PORTUGALIAE MATHEMATICA
Vol. 57 Fasc. 2 – 2000

# UNIFORM CONVERGENCE RESULTS FOR CERTAIN TWO-DIMENSIONAL CAUCHY PRINCIPAL VALUE INTEGRALS *

E. Santi

**Abstract:** A general uniform convergence theorem for numerical integration of certain two-dimensional Cauchy principal value integrals is proved. A special instance of this theorem is given as corollary.

## 1 – Introduction

In this paper we study the uniform convergence with respect to the parameters $\vartheta$ and $\zeta$, of numerical methods for evaluating the Cauchy principal value (CPV) integral

$$J(f;\vartheta,\zeta) := \fint_{-1}^{1} \fint_{-1}^{1} w_1(x)\, w_2(x)\, \frac{f(x,\widetilde{x})}{(x-\vartheta)\,(\widetilde{x}-\zeta)}\, dx\, d\widetilde{x} , \qquad \vartheta \in (-1,1) , \quad \zeta \in (-1,1) , \tag{1.1}$$

where $w_1,\ w_2$ are the Jacobi weight functions

$$w_1(x) := (1-x)^{\alpha_1}\,(1+x)^{\beta_1} , \qquad w_2(\widetilde{x}) := (1-\widetilde{x})^{\alpha_2}\,(1+\widetilde{x})^{\beta_2} , \qquad \alpha_i,\, \beta_i > -1, \quad i=1,2 . \tag{1.2}$$

In some recent papers [2, 3, 8], integrals of type (1.1), has been approximated by cubature rules based on quasi-uniform tensor product spline spaces.

---

*Received*: July 29, 1998; *Revised*: February 18, 1999.
*AMS Subject Classification*: 65D32, 41A15, 41A63.
*Keywords*: Cubatures; Cauchy principal value integrals; Splines.
* Work sponsored by M.U.R.S.T. and C.N.R. of Italy.

Supposing that $f \in H_p(\mu,\mu)$, $0 < \mu \le 1$, $0 \le p < m-1$, where $m$ denotes the order of the splines, in [2, 3] the authors proved a convergence theorem for cubature rules obtained by substituting the integrand function $f$ with quasi-interpolating or nodal spline operators; they gave besides, an upper bound for the remainder term. By subtracting out the singularities, in [8], it has been proved the uniform convergence of cubature rules based on quasi-interpolating spline-projectors for $(\vartheta,\zeta)$ belonging to any closed region contained in the integration domain.

Since in many applications it is necessary to have at one's disposal rules uniformly converging for all $(\vartheta,\zeta) \in (-1,1)\times(-1,1)$, the main object of this paper is to give general conditions for obtaining such rules. We shall do that in Section 2, proving the general convergence theorem 1.

In Section 3 we shall apply the obtained results to a particular approximation operator and we shall give the relative convergence results as corollary of the above general convergence theorem.

## 2 – A general uniform convergence theorem

In this section we shall state and prove a general uniform convergence theorem for two-dimensional CPV integrals (1.1), that can be written in the form

$$\begin{aligned} J(f;\,\vartheta,\zeta) \;=\; & ⨍_{-1}^{1} ⨍_{-1}^{1} w_1(x)\, w_2(\widetilde{x})\, \frac{f(x,\widetilde{x}) - f(\vartheta,\zeta)}{(x-\vartheta)\,(\widetilde{x}-\zeta)}\, dx\, d\widetilde{x} \\ & + f(\vartheta,\zeta)\, W_1(\vartheta)\, W_2(\zeta)\ . \end{aligned} \tag{2.1}$$

where

$$W_1(\vartheta) = ⨍_{-1}^{1} \frac{w_1(x)}{x-\vartheta}\, dx\,, \qquad W_2(\zeta) = ⨍_{-1}^{1} \frac{w_2(\widetilde{x})}{\widetilde{x}-\zeta}\, d\widetilde{x}\ . \tag{2.2}$$

We denote $\Omega = I\times\widetilde{I}$, with $I = [-1,1]$, $\widetilde{I} = [-1,1]$, and we define some partitions $X_N = \{x_i\}_{i=1}^{N}$, $\widetilde{X}_{\widetilde{N}} = \{\widetilde{x}_{\widetilde{i}}\}_{\widetilde{i}=1}^{\widetilde{N}}$ of $I$ and $\widetilde{I}$ respectively. Connected with the sequence of tensor product partitions $\{X_N\times\widetilde{X}_{\widetilde{N}}\colon\, N = N_1, N_2, ...;\ \widetilde{N} = \widetilde{N}_1, \widetilde{N}_2, ...\}$, we consider a sequence of real positive numbers $\{\Delta_{N\widetilde{N}}\}$ such that

$$\lim_{\substack{N\to\infty \\ \widetilde{N}\to\infty}} \Delta_{N\widetilde{N}} \;=\; 0\ . \tag{2.3}$$

Let $\Phi = \{\varphi_{i\widetilde{i}}(x,\widetilde{x})\}_{i=1,2,\cdots N}^{\widetilde{i}=1,2,\cdots \widetilde{N}}$ be a set of basis functions real continuous in $\Omega$ and let $F_{N\widetilde{N}}$ be a linear operator defined by $\Phi$ and approximating to $f$.

We assume

$$r_{N\widetilde{N}}(x,\widetilde{x}) = f(x,\widetilde{x}) - F_{N\widetilde{N}}(x,\widetilde{x}) \ . \tag{2.4}$$

Denoting by

$$S_{N\widetilde{N}}(x) = \int_{-1}^{1} w_2(\widetilde{x})\, \frac{r_{N\widetilde{N}}(x,\widetilde{x}) - r_{N\widetilde{N}}(x,\zeta)}{\widetilde{x}-\zeta}\, d\widetilde{x} \ , \tag{2.5}$$

$$T_{N\widetilde{N}}(\widetilde{x}) = \int_{-1}^{1} w_1(x)\, \frac{r_{N\widetilde{N}}(x,\widetilde{x}) - r_{N\widetilde{N}}(\vartheta,\widetilde{x})}{x-\vartheta}\, dx \ , \tag{2.6}$$

and considering the cubature rule

$$\begin{aligned} J_{N\widetilde{N}}(f;\vartheta,\zeta) = & ⨍_{-1}^{1} ⨍_{-1}^{1} w_1(x)\, w_2(\widetilde{x})\, \frac{F_{N\widetilde{N}}(x,\widetilde{x}) - F_{N\widetilde{N}}(\vartheta,\zeta)}{(x-\vartheta)\,(\widetilde{x}-\zeta)}\, dx\, d\widetilde{x} \\ & + f(\vartheta,\zeta)\, W_1(\vartheta)\, W_2(\zeta) \ , \end{aligned} \tag{2.7}$$

the error term $E_{N\widetilde{N}}(f;\vartheta,\zeta) = J(f;\vartheta,\zeta) - J_{N\widetilde{N}}(f;\vartheta,\zeta)$, given by

$$E_{N\widetilde{N}}(f;\vartheta,\zeta) = ⨍_{-1}^{1} ⨍_{-1}^{1} w_1(x)\, w_2(\widetilde{x})\, \frac{r_{N\widetilde{N}}(x,\widetilde{x}) - r_{N\widetilde{N}}(\vartheta,\zeta)}{(x-\vartheta)\,(\widetilde{x}-\zeta)}\, dx\, d\widetilde{x} \ , \tag{2.8}$$

can be written in the form [8],

$$\begin{aligned} E_{N\widetilde{N}}(f;\vartheta,\zeta) = & \int_{-1}^{1} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \\ & + W_1(\vartheta)\, S_{N\widetilde{N}}(\vartheta) + W_2(\zeta)\, T_{N\widetilde{N}}(\zeta) \ . \end{aligned} \tag{2.9}$$

We say that $f \in H(\mu,\mu)$, $0 < \mu \le 1$, if $f$ is a continuous function in $\Omega$ such that for all $(x_1,\widetilde{x}_1),(x_2,\widetilde{x}_2) \in \Omega$ there results $|f(x_1,\widetilde{x}_1) - f(x_2,\widetilde{x}_2)| \le C[|x_1 - x_2|^\mu + |\widetilde{x}_1 - \widetilde{x}_2|^\mu]$, with $C$ real constant.

For proving theorem 1 below, we need some lemmas [6].

**Lemma 1.** *Let $g \in H(\sigma,\sigma)$ in $\Omega$, $0 \le \sigma < 1$. The function*

$$\varphi(x,\widetilde{x}) = \frac{g(x,\widetilde{x}) - g(x,\widetilde{x}_0)}{|x - \widetilde{x}_0|^{\bar{\epsilon}}} \ , \qquad \forall\, \bar{\epsilon}, \quad 0 < \bar{\epsilon} < \sigma \ ,$$

*satisfies the $H(\sigma - \bar{\epsilon})$ condition for the variable $x$ uniformly with respect to $\widetilde{x}$, and $H(\sigma - \bar{\epsilon})$ condition for the variable $\widetilde{x}$ uniformly with respect to $x$.* ∎

If we define the functions

$$s(x) \;=\; \int_{-1}^{1} w_2(\widetilde{x})\, \frac{g(x,\widetilde{x})-g(x,\widetilde{x}_0)}{\widetilde{x}-\widetilde{x}_0}\; d\widetilde{x} \tag{2.10}$$

and

$$t(\widetilde{x}) \;=\; \int_{-1}^{1} w_1(x)\, \frac{g(x,\widetilde{x})-g(x_0,\widetilde{x})}{x-x_0}\; dx\ , \tag{2.11}$$

we have:

**Lemma 2.** *Suppose $g \in H(\sigma,\sigma)$ in $\Omega$, $0 \le \sigma < 1$. The functions $s$ and $t$, defined in (2.10), (2.11), satisfy a Hölder condition of order $\sigma-\bar\epsilon$, where $\bar\epsilon$ is an arbitrary real number such that $0<\bar\epsilon<\sigma$, i.e.*

$$|s(x)-s(x_0)| \;\le\; K_0\,|x-x_0|^{\sigma-\bar\epsilon}\ , \tag{2.12}$$

$$|t(\widetilde{x})-t(\widetilde{x}_0)| \;\le\; K_1\,|\widetilde{x}-\widetilde{x}_0|^{\sigma-\bar\epsilon}\ . \ \blacksquare \tag{2.13}$$

**Theorem 1.** *Let $f \in H(\mu,\mu)$ in $\Omega$, and assume that the approximation $F_{N\widetilde{N}}$ to $f$ is such that*

**(i)** $r_{N\widetilde{N}}(x,\pm1)=0,\ \forall\, x\in I,\ \ r_{N\widetilde{N}}(\pm1,\widetilde{x})=0,\ \forall\, \widetilde{x}\in \widetilde{I}$,

**(ii)** $\|r_{N\widetilde{N}}\|_\infty = O(\Delta^\nu_{N\widetilde{N}}),\ \ 0<\nu\le\mu$,

**(iii)** $r_{N\widetilde{N}} \in H(\sigma,\sigma),\ \ 0<\sigma\le\mu$.

*If*

$$\rho+\gamma-\bar\epsilon \;>\; 0 \tag{2.14}$$

*where $\rho := \min(\sigma,\nu)$, $\gamma := \min(\alpha_1,\alpha_2,\beta_1,\beta_2,0)$ and $\bar\epsilon$ is a positive real number as small as we like, then*

$$E_{N\widetilde{N}}(f;\vartheta,\zeta) \to 0 \quad \text{as}\ \ N\to\infty,\ \widetilde{N}\to\infty\ , \quad \text{uniformly for all}\ \ (\vartheta,\zeta)\in(-1,1)\times(-1,1)\ . \tag{2.15}$$

**Proof:** We can write:

$$\begin{aligned} E_{N\widetilde{N}}(f;\vartheta,\zeta) \;&=\; \int_{-1}^{1} w_1(x)\, \tfrac{S_{N\widetilde{N}}(x)-S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\; dx \;+\; W_1(\vartheta)\, S_{N\widetilde{N}}(\vartheta) \;+\; W_2(\zeta)\, T_{N\widetilde{N}}(\zeta) \\ &=\; T_1+T_2+T_3\ . \end{aligned}$$

Taking into account the condition (iii), and assuming in lemma 2 $g(x,\widetilde{x}) = r_{N\widetilde{N}}(x,\widetilde{x})$, $x_0=\vartheta$, $\widetilde{x}_0=\zeta$, the functions $S_{N\widetilde{N}}(x)$, $T_{N\widetilde{N}}(\widetilde{x})$ defined in (2.5), (2.6), satisfy (2.12) and (2.13) respectively. Furthermore, for $0 < \epsilon^* < \sigma$, by (ii) and lemma 5 in [6], there results:

$$|S_{N\widetilde{N}}(x)| \le \bar{C}\,\Delta_{N\widetilde{N}}^{\nu(1-\frac{\epsilon^*}{\sigma})} \quad \forall\,\zeta \in (-1,1)\ , \tag{2.16}$$

$$|T_{N\widetilde{N}}(\widetilde{x})| \le \bar{C}_1\,\Delta_{N\widetilde{N}}^{\nu(1-\frac{\epsilon^*}{\sigma})} \quad \forall\,\vartheta \in (-1,1)\ . \tag{2.17}$$

Consider first $T_2$. Because (i), in a neighbourhood of $x = 1$, we have

$$S_{N\widetilde{N}}(\vartheta) = \int_{-1}^{1} w_2(\widetilde{x})\,\frac{\Big[r_{N\widetilde{N}}(\vartheta,\widetilde{x}) - r_{N\widetilde{N}}(\vartheta,\zeta)\Big] - \Big[r_{N\widetilde{N}}(1,\widetilde{x}) - r_{N\widetilde{N}}(1,\zeta)\Big]}{\widetilde{x}-\zeta}\,d\widetilde{x}$$

then, by condition (iii) and lemma 1, there results

$$|S_{N\widetilde{N}}(\vartheta)| \le C\,|\vartheta-1|^{\sigma-\bar{\epsilon}} \int_{-1}^{1} \frac{w_2(\widetilde{x})}{|\widetilde{x}-\zeta|^{1-\bar{\epsilon}}}\,d\widetilde{x} = O(1-\vartheta)^{\sigma-\bar{\epsilon}}\ . \tag{2.18}$$

Besides, in a neighbourhood of $x = 1$, by §4.62 of [10],

$$W_1(\vartheta) = \begin{cases} O\Big((1-\vartheta)^{\alpha_1}\Big) + c & \text{if } \alpha_1 \text{ is not an integer,} \\ O\Big(|\log(1-\vartheta)|\Big) & \text{if } \alpha_1 \text{ is an integer .} \end{cases} \tag{2.19}$$

Hence, we can find $\delta > 0$ sufficiently small so that for all $\vartheta \in [1-\delta, 1]$ and $\forall\,\epsilon > 0$, $T_2 = O((1-\vartheta)^{\sigma-\bar{\epsilon}+\alpha_1}\,|\log(1-\vartheta)|) < \epsilon$ uniformly in $\vartheta$ if (2.14) holds.
Similarly, we can find $\bar{\delta} > 0$ such that for all $\vartheta \in [-1, -1+\bar{\delta}]$

$$T_2 = O\Big((1+\vartheta)^{\sigma-\bar{\epsilon}+\beta_1}\,|\log(1+\vartheta)|\Big) < \epsilon \quad \text{uniformly in } \vartheta\ .$$

Finally, since $W_1(\vartheta) = O(1)$ in $[-1+\bar{\delta}, 1-\delta]$ and $\|S_{N\widetilde{N}}\|_\infty = o(1)$ as $N \to \infty$, $\widetilde{N} \to \infty$, we conclude that

$$T_2 = o(1) \quad \text{as } N \to \infty,\ \widetilde{N} \to \infty\ , \quad \text{uniformly for all } (\vartheta,\zeta) \in (-1,1)\times(-1,1)\ .$$

In the same way we can prove that

$$T_3 = o(1) \quad \text{as } N \to \infty,\ \widetilde{N} \to \infty\ , \quad \text{uniformly for all } (\vartheta,\zeta) \in (-1,1)\times(-1,1)\ .$$

Consider now

$$
\begin{aligned}
|T_1| &= \left|\int_{-1}^{1} w_1(x)\,\frac{S_{N\widetilde{N}}(x)-S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\;dx\right| \\
&\le \left|\int\limits_U w_1(x)\,\frac{S_{N\widetilde{N}}(x)-S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\;dx\right| \\
&\quad + \left|\int\limits_{\substack{|x-\vartheta|\ge\Delta_{N\widetilde{N}}\\ x\notin U}} w_1(x)\,\frac{S_{N\widetilde{N}}(x)-S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\;dx\right| \\
&\quad + \left|\int\limits_{\substack{|x-\vartheta|\le\Delta_{N\widetilde{N}}\\ x\notin U}} w_1(x)\,\frac{S_{N\widetilde{N}}(x)-S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\;dx\right| \\
&= T_{11}+T_{12}+T_{13}
\end{aligned}
$$

where $U := [-1,-1+\widetilde{s}] \cup [1-\bar{s},1]$, for some $\widetilde{s}$, $\bar{s}$ to be determined below. There results

$$
\begin{aligned}
\left|\int_{-1}^{-1+\widetilde{s}} w_1(x)\,\tfrac{S_{N\widetilde{N}}(x)-S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\;dx\right| &\le K_0 \int_{-1}^{-1+\widetilde{s}} w_1(x)\,|x-\vartheta|^{\sigma-\bar{\epsilon}-1}\,dx \\
&= O\left(\int_{-1}^{-1+\widetilde{s}} (1+x)^{\gamma+\sigma+\bar{\epsilon}-1}\;dx\right) \;<\; \epsilon
\end{aligned}
$$

for $\widetilde{s}$ sufficiently small. Similarly we have

$$
\left|\int_{1-\bar{s}}^{1} w_1(x)\,\frac{S_{N\widetilde{N}}(x)-S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\;dx\right| < \epsilon
$$

for $\bar{s}$ sufficiently small, and then

$$
T_{11} < 2\,\epsilon\,, \qquad \forall\,\epsilon>0\;. \tag{2.20}
$$

Considering the term $T_{12}$, we have

$$(2.21) \qquad \begin{aligned} T_{12} &= \left| \int\limits_{\substack{|x-\vartheta|\geq \Delta_{N\widetilde{N}} \\ x\notin U}} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| \\ &= O\left( \Delta_{N\widetilde{N}}^{\nu(1-\frac{\epsilon^*}{\sigma})} \ln(\Delta_{N\widetilde{N}}) \right) = o(1) \qquad \text{as } N\to\infty,\ \widetilde{N}\to\infty\ . \end{aligned}$$

Finally, by (2.12), (2.13),

$$(2.22) \qquad \begin{aligned} T_{13} &= \left| \int\limits_{\substack{|x-\vartheta|\leq \Delta_{N\widetilde{N}} \\ x\notin U}} w_1(x)\, \frac{S_{N\widetilde{N}}(x) - S_{N\widetilde{N}}(\vartheta)}{x-\vartheta}\, dx \right| \\ &\leq C_2 \int\limits_{\substack{|x-\vartheta|\leq \Delta_{N\widetilde{N}} \\ x\notin U}} |x-\vartheta|^{\sigma-\bar{\epsilon}-1}\, dx = o(1) \end{aligned}$$

as $N\to\infty$, $\widetilde{N}\to\infty$ uniformly $\forall\,(\vartheta,\zeta)\in(-1,1)\times(-1,1)$.
The thesis follows considering that $T_1$, $T_2$, $T_3$ can be made arbitrarily small, uniformly in $(\vartheta,\zeta)\in(-1,1)\times(-1,1)$, as $N\to\infty$, $\widetilde{N}\to\infty$. ∎

We remark that the theorem 1 extends to the two-dimensional case the results of theorem 1 in [7].

## 3 – Particular example of theorem 1

In this section we derive uniform convergence results for the below defined approximation $\bar{W}_{N\widetilde{N}}$ to $f$, which we state as corollary.

Consider the sequence of two-dimensional nodal spline operators studied in [3]:

$$(3.1) \qquad \begin{gathered} W^*_{N\widetilde{N}}(f;x,\widetilde{x}) = \sum_{i=p_j}^{q_j} \sum_{\widetilde{i}=\widetilde{p}_{\widetilde{j}}}^{\widetilde{q}_{\widetilde{j}}} \omega_{m\widetilde{m}i\widetilde{i}}(x,\widetilde{x})\; f(\tau_j,\widetilde{\tau}_{\widetilde{j}})\ , \\ (x,\widetilde{x})\in[\tau_j,\tau_{j+1}]\times[\widetilde{\tau}_{\widetilde{j}},\widetilde{\tau}_{\widetilde{j}+1}]\,, \quad j=0,1,...N-1\,, \quad \widetilde{j}=0,1,...,\widetilde{N}-1\ . \end{gathered}$$

For constructing (3.1), we firstly need to consider the partitions

$$X_{mN} := \Big\{-1 \equiv x_0 < x_1 < \cdots < x_{(m-1)N} \equiv 1\Big\} ,$$

$$\widetilde{X}_{\widetilde{m}\widetilde{N}} := \Big\{-1 \equiv \widetilde{x}_0 < \widetilde{x}_1 < \cdots < \widetilde{x}_{(\widetilde{m}-1)\widetilde{N}} \equiv 1\Big\}$$

of $I$, $\widetilde{I}$ respectively. The nodes $\tau_i$, $\widetilde{\tau}_{\widetilde{i}}$ are defined by

$$\tau_i := x_{(m-1)i}\,, \qquad 0 \le i \le N, \quad \text{with } N \ge m-1 ,$$

$$\widetilde{\tau}_{\widetilde{i}} := \widetilde{x}_{(\widetilde{m}-1)\widetilde{i}}\,, \qquad 0 \le \widetilde{i} \le \widetilde{N}, \quad \text{with } \widetilde{N} \ge \widetilde{m}-1 ,$$

and the functions $\omega_{m\widetilde{m}i\widetilde{i}}$ are such that $\omega_{m\widetilde{m}i\widetilde{i}} = \omega_{mi}(x) \times \omega_{\widetilde{m}\widetilde{i}}(\widetilde{x})$ where

$$\omega_{mi}(x) := \begin{cases} \prod\limits_{\substack{k=0\\k\neq i}}^{m} \dfrac{x-\tau_k}{\tau_i-\tau_k} & x \in [-1, \tau_{i_1-1}],\ (i \le m-1) , \\[2ex] s_i(x) & x \in [\tau_{i_1-1}, \tau_{N-i_0+1}],\ (N \ge m) , \\[2ex] \prod\limits_{\substack{k=0\\k\neq N-i}}^{m} \dfrac{x-\tau_{N-k}}{\tau_i-\tau_{N-k}} & x \in [\tau_{N-i_0+1}, 1],\ (i \ge N-(m-1)) , \end{cases}$$

with $i_0 := [\frac{m}{2}] + 1$, $i_1 := m - [\frac{m}{2}]$. For the functions $s_i$ we have

$$s_i(x) := \sum_{r=0}^{m} \sum_{t=-i_0}^{-i_0+m-1} \alpha_{irt}\, B^m_{(m-1)(i+t)+r}(x)\,, \qquad i = 0, 1, ..., N ,$$

where $\{B_i^m(x)\}_{i=1-m}^{(m-1)N-1}$ are the normalized B-splines of order $m$.

We recall that

$$0 < B_i^m(x) \le 1 \ \text{ if } x \in (x_i, x_{i+m})\,, \qquad B_i^m(x) = 0 \ \text{ otherwise} ,$$

except that $B^m_{1-m}(-1) = 1$, $B_{(m-1)N-1}(1) = 1$.

We besides assume

$$p_j := \begin{cases} 0 & \text{if } j = 0, 1, ..., i_1-2 , \\ j - i_1 + 1 & \text{if } j = i_1-1, ..., N-i_0 , \\ N - m + 1 & \text{if } j = N-i_0+1, ..., N-1 , \end{cases}$$

$$q_j := \begin{cases} m - 1 & \text{if } j = 0, 1, ..., i_1-2 , \\ j + i_0 & \text{if } j = i_1-1, ..., N-i_0 , \\ N & \text{if } j = N-i_0+1, ..., N-1 . \end{cases}$$

Likewise we define $\widetilde{\omega}_{\widetilde{mi}}(\widetilde{x})$, $\widetilde{p}_{\widetilde{j}}$, $\widetilde{q}_{\widetilde{j}}$. Without loss of generality, we can assume $m = \widetilde{m}$, i.e. we use splines of the same order on both axes.

We remark that $W^*_{N\widetilde{N}}$ is a spline operator with the following properties:

(**a**) $W^*_{N\widetilde{N}}$ is local, in the sense that $W^*_{N\widetilde{N}}(f; x, \widetilde{x})$ depends only on the values of $f$ in a small neighborhood of $(x, \widetilde{x})$;

(**b**) $W^*_{N\widetilde{N}}$ satisfies the relations: $W^*_{N\widetilde{N}}(f; \tau_i, \widetilde{\tau}_{\widetilde{i}}) = f(\tau_i, \widetilde{\tau}_{\widetilde{i}})$;

(**c**) $W^*_{N\widetilde{N}}$ has the optimal order polynomial reproduction property, that means $W^*_{N\widetilde{N}} p = p$ for all $p \in \mathcal{P}^2_m$, where $\mathcal{P}^2_m$ is the set of bivariate polynomials of total order $m$.

We say that the collection of product partitions

$$\left\{X_{mN} \times \widetilde{X}_{\widetilde{m}\widetilde{N}}, \quad N = N_1, N_2, ..., \ \widetilde{N} = \widetilde{N}_1, \widetilde{N}_2, ...\right\}$$

of $\Omega = I \times \widetilde{I}$, is quasi-uniform (q.u.) if there exists a positive constant $A$ such that $\Delta_i/\delta_j \le A$ for $i$ and $j$ equal to $N$ or $\widetilde{N}$, where

$$\begin{aligned} \Delta_N &= \max_{0 \le i \le (m-1)N-1}(x_{i+1} - x_i)\,, \qquad & \Delta_{\widetilde{N}} &= \max_{0 \le \widetilde{i} \le (m-1)\widetilde{N}-1}(\widetilde{x}_{\widetilde{i}+1} - \widetilde{x}_{\widetilde{i}})\,, \\ \delta_N &= \min_{0 \le i \le (m-1)N-1}(x_{i+1} - x_i)\,, \qquad & \delta_{\widetilde{N}} &= \min_{0 \le \widetilde{i} \le (m-1)\widetilde{N}-1}(\widetilde{x}_{\widetilde{i}+1} - \widetilde{x}_{\widetilde{i}})\,, \end{aligned}$$

and we shall call a sequence of spline spaces quasi-uniform if they are based on a sequence of q.u. partitions.

We define $\Delta_{N\widetilde{N}} = \Delta_N + \Delta_{\widetilde{N}}$, and suppose that

$$\Delta_N \to 0 \ \text{ as } \ N \to \infty\,, \qquad \Delta_{\widetilde{N}} \to 0 \ \text{ as } \ \widetilde{N} \to \infty\,. \tag{3.2}$$

We shall use the results in [3] for deducing the following

**Proposition 1.** *Suppose $f \in C(\Omega)$, for any sequence of q.u. nodal spline spaces $\{W^*_{N\widetilde{N}}\}$, there results:*

$$\|f - W^*_{N\widetilde{N}}\|_\infty \le K\,\omega(f; \Delta_{N\widetilde{N}}; \Omega) \tag{3.3}$$

*where $K$ is a constant depending only on $m$ and $A$, and*

$$\omega(W^*_{N\widetilde{N}}; \Delta_{N\widetilde{N}}; \Omega) = O\Big(\omega(f; \Delta_{N\widetilde{N}}; \Omega)\Big) \tag{3.4}$$

*with*

$$\omega(\phi; \Delta; \Theta) = \max_{\substack{|h|, |\widetilde{h}| \le \Delta \\ (x,\widetilde{x}), (x+h, \widetilde{x}+\widetilde{h}) \in \Theta}} \left|\phi(x+h, \widetilde{x}+\widetilde{h}) - \phi(x, \widetilde{x})\right|\,. \ \blacksquare$$

Using the method introduced in [5], we modify the operator (3.1) as

$$
\begin{aligned}
\bar{W}_{N\widetilde{N}}(f;x,\widetilde{x}) \;=\; & W^*_{N\widetilde{N}}(f;x,\widetilde{x}) \\
& + \left[f(-1,\widetilde{x}) - W^*_{N\widetilde{N}}(f;-1,\widetilde{x})\right] B^m_{1-m}(x) \\
& + \left[f(1,\widetilde{x}) - W^*_{N\widetilde{N}}(f;1,\widetilde{x})\right] B^m_{(m-1)N-1}(x) \\
& + \left[f(x,-1) - W^*_{N\widetilde{N}}(f;x,-1)\right] \widetilde{B}^m_{1-m}(\widetilde{x}) \\
& + \left[f(x,1) - W^*_{N\widetilde{N}}(f;x,1)\right] \widetilde{B}^m_{(m-1)\widetilde{N}-1}(\widetilde{x}) \;,
\end{aligned} \tag{3.5}
$$

and we prove the following

**Lemma 3.** *Let $f \in H(\mu,\mu)$ in $\Omega$, and let $\{\bar{W}_{N\widetilde{N}}\}$ be a sequence of q.u. spline operators defined in (3.5). There results:*

(**1**) $r_{N\widetilde{N}}(\mp 1,\widetilde{x}) = 0,\ \forall\, \widetilde{x} \in \widetilde{I},\ \ r_{N\widetilde{N}}(x,\mp 1) = 0,\ \forall\, x \in I;$

(**2**) $\|r_{N\widetilde{N}}\|_\infty = O(\Delta^\mu_{N\widetilde{N}});$

(**3**) $r_{N\widetilde{N}} \in H(\mu,\mu)$ *in* $\Omega$

*with* $r_{N\widetilde{N}}(x,\widetilde{x}) = f(x,\widetilde{x}) - \bar{W}_{N\widetilde{N}}(x,\widetilde{x})$.

**Proof:** Property (1) derives by the definition (3.5) and by the interpolating property of $W^*_{N\widetilde{N}}$. From the definition (3.5) we have $\|r_{N\widetilde{N}}\|_\infty \le 3\,\|f - W^*_{N\widetilde{N}}\|$ and by (3.3) property (2) holds. Property (3) follows considering that for $(x_1,\widetilde{x}_1),(x_2,\widetilde{x}_2) \in \Omega$ we can write

$$
\begin{aligned}
& \left| r_{N\widetilde{N}}(x_1,\widetilde{x}_1) - r_{N\widetilde{N}}(x_2,\widetilde{x}_2) \right| \le \\
& \le \left| f_{N\widetilde{N}}(x_1,\widetilde{x}_1) - f_{N\widetilde{N}}(x_2,\widetilde{x}_2) \right| + \left| \bar{W}_{N\widetilde{N}}(x_1,\widetilde{x}_1) - \bar{W}_{N\widetilde{N}}(x_2,\widetilde{x}_2) \right| .
\end{aligned} \tag{3.6}
$$

Then, by the definition of $\bar{W}_{N\widetilde{N}}$, using (3.3) and (3.4), it is possible to prove that in each of the different cases, according to the positions of the points $(x_1,\widetilde{x}_1)$, $(x_2,\widetilde{x}_2)$ in $\Omega$, there results [9]:

$$
\left| \bar{W}_{N\widetilde{N}}(f;x_1,\widetilde{x}_1) - \bar{W}_{N\widetilde{N}}(f;x_2,\widetilde{x}_2) \right| \le \bar{C}\,\omega(f;\Delta_{N\widetilde{N}};\Omega) .
$$

Therefore the claim follows by (3.6) and the hypothesis on $f$. ∎

**Corollary 1.** *Let $f \in H(\mu,\mu)$ in $\Omega$ and let $\{F_{N\widetilde{N}}\}$ be a sequence of q.u. spline spaces $\{\bar{W}_{N\widetilde{N}}\}$ defined in (3.5). Assume that (3.2) holds, then*

$$
E_{N\widetilde{N}}(f;\vartheta,\zeta) \to 0 \ \text{ as } N,\widetilde{N} \to \infty \quad \textit{uniformly in } (\vartheta,\zeta) \in (-1,1)\times(-1,1) .
$$

**Proof:** Condition (i) of theorem 1 follows by (1) of lemma 3. By conditions (2) and (3) of the same lemma, condition (ii) (with $\Delta_{N\widetilde{N}} = \Delta_N + \widetilde{\Delta}_{\widetilde{N}}$, $\nu = \mu$) and condition (iii) (with $\sigma = \mu$) hold. ∎

*ACKNOWLEDGEMENTS* – The author wishes to thank the referee for his insightful comments.

## REFERENCES

[**1**] Dagnino, C. and Santi, E. – Numerical evaluation of Cauchy principal value integrals by means of nodal spline approximation, *Revue Anal. Numér. Theor. Approx.,* 27 (1998), 59–69.

[**2**] Dagnino, C.; Perotto, S. and Santi, E. – Product formulas based on spline approximation for the numerical evaluation of certain 2D CPV integrals, *Approx. Optim.* (D.D. Stancu, G. Coman, W. Breckner and P. Blaga, Eds.), vol. I (1996), 241–250.

[**3**] Dagnino, C.; Perotto, S. and Santi, E. – Convergence of rules based on nodal splines for the numerical evaluation of certain 2D Cauchy principal value integrals, *J. Comp. Appl. Math.,* 89 (1998), 225–235.

[**4**] Dagnino, C. and Rabinowitz, P. – Product integration of singular integrands using quasi-interpolatory splines, *Comp. Math. Appl.,* 33 (1997), 59–67.

[**5**] Demichelis, V. – Uniform convergence for Cauchy principal value integrals of modified quasi-interpolatory splines, *Intern. J. Computer Math.,* 53 (1994), 189–196.

[**6**] Monegato, G. – Convergence of product formulas for the numerical evaluation of certain two-dimensional Cauchy principal value integrals, *Numer. Math.,* 43 (1984), 161–173.

[**7**] Rabinowitz, P. – Uniform convergence results for Cauchy principal value integrals, *Math. Comp.,* 56 (1991), 731–740.

[**8**] Santi, E. and Cimoroni, M.G. – On the convergence of projector-splines for the numerical evaluation of certain two-dimensional CPV integrals, submitted for publication.

[**9**] Santi, E. – On the properties of the modified two-dimensional nodal spline operator, *Int. Report Dip. di Energetica, Università dell'Aquila*, 1998.

[**10**] Szegö, G. – *Orthogonal polynomials*, in "Amer. Math. Soc. Colloq. Publ.", vol. 23, Amer. Math. Soc., Providence, RI, 1975.

Elisabetta Santi,
Dipartimento di Energetica – Università dell'Aquila,
Localita' Monteluco, 67040 Roio Poggio (AQ) – ITALY
E-mail: esanti@dsiaq1.ing.univaq.it