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# CERTAIN ALGEBRAIC SURFACES WITH NON-REDUCED MODULI SPACE

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**Abstract:** We show that all the even canonical surfaces in a certain area of the zone of existence have non-reduced Kuranishi space. In particular, for a given irregularity, the existence of surfaces of general type with non-reduced moduli space is shown.

## 0 – Introduction

In the study of surfaces of general type, the structure of moduli spaces is one of the main objects. Unlike the case of curves, the moduli space happens to be non-reduced. Indeed, Catanese [C] showed that some weighted projective hypersurfaces have non-reduced moduli, unifying the previously known examples due to Horikawa [Ho1] and Miranda [M] (see also [Ho2]). We found another kind of examples along the lines $K^2 = 3\,p_g - 6$, $q = 0$ [K1] and $K^2 = 3\,p_g + 1$, $q = 1$ [K2], which are even and canonical. Here we call a non-singular projective surface $S$

– an *even* surface if $K_S$ is divisible by 2 in the Picard group,

– a *canonical* surface if the canonical map is birational onto its image.

The purpose of the present article is to extend and unify the latter examples by showing that all the even canonical surfaces in a certain area of the zone of the existence have non-reduced moduli. Namely, we shall show the following:

**Theorem.** *Let $S$ be an even canonical surface with*

$$3\,\chi(\mathcal{O}_S) + 10\Big(q(S) - 1\Big) \;<\; K_S^2 \;<\; \frac{16}{5}\Big(\chi(\mathcal{O}_S) + 3\Big(q(S) - 1\Big)\Big)\;.$$

*Suppose further*

(**1**) $p_q(S) \geq 12$ *when* $q(S) = 0$,

(**2**) $p_g(S) \geq 13\,(q(S) - 1)$ *when* $q(S) > 0$.

*Then the Kuranishi space of $S$ is not reduced.*

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Similarly as in [K1] and [K2], the proof is an easy application of the well-known result of Burns–Wahl [BW], which roughly says that $(-2)$-curves are obstruction for the Kuranishi space to be smooth. So our task is reduced to showing the existence of a $(-2)$-curve. In §3, we will construct explicit examples which satisfy the requirements in Theorem. In particular, it will show the existence of surfaces with non-reduced moduli and with given irregularity, which seems to be a new result. Appendix will be devoted to describing a certain type of degenerate fibres in pencils of non-hyperelliptic curves of genus 3 appeared in our example.

## 1 – Even surfaces with a fibration

A non-singular projective surface $S$ is called an even surface if there exists a line bundle $L$, which will be called a semi-canonical bundle, satisfying $K_S = 2\,L$. Then $S$ is automatically minimal and $L$ is nef if $S$ is of general type.

Let $S$ be an even surface of general type, and assume that we have a fibration $f:\ S \to B$ over a non-singular projective curve $B$ of genus $b$. We denote by $F$ a general fibre of $f$ and put $g = g(F)$. Note that we have $g \geq 2$, since $S$ is of general type. By [Ha], $f_*\mathcal{O}_S(L)$ is locally free. Since $K_S = 2\,L$, we have $(R^1 f_*\mathcal{O}_S(L))^\vee \simeq f_*\mathcal{O}_S(L) \otimes \omega_B^{\otimes(-1)}$ by the relative duality theorem. Hence, using Leray spectral sequence, we get

$$\chi(L) = 2\,\chi(f_*L) - \operatorname{length}(R^1 f_*L)_{\mathrm{tor}}\ ,$$

where $(R^1 f_*L)_{\mathrm{tor}}$ denotes the torsion part of $R^1 f_*L$. We want to know when $R^1 f_*L$ is locally free.

Put $F_t := f^*t$ and $L_t = L|_{F_t}$ for $t \in B$. From the usual multiplication map

$$H^0(F_t, L_t) \otimes H^0(F_t, L_t) \;\to\; H^0(F_t, 2L_t) \simeq H^0(F_t, K_{F_t})\ ,$$

and Hopf's Lemma, we know that $g = h^0(K_{F_t}) \geq 2\,h^0(F_t, L_t) - 1$, that is, $h^0(F_t, L_t) \leq (g+1)/2$ for all $t \in B$. furthermore, since $L_t$ is a theta characteristic, we can apply a remarkable theorem of Sorger [S] to see that $h^0(F_t, L_t)$ is constant modulo 2. Therefore, we get:

**Lemma 1.1.** *In the above situation, if $h^0(S, L) \neq 0$ and $g \leq 4$, then $h^0(F_t, L_t)$ (hence also $h^1(F_t, L_t)$) is constant. In particular, if $K_S^2 < 8\,\chi(\mathcal{O}_S)$ and $g \leq 4$, then $\chi(L) = 2\,\chi(f_*L)$.*

**Proof:** It remains to show the last assertion. By the Riemann–Roch theorem and the Serre duality, we have

$$\chi(L) \;=\; 2\,h^0(L) - h^1(L) \;=\; -\frac{L^2}{2} + \chi(\mathcal{O}_S)$$

since $K_S = 2\,L$. Hence, if $K_S^2 < 8\,\chi(\mathcal{O}_S)$, then $h^0(L) > 0$. If $S$ has a fibration $f\colon\ S \to B$ of genus $g \leq 4$, then $R^1 f_*L$ is locally free, since $h^1(L_t)$ is constant. ∎

**Lemma 1.2.** *Let $S$ be an even surface with a fibration $f\colon S \to B$ of genus $g \geq 2$. Let $F$ be a fibre and let $F = D_1 + D_2$ be any decomposition with $D_1 > 0$, $D_2 > 0$. Then $D_i^2$ $(i = 1, 2)$ and $D_1 D_2$ are even integers.*

*In particular, if $f$ has no multiple fibres, then every fibre of $f$ is numerically 2-connected, and the natural sheaf homomorphism $f^* f_* \omega_{S/B} \to \omega_{S/B}$ is surjective.*

**Proof:** For any divisor $D$ on $S$, we know that $K_S D + D^2$ is an even integer. Since $K_S = 2\,L$, we see that $D^2$ must be even.

Let $F$ be a fibre of $f$ and let $F = D_1 + D_2$ be a decomposition as above. Then we have $0 = FD_1 = D_1^2 + D_1 D_2$. Since $D_1^2$ is even, so is $D_1 D_2$. Assume that $F$ is not a multiple fibre. Then it is at least numerically 1-connected. But, as we have seen above, we cannot have $D_1 D_2 = 1$ for any effective decomposition $F = D_1 + D_2$. Hence $F$ is numerically 2-connected. Then it is well-known (see e.g. [CF]) that $K_F$ is generated by its global sections. Hence the last assertion follows. ∎

## 2 – Proof of the main result

Let $S$ be an even surface of general type, $L$ a semi-canonical bundle and $f\colon S \to B$ a non-hyperelliptic fibration of genus 3. Then it is known that the numerical characters of $S$ satisfy

$$K_S^2 \;\geq\; 3\,\chi(\mathcal{O}_S) + 10\,(b-1)\ ,$$

see e.g. [K2], [CC], [R]. We assume that $h^0(S, L) > 0$. Recall that this is automatically satisfied when $K_S^2 < 8\,\chi(\mathcal{O}_S)$. Since $LF = 2$ and $F$ is non-hyperelliptic, we have $h^0(F, L) = 1$. It follows that the rational map induced from $|L|$ factors through $B$ and that $\mathcal{L} := f_* \mathcal{O}_S(L)$ is a line bundle. We have an effective divisor $Z$ such that $L = f^*\mathcal{L} + Z$. Let $Z_v$ be the maximal subdivisor of $Z$ with $Z_v F = 0$, and put $Z_h = Z - Z_v$. Since $LF = 2$, there are three possibilities for $Z_h$:

(**i**) $Z_h = 2\,G$ with $G$ irreducible, $GF = 1$;

(**ii**) $Z_h = G_1 + G_2$ with $G_i$ irreducible, $G_1 \neq G_2$, $G_i F = 1$;

(**iii**) $Z_h$ is irreducible.

Recall that we have

$$\deg \mathcal{L} \;=\; -\frac{L^2}{4} + \frac{\chi(\mathcal{O}_S)}{2} + b - 1\ .$$

from the proof of Lemma 1.1.

**Lemma 2.1.** *If $K_S^2 < (16/5)\,(\chi(\mathcal{O}_S) + 3\,(b-1))$, then $Z_h$ must be of type (i).*

**Proof:** Suppose that (ii) or (iii) is the case. Since each irreducible component of $Z_h$ has multiplicity one, it is easy to see that $K_{S/B}+Z$ is nef (see [K3, Part III]). Hence we have $(K_{S/B}+Z)\,Z \geq 0$, which is equivalent to $K_S^2 \geq (16/5)\,(\chi(\mathcal{O}_S) + 3\,(b-1))$. ∎

**Lemma 2.2.** *Assume that $Z_h$ is of type (i). If $Z_v = 0$, then $K_S^2 = 3\,\chi(\mathcal{O}_S) + 10\,(b-1)$.*

**Proof:** Write $Z = Z_h = 2\,G$ as in (i). Since $G$ is a section of $f$, we get

$$2\,b - 2 \;=\; G(K_S + G) \;=\; 5\,G^2 + 2\deg(\mathcal{L})\ .$$

On the other hand, since $Z = 2\,G$, we have

$$4\,G^2 \;=\; Z^2 \;=\; (L - f^*\mathcal{L})^2 \;=\; L^2 - 4\deg(\mathcal{L})\ .$$

Eliminating $G^2$, we get

$$5\,L^2 = \;12\deg(\mathcal{L}) + 8\,(b-1)\ ,$$

which is equivalent to $K_S^2 = 3\,\chi(\mathcal{O}_S) + 10\,(b-1)$. ∎

**Lemma 2.3.** *Assume that $Z_h$ is of type (i). then $Z_v$ consists of $(-2)$-curves In particular, $Z_v = 0$ if $K_S$ is ample.*

**Proof:** Since $G$ is a section, $f$ cannot have multiple fibres. Hence any fibre $F$ is numerically 2-connected by Lemma 1.2. Let $s$ be a non-zero element in $H^0(F, L)$, and let $F_1$ be the maximal subdivisor of $F$ on which $s$ vanishes identically. Put $F_2 = F - F_1$. then $s$ can be regarded as an element in $H^0(F_2, L - F_1)$ which does not vanish identically on any components of $F_2$. In particular, we must have $(L - F_1)\,F_2 \geq 0$, or $LF_2 \geq F_1F_2$. If $F_1 \neq 0$, then $F_1F_2 \geq 2$ by the numerical 2-connectedness of $F$. Therefore, $LF_2 \geq 2$. On the other hand, we have $LF_2 \leq LF = 2$. These together imply $LF_1 = 0$, which in turn means that $F_1$ consists of $(-2)$-curves since $K_S = 2\,L$. ∎

**Remark.** In the appendix, we shall show the existence of a particular degenerate fibre which contributes to the Castelnuovo–Horikawa index and which is of hyperelliptic type. Let $F$ be such a hyperelliptic fibre. Note that $F$ is not irreducible, because, otherwise, $G \cap F$ would be a Weierstrass point while $K_F = 2\,L|_F$ and $h^0(F, Z) = 1$. Then $F$ necessarily contains a part of $Z_v$, that is, a $(-2)$-curve. □

Summing up, we get:

**Proposition 2.4.** *Let $S$ be an even surface of general type with a non-hyperelliptic genus 3 fibration $f : S \to B$. If the numerical characters of $S$ satisfy*

$$3\,\chi(\mathcal{O}_S) + 10\,(b-1) \;<\; K_S^2 \;<\; \frac{16}{5}\Big(\chi(\mathcal{O}_S) + 3\,(b-1)\Big)\;,$$

*then $K_S$ cannot be ample.* ∎

Now, we show our main result, i.e. Theorem in Introduction.

**Proof of Theorem:** From the structure theorem of even canonical surfaces with small $K^2$ [K3, Parts I and III], we know that $S$ has a non-hyperelliptic fibration of genus 3, $f\colon S \to B$, over a curve $B$ of genus $q(S)$, under the numerical hypothesis of Theorem. Such $f$ is obtained as the Albanese map when $S$ is irregular, and as the semi-canonical map when $S$ is regular. Then, by Proposition 2.4, $K_S$ is non-ample, or equivalently, there exists a $(-2)$-curve on $S$.

Note that being an even surface does not depend on the complex structure but on the topology of the underlying differentiable manifold, since it is equivalent to the vanishing of the second Stiefel–Whitney class. Hence any deformation of $S$ is again an even surface. Also, since the condition that the canonical map is birational onto the image is an open condition, any small deformation of $S$ is again a canonical surface.

Let $M$ be the Kuranishi space of $S$, and let $\{S_t\}$, $t \in M$, be the Kuranishi family. We remark that $f$ can be naturally extended to the family: $\{f_t\colon S_t \to B_t\}$. From the above observation, we see that every $S_t$ is an even canonical surface and it has a $(-2)$-curve contained in a fibre of $f_t$. Let $\Theta_S$ denote the tangent sheaf of $S$. Then $H^1(S, \Theta_S)$ can be identified with the Zariski tangent space of $M$ at $S$. recall that Burns–Wahl's theorem [BW] implies that a general vector in $H^1(S, \Theta_S)$ kills every $(-2)$-curve on $S$. Since every nearby $S_t$ must have a $(-2)$-curve, we see that $h^1(\Theta_S)$ is strictly greater than $\dim_{\mathbb{C}} M$, implying that $M$ is singular at $S$. Since the same is true around any nearby $S_t$, we conclude that $M$ is everywhere singular, that is, non-reduced. ∎

## 3 – Examples

In this section, we construct even canonical surfaces as in Theorem.

We take positive integers $m$, $n$ and $\epsilon$ satisfying:

$$m - 3\,n + 2\,\epsilon + 1 \geq 0\,, \qquad \epsilon < 2\,n\,, \qquad m + 3\,\epsilon < 8\,n\;. \tag{3.1}$$

Let $B$ be a hyperelliptic curve of genus $b := m - 3n + 2\epsilon + 1$, and let $\Xi$ be a $g_2^1$ on $B$. [Since $B$ can be expressed as a branched double covering of $\mathbb{P}^1$ when $b \le 1$, we regard $B$ as a hyperelliptic curve by abuse of terminology]. Choose a point $P$ with $2P \in |\Xi|$ and let $\Xi'$ be $[\epsilon/2]\,\Xi + (\epsilon - 2\,[\epsilon/2])P$, where $[x]$ means the integer part of a real number $x$. Then $2\,\Xi' = \epsilon\,\Xi$. We take a general section $\tau \in H^0(B, \Xi')$ and assume that the divisor $(\tau)$ consists of $\epsilon$ distinct points.

Put

$$L_0 = (m + 2\,n + \epsilon)\,\Xi\,, \qquad L_1 = (m - n + \epsilon)\,\Xi + \Xi'\,, \qquad L_2 = (m - 2\,n + 2\,\epsilon)\,\Xi$$

and consider the $\mathbb{P}^2$-bundle $\pi\colon \mathbb{P}_B(L_0 \oplus L_1 \oplus L_2) \to B$. Let $H$ be a tautological divisor with $\pi_*\mathcal{O}(H) = L_0 \oplus L_1 \oplus L_2$. Then we can find sections $X_i$ of $[H - \pi^* L_i]$, $0 \le i \le 2$, such that $(X_0, X_1, X_2)$ forms a system of homogeneous coordinates on any fibre of $\pi$, where $[D]$ means the line bundle associated to a divisor $D$. Let $w$ be the fibre coordinate of the line bundle $W = [2\,H - \pi^*(2\,L_1 + \Xi')]$ over the $\mathbb{P}^2$-bundle.

Let $S'$ be a surface defined by the following equation in the total space of $W$:

$$\tau\, w = \Phi_1\,, \qquad w^2 = X_0\,\Phi_2\,, \tag{3.2}$$

where $\Phi_1$ and $\Phi_2$ are general sections of $[2H - 2\,\pi^* L_1]$ and $[3H - 3\,\pi^* L_2]$, respectively. Note that we can write $\Phi_i$'s as

$$\Phi_1 \;=\; X_0(\phi_0 X_0 + \phi_1 X_1 + \phi_2 X_2) + a_1 X_1^2\,, \qquad \Phi_2 \;=\; a_2 X_2^2 + \cdots$$

with non-zero constants $a_1$, $a_2$ and, general sections $\phi_0$, $\phi_1$ and $\phi_2$ of $2L_0 - 2L_1$, $L_0 - L_1$ and $L_0 + L_2 - 2L_1$, respectively. Then it is easy to see that $S'$ is smooth except at points given by $\tau = w = X_0 = X_1 = 0$.

**Claim 3.1.** *These singular points are rational double points of type $A_1$.*

**Proof:** Let $t$ be a local parameter around a zero of $\tau$. The second equation in (3.2) shows that $X_0 \equiv w^2$ modulo unit functions locally. Hence if we put $\overline{x} = X_1/X_2$, $\overline{w} = w$ (modulo unit functions), then the first equation in (3.2) gives us a local analytic equation of the form

$$t\,\overline{w} \;=\; \overline{w}^2 + \overline{x}^2\;. \tag{3.3}$$

Therefore, it is a rational double point of type $A_1$. ∎

We let $S$ be the minimal resolution of $S'$. Then $S$ has $(-2)$-curves $E_i$, $1 \le i \le \epsilon$. It remains to show that $S$ is indeed an even canonical surface.

It is not so difficult to see that a canonical divisor of $S'$ is induced from $(X_0) + (m - 2\,n + \epsilon)\,\pi^*\,\Xi$. Since $\Xi$ is divisible by two from our choice of $\Xi$, it suffices to see that so is the pull-back $\widetilde{Z}$ of $(X_0)$. For this purpose, note that $S$ can be regarded as the minimal resolution of the surface defined in $\mathbb{P}_B(L_0 \oplus L_1 \oplus L_2)$ by

$$\Phi_1^2 \;=\; \tau^2 X_0\,\Phi_2\ ,$$

the equation obtained by eliminating $w$ from (3.2). This surface contains the section $\overline{B}$ defined by $X_0 = X_1 = 0$, and, near a generic point of $\overline{B}$, $X_0 = 0$ defines $4\,\overline{B}$. Therefore, we see that $\widetilde{Z}$ is of the form

$$\widetilde{Z} \;=\; 4\,G + (\text{divisor supported on } \textstyle\bigcup E_i)\ ,$$

where $G$ is the proper transform on $S$ of $\overline{B}$.

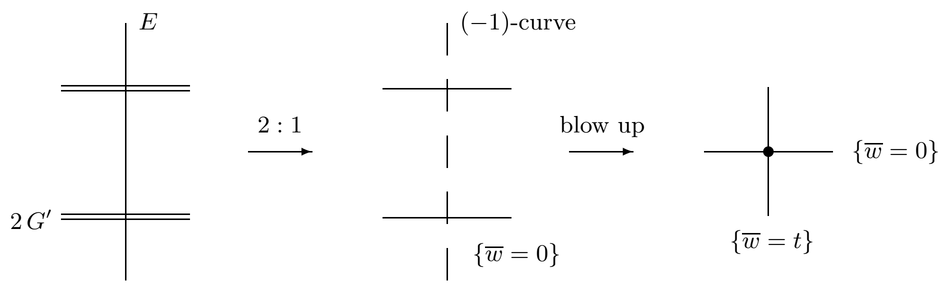


**Figure 1**

**Claim 3.2.** *The above "correction term" is* $2\sum_{i=1}^{\epsilon} E_i$.

**Proof:** The easiest way to see this is as follows: Regard (3.3) as a double covering of the $(t, \overline{w})$-plane with branch locus $\{t\,\overline{w} - \overline{w}^2 = 0\}$. Then, in its canonical (even) resolution, one easily see that the pull-back of $\{\overline{w} = 0\}$ is of the form $E + 2\,G'$, where $E$ is the exceptional $(-2)$-curve and $G'$ is the ramification divisor over $\{\overline{w} = 0\}$ (see Figure 1). Since the divisor $(\overline{w}^2)$ corresponds to $(X_0)$ locally, we conclude that $\widetilde{Z}$ is of the form $\widetilde{Z} = 4\,G + 2\sum_{i=1}^{\epsilon} E_i$. ∎

Hence $S$ is an even surface with a semi-canonical bundle $L = [2\,G + \sum_{i=1}^{\epsilon} E_i + (m + 2\,n + \epsilon)\,F]$, where $F$ is the fibre over $P$ of the induced natural fibration $f : S \to B$. It may be clear that $S$ is a canonical surface, since $f_*\,\omega_S \simeq L_0 \oplus L_1 \oplus L_2$. Furthermore, the numerical invariants of $S$ can be easily computed by the standard method as

$$\begin{cases} \chi(\mathcal{O}_S) \;=\; 2\,m + 10\,n + \epsilon \;, \\ q(S) \;=\; m - 3\,n + 2\,\epsilon + 1 \;, \\ K_S^2 \;=\; 3\,\chi(\mathcal{O}_S) + 10\,\Big(q(S) - 1\Big) + \epsilon \;. \end{cases}$$

**Remark.** The result may not be extended beyond the line $K^2 = (16/5)\,(\chi + 3\,(q-1))$, since we can construct, as above, an even canonical surface with ample $K$ on this line. □

## 4 – Appendix

The contents here may be well-known to the experts. But we include it mainly for two reasons. The first is to show singular fibres in our examples in §3 are natural ones. The second is to supplement important preprints [Ho3] and [R] both of which may never appear in a journal. Our consideration goes along a line in [R], the use of the relative canonical algebra originated from [CC].

Let $f\colon S \to B$ be a relatively minimal, non-hyperelliptic fibration of genus 3, $\omega_{S/B}$ the relative dualizing sheaf as usual. Then the multiplication map $\mathrm{Sym}^2(f_*\,\omega_{S/B}) \to f_*(\omega_{S/B}^{\otimes 2})$ is injective and generically surjective with cokernel a torsion sheaf $\mathcal{T}$ on $B$. For any $P \in \mathrm{Supp}(\mathcal{T})$, we put $\mathrm{Ind}(f^{-1}P) := \mathrm{length}(\mathcal{T})_P$ and call it the Castelnuovo–Horikawa index of $f^{-1}P$. Then we have

$$K_{S/B}^2 \;=\; 3\deg(f_*\,\omega_{S/B}) + \sum_{P \in \mathrm{Supp}(\mathcal{T})} \mathrm{Ind}(f^{-1}P) \;.$$

See [CC], [R] (or [K2]) for detail.

We fix a point $P \in \mathrm{Supp}(\mathcal{T})$ and put $F = f^{-1}P$. We shall describe the relative canonical model of $S$ around $F$, assuming that $|K_F|$ is free from base points. Let $\mathcal{E}$ be the kernel of the evaluation map $H^0(K_F) \otimes \mathcal{O}_F \to \mathcal{O}_F(K_F)$ so that we have a short exact sequence of locally free sheaves on $F$:

$$0 \;\to\; \mathcal{E} \;\to\; H^0(K_F) \otimes \mathcal{O}_F \;\to\; \mathcal{O}_F(K_F) \;\to\; 0 \;. \tag{A.1}$$

It follows that $\mathrm{rk}(\mathcal{E}) = 2$, $\bigwedge^2 \mathcal{E} \simeq \mathcal{O}_F(-K_F)$ and $H^0(\mathcal{E}) = 0$.

We claim that the multiplication map $\mu_m\colon H^0(K_F) \otimes H^0(m\,K_F) \to H^0((m+1)\,K_F)$ is surjective when $m \geq 2$. By tensoring (A.1) with $\mathcal{O}_F(m\,K_F)$, one sees that $\mu_m$ is surjective if $H^1(\mathcal{E}(m\,K_F)) = 0$. We have

$$H^1(\mathcal{E}(m\,K_F))^\vee \simeq H^0(\mathcal{E}^\vee((1-m)\,K_F)) \simeq H^0(\mathcal{E}((2-m)\,K_F)) \;,$$

since $\mathcal{E}^\vee \simeq \mathcal{E}(K_F)$. Hence the surjectivity of $\mu_2$ follows from the fact that $H^0(\mathcal{E})=0$. When $m\geq 3$, we use the inclusion $H^0(\mathcal{E}((2-m)K_F))\hookrightarrow H^0(K_F)\otimes H^0((2-m)K_F)$ coming from (A.1) tensored with $\mathcal{O}_F((2-m)K_F)$. Since we have $H^0((2-m)K_F)=0$ for $m \geq 3$, we get $H^0(\mathcal{E}((2-m)\,K_F)) = 0$ and the surjectivity of $\mu_m$.

Now, let $\{x_0, x_1, x_2\}$ be a basis for $H^0(K_F)$. By the free pencil trick, one knows that the rank of $\mu_1$ is at least 5. Since $P \in \mathrm{Supp}(\mathcal{T})$ and $h^0(2K_F) = 6$, we in fact have $\mathrm{rk}(\mu_1) = 5$. This implies that there are a quadric relation $A_2(x) = 0$ among the $x_i$ and a non-zero element $y \in H^0(2K_F)$ not contained in the image of $\mu_1$. Since $\mu_3$ is surjective, $y^2$ can be expressed as a linear combination of the other elements in $H^0(4K_F)$. Hence we have a non-trivial relation of the form $y^2 + B_2(x)\,y + B_4(x) = 0$, where the $B_j(x)$ are homogeneous form of degree $j$ in the $x_i$. It is easy to see that there are no further relations among $x_0$, $x_1$, $x_2$ and $y$. It follows that the canonical model $\mathrm{Proj}(\bigoplus H^0(m\,K_F))$ of $F$ is defined in the weighted projective space $\mathbb{P}(1,1,1,2)$ by

$$\text{(A.2)} \qquad A_2(x) = 0\,, \quad y^2 + B_2(x)\,y + B_4(x) = 0$$

similarly as non-singular hyperelliptic curves of genus 3. But one should note that the conic $\{A_2(x) = 0\}$ can be a pair of lines in the present case.

Since the canonical model of $F$ is a weighted complete intersection of the above type, we only need to add a parameter to the coefficients for describing the relative canonical model $X$ of $S$ near $F$. To be more precise, let $\Delta$ be a sufficiently small open disc around $P$ and $t$ a local parameter on $\Delta$, $P = \{t = 0\}$. Then there exists a positive integer $k$ such that $X$ is defined in $\Delta\times\mathbb{P}(1,1,1,2)$ by

$$\text{(A.3)} \qquad t^k\,u(t)\,y - A_2(t,x) \;=\; 0\,, \quad v(t)\,y^2 + B_2(t,x)\,y + B_4(t,x) \;=\; 0$$

where $u(t)$, $v(t)$ are unit functions in $t$, and $A_2(t,x)$, $B_j(t,x)$ are homogeneous forms in the $x_i$ with coefficients in $\mathbb{C}\{t\}$ satisfying $A_2(0,x) = A_2(x)$, $B_2(0,x) = B_2(x)$. (The presence of $y$ in the first equation is required, because we are considering a non-hyperelliptic fibration.) From (A.3), one sees immediately that the canonical image locally around $F$ is given

$$v(t)\,A_2(t,x)^2 + u(t)\,B_2(t,x)\,t^k\,A_2(t,x) + u(t)^2\,B_4(t,x)\,t^{2k} \;=\; 0$$

and can determine the Castelnuovo–Horikawa index as $\mathrm{Ind}(F) = k$. Such a singular fibre is called of type $(\mathrm{I}_k)$ in [Ho3].

In view of [R], it would be interesting to consider the Morsification of singular fibres of type $(\mathrm{I}_k)$. Modulo singular fibres arising from rational double points, we see that a singular fibre of type $(\mathrm{I}_k)$ is morsified to $k$ fibres of type $(\mathrm{I}_1)$, because we can obtain deformations of the singular fibre germ by replacing $t^k$ by $t(t-\alpha_1)\cdots(t-\alpha_{k-1})$ in the first equation of (A.3), where $(\alpha_1, ..., \alpha_{k-1})$ moves in a neighbourhood of the origin in $\mathbb{C}^{k-1}$. By perturbing the other coefficients if necessary, one may further assume that the support of each fibre of type $(\mathrm{I}_1)$ thus obtained is a non-singular hyperelliptic curve of genus 3, an atomic fibre in [R]. But our example in §3 implies that such a 'smoothing' is not necessarily available in the global situation.

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