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UNISERIAL GROUPS

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Abstract: An abelian group G is E-cyclic (uniserial) if G is a cyclic (uniserial) module over its endomorphism ring $E(G)$. In this note, abelian groups G which are uniserial R-modules, for R a subring of $E(G)$, are studied. Results of J. Hausen on E-uniserial groups are generalized to R-uniserial groups. It will be shown that if G is a finite rank reduced torsion free group, R is a commutative subring of $E(G)$, satisfying $RG = G$, and G is R-uniserial, then R is a domain whose lattice of ideals is totally ordered, and G is the additive group of a ring isomorphic to R.

Notation.

G an abelian group.

$E = E(G)$ the ring of endomorphisms of G.

R a subring of E.

1_G the identity map on G.

$I(G)$ the subgroup of E^+ generated by $\{\phi(g) \mid \phi \in \mathrm{Hom}(G, E^+)\}$. □

Definitions. G is *R-cyclic* if there exists $e \in G$ such that $G = R\,e$. If the lattice of R-submodules of G is totally ordered with respect to inclusion, then G is said to be *R-uniserial*. A ring is a *TOLI-ring* if its lattice of ideals is totally ordered with respect to inclusion. A torsion free group G is said to be *strongly R-irreducible* if G/H is a bounded group for every R-submodule $H \neq 0$ of G. If G is strongly E-irreducible, then G is called a *strongly irreducible* group. A torsion free group G is *p-local*, p a prime, if $p\,G \neq G$, but $q\,G = G$ for every prime $q \neq p$. □

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Notation and terminology will follow [1, 4].

If R is "small" then there may be very few R-cyclic or R-uniserial groups. For example, if R is the subring of E generated by 1_G, then the R-submodules of G are precisely the subgroups of G. It is readily seen that in this case, G is R-cyclic if and only if G is cyclic, and G is R-uniserial if and only if $G = Z(p^k)$, $0 \leq k \leq \infty$. On the other end of the scale, $Z \oplus G$ is E-cyclic for every abelian group G. There are also many E-uniserial groups.

The following two results concerning torsion free uniserial groups are due to Jutta Hausen, [5].

Proposition 1. *Let G be a reduced torsion free E-uniserial group. Then*

(**i**) *G is p-local for some prime p, and*

(**ii**) *G is strongly irreducible.*

If G has finite rank then

(**iii**) *G is E-cyclic.*

Proof: [5, Theorem 3.1]. ∎

Proposition 2. *Let G be a strongly indecomposable finite rank torsion free group. Then G is E-uniserial if and only if G is the additive group of a TOLI-ring.*

Proof: [5, Corollary 4.2]. ∎

The following observation is obvious.

Observation 3. Let $R \leq S$ be subrings of E. If G is R-cyclic (uniserial) then G is S-cyclic (uniserial). □

Essentially the same arguments used in [5] to prove Proposition 1 yield the following:

Theorem 4. *Let G be a reduced torsion free R-uniserial group. Then*

(**i**) *G is p-local for some prime p, and*

(**ii**) *G is strongly R-irreducible.*

If G has finite rank, and $RG = G$ then

(**iii**) *G is R-cyclic.*

Proof: G is E-cyclic by Observation 3, so (i) follows from Proposition 1. Let $H \neq 0$ be an R-submodule of G. Since G is reduced and p-local, there exists a positive integer n such that $H \not\subseteq p^n G$. Since both H and $p^n G$ are R-submodules of G, it follows that $p^n G \subset H$, and so G/H is bounded.

Suppose that G has finite rank, that $RG = G$, and let $e \in G$ such that $R\,e \not\subseteq p\,G$. Then $p\,G \subset R\,e$, and so $G/R\,e$ is a finite p-group. Choose e so that $|G/R\,e|$ is minimal. if there exists $x \in G$ such that $R\,x \not\subseteq R\,e$ then $R\,e \subset R\,x$, and so $|G/R\,x| < |G/R\,e|$, a contradiction. ∎

Corollary 5. *Let G be a reduced torsion free group, and let R be commutative. If G is R-uniserial then every $\alpha \in R$, $\alpha \neq 0$ is a monomorphism.*

Proof: Since $\ker \alpha$ is a pure subgroup of G, and $\ker \alpha$ is an R-submodule of G, Theorem 4 (ii) yields that $\ker \alpha = 0$. ∎

Theorem 6. *Let G be a reduced finite rank torsion free group, let R be commutative, and let $RG = G$. If G is R-uniserial then G is the additive group of a ring $S \simeq R$, and the ring R satisfies the following properties:*

- **(i)** *R is a maximal commutative subring of E,*
- **(ii)** *R is an integral domain,*
- **(iii)** *R is a TOLI-ring.*

Proof: By Theorem 4 (iii), there exists $e \in G$ such that $G = R\,e$. For $g \in G$, there exists, by Corollary 5, a unique $\alpha \in R$ such that $g = \alpha\,e$. For $g_1, g_2 \in G$ define $g_1 \cdot g_2 = \alpha_1\,\alpha_2\,e$, where $\alpha_1, \alpha_2 \in R$ satisfy $\alpha_1\,e = g_1$ and $\alpha_2\,e = g_2$. These products induce a ring structure S on G. The map $\phi : R \to S$ defined by $\phi(\alpha) = \alpha\,e$ for all $\alpha \in R$ is well defined, and is an isomorphism. Suppose that R' is a commutative subring of E satisfying $R' \supseteq R$. Then G is R'-uniserial, and R'-cyclic by Observation 3. Let $\alpha' \in R'$. Since $R'e = R\,e$, there exists $\alpha \in R$ such that $\alpha' e = \alpha\,e$, and so $\alpha' = \alpha$ by Corollary 5. Therefore R satisfies condition (i). Corollary 5 clearly implies that R has no zero-divisors. The subring of E generated by R and 1_G is commutative, so $1_G \in R$ by (i), i.e., (ii) is satisfied. Let $A \trianglelefteq S$, let $a \in A$, and let $\alpha \in R$. It is readily seen that $\alpha\,a = \alpha(e) \cdot a \in A$. Therefore every ideal in S is an R-submodule of G. Since G is R-uniserial it follows that S and R are TOLI-rings. ∎

Question. The following question was suggested by the referee:

If either or both of the conditions, **1**) R is commutative, 2) $RG = G$, are removed from the statement of Theorem 6, does some weakened version of the theorem remain valid? □

Fried, [3], studied subgroups of G which are ideals in every ring S with $S^+ = G$. He proved the following:

Proposition 7.

(i) $I(G) \trianglelefteq E$.

(ii) *A subgroup $H \leq G$ is an ideal in every ring S satisfying $S^+ = G$ if and only if H is an $I(G)$-submodule of G.*

Proof: [2, Lemma 5.1.1 and Theorem 5.1.2]. ∎

Clearly, if G is the additive group of a TOLI-ring, then G is $I(G)$-uniserial. This combined with Proposition 2 and Observation 3 yields:

Corollary 8. *Let G be a strongly indecomposable finite rank torsion free group. The following are equivalent:*

(i) *G is $I(G)$-uniserial.*

(ii) *G is E-uniserial.*

(iii) *G is the additive group of a TOLI-ring.* ∎

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