department at Western Kentucky University for release time during the Spring 1998 semester, during which this work was completed.

REFERENCES

[1] Chang, D.K. and Rao, M.M. – *Bimeasures and Nonstationary Processes*, "Real and Stochastic Analysis", p. 7–118, John Wiley and Sons, New York, 1986.

[2] Cramér, H. – A Contribution to the Theory of Stochastic Processes, *Proc. Second Berkeley Symp. Math. Statist. Prob.,* 2 (1952), 55–77.

[3] Gihman, I.I. and Skorohod, A.V. – *The Theory of Stochastic Processes I*, Springer-Verlag, New York, 1974.

[4] Hardy, G.H. – *Divergent Series*, Oxford University Press, London, 1949.

[5] KAKIHARA, Y. – *Multidimensional Second Order Stochastic Processes*, World Scientific Press, Singapore, 1997.

[6] RAO, M.M. – *Covariance Analysis of Non Stationary Time Series*, in "Developments in Statistics", Volume 1, p. 171–275, 1978.

[7] RAO, M.M. – Harmonizable Processes: Structure Theory, *L'Enseign Math.,* 28 (1984), 295–351.

[8] SWIFT, R.J. – *Some Aspects of Harmonizable Processes and Fields*, in "Real and Stochastic Analysis: Recent Advances" (M.M. Rao, Ed.), p. 303–365, CRC Press, Boca Raton, 1997.

[9] YOSHIZAWA, T. – *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Appl. Math. Ser., 14, Springer-Verlag, New York, 1975.

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