PORTUGALIAE MATHEMATICA
Vol. 58 Fasc. 2 – 2001
Nova Série

ON TOTALLY REAL SUBMANIFOLDS IN A NEARLY KÄHLER MANIFOLD *

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Abstract: Let M^m be a totally real submanifold of a nearly Kähler manifold $\bar{M}^{2m}$. We prove an important relationship between the covariant differential of the second fundamental form of M^m and that of the almost complex structure of $\bar{M}^{2m}$. And we show an application to the pinching problem on the square of the length of the second fundamental form of M^m.

0 – Introduction

Let $(\bar{M}^{2m}, g, J)$ be an almost Hermitian manifold with Riemannian metric g and almost complex structure J. $\bar{M}^{2m}$ is called a *nearly Kähler* manifold if the almost complex structure J satisfies $g(JX, JY) = g(X, Y)$ and $(\bar{\nabla}_X J)(X) = 0$, for any tangent vector fields X and Y on $\bar{M}^{2m}$. A Kähler manifold is a nearly Kähler manifold.

The canonical example of non-Kähler nearly Kähler manifold is the six dimensional standard unit sphere S^6. There are many other non-Kähler nearly Kähler manifolds such as $\mathbb{R}P^7 \times \mathbb{R}P^7$, $F_4/A_2 \times A_2$ and $U(4)/U(2) \times U(1) \times U(1)$ etc. A. Gray in [5] stated that a 3-symmetric space with a naturally reductive G-invariant pseudo-Riemannian metric is a nearly Kähler manifold. Many interesting theorems about the topology and the geometry of nearly Kähler manifolds have been proved by many authors (cf. e.g. [4], [6], [8], [10], [11] and [12] etc.).

Received: June 29, 2000.
1980 *Mathematics Subject Classification*: Primary 53C42; Secondary 32v40.
Keywords and Phrases: Nearly Kähler manifold; Totally real submanifold.
* Work done under partial support by SRF for ROCS, SEM.
* Work done under support by Fundação da Universidade de Lisboa through CMAF, FCT and Programa PRAXIS XXI.

The geometry of submanifolds in an almost Hermitian manifold is a very active topic in the theory of submanifolds. There have been many results on geometry of submanifolds in a Kähler manifold.

The theory of submanifolds in a nearly Kähler manifold say $\bar{M}^{2m}$ was studied by many authors. Let M^m be a totally real submanifold of $\bar{M}^{2m}$. Denote by A_ξ the shape operator on the tangent bundle TM of M in the direction of a unit normal vector field ξ in the normal bundle NM. When $\bar{M}$ is a complex space form, Chen–Ogiue [1] proved that

$$A_{JX}(Y) = A_{JY}(X) ,$$

for any two vector fields X and Y tangent to M. Ejiri [3] showed that the same property holds for a 3-dimensional totally real submanifold in S^6. In section 1, we shall prove that the same property also holds for a totally real submanifold in a nearly Kähler manifold.

We define a skew-symmetric tensor field G of type $(1,2)$ by

$$G(X,Y) = (\bar{\nabla}_X J)\, Y ,$$

where X and Y are vector fields on $\bar{M}^{2m}$ and $\bar{\nabla}$ is the Levi–Civita connection on $\bar{M}^{2m}$. Ejiri [3] proved that, for a 3-dimensional totally real submanifold M^3 in S^6, $G(X,Y)$ is orthogonal to M^3 for any vector fields X and Y tangent to M^3. By applying this fact, he proved that such M^3 is orientable and minimal. In section 2, we shall prove that, for any totally real submanifold M^m in $\bar{M}^{2m}$, $G(X,Y)$ is orthogonal to M^n for any vector fields X and Y tangent to M^m.

Denote by σ the second fundamental form of M^m. We proceed to show an important relationship between σ and G. Precisely, we shall prove

Proposition 1.4. *Let M^m be a Lagrangian submanifold of a nearly Kähler manifold $(\bar{M}^{2m}, g, J)$. Let σ be the second fundamental form of M^m. Define a skew-symmetric tensor field G of type $(1,2)$ by $G(X,Y) = (\bar{\nabla}_X J)Y$ for any vector fields X and Y tangent to M^m, where $\bar{\nabla}$ is the Levi–Civita connection with respect to g. Then*

$$\begin{aligned} g\Big(\nabla\sigma(X,Y,Z),\, JW\Big) &= g\Big(\nabla\sigma(X,Y,W),\, JZ\Big) + g\Big(\sigma(Y,Z),\, G(W,X)\Big) \\ &\quad - g\Big(\sigma(Y,W),\, G(Z,X)\Big) \end{aligned}$$

for any vector fields X, Y, Z and W tangent to M^m.

As an application, we shall prove a pinching theorem on the square of the length of the second fundamental form of a 3-dimensional totally real submanifold M^3 in S^6. Precisely, we shall prove the following

Proposition 2.1. *Let M^3 be a closed 3-dimensional totally real submanifold of S^6. Denote by S the square of the length of the second fundamental form of M^3. If $S < 5/2$, then M^3 is totally geodesic.*

Remark 0.1. Simons [9] and Chern–do Carmo–Kobayashi [2] proved that, for a closed n-dimensional minimal submanifold M^n in an $(n+p)$-dimensional unit sphere S^{n+p}, M^n is totally geodesic if $S < n/(2-1/p)$. Moreover $S = n/(2-1/p)$ when and only when $p=1$ or $n=p=2$. A.M. Li and J.M. Li in [7] improved this result. They proved that if $p \geq 2$ and $S < 2n/3$, then M^n is totally geodesic. Moreover, $S = 2n/3$ when and only when $n = p = 2$. Therefore we wonder whether or not this pinching constant could be improved when $n \geq 3$ and $p \geq 2$. Proposition 2.1 give an affirmative answer to this expectation because $5/2 > 2 = 3 \times (2/3)$. □

1 – Geometry of totally real submanifolds in a nearly Kähler manifold

Let $(\bar{M}^{2m}, g, J)$ be an almost Hermitian manifold with Riemannian metric g and almost complex structure J. Let $\{e_1, e_2, ..., e_{2m}\}$ be a local orthonormal frame field on the tangent bundle $T\bar{M}^{2m}$. Let $\{\omega_1, \omega_2, ..., \omega_{2m}\}$ be its coframe field and (ω_{ab}) be the Levi–Civita connection form associated with this coframe field. Then we have the Cartan structure equations:

$$\begin{cases} \bar{\nabla} e_a = \omega_{ab}\, e_b \ ; \\ d\omega_a = \omega_{ab} \wedge \omega_b \ ; \\ d\omega_{ab} = \omega_{ac} \wedge \omega_{cb} + \bar{\Omega}_{ab} \ , \end{cases} \qquad \begin{array}{l} \omega_{ab} + \omega_{ba} = 0 \ ; \\ \bar{\Omega}_{ab} + \bar{\Omega}_{ba} = 0 \ ; \end{array} \tag{1.1}$$

where (Ω_{ab}) is the curvature form. The almost complex structure J of $\bar{M}$ satisfies:

(**1**) $J\colon T_x\bar{M} \to T_x\bar{M}$ is linear;

(**2**) $J(JX) = -X$;

(**3**) $g(JX, JY) = g(X, Y)$;

for any X, Y in $T_x\bar{M}$. Under the above orthonormal frame field and associated coframe field, J can be expressed as $J = J_{ab}\, \omega_a\, e_b$. In this case, $J(X) = J_{ab}\, X_a\, e_b$ for any $X = X_a\, e_a \in T_x\bar{M}$. Moreover conditions (2) and (3) can be represented by

$$J_{ac}\, J_{bc} = \delta_{ab} \ , \tag{1.2}$$

$$J_{ac}\, J_{cb} = -\delta_{ab} \ , \tag{1.3}$$

where (δ_{ab}) is the Kronecker symbol. From (1.2) and (1.3) we have

$$J_{ab} + J_{ba} = 0 \ . \tag{1.4}$$

Put $\tilde{J} = \frac{1}{2} J_{ab}\, \omega_a \wedge \omega_b$. From (1.4) it follows that $\tilde{J}$ is a 2-form which is called the *fundamental form associated the almost complex structure* J.

The covariant differential of (J_{ab}), say $(J_{ab,c})$, is defined to be

$$\bar{\nabla} J_{ab} := J_{ab,c}\, \omega_c = dJ_{ab} + J_{cb}\, \omega_{ca} + J_{ac}\, \omega_{cb} \ . \tag{1.5}$$

It follows from (1.3) that

$$J_{ae,c}\, J_{eb} + J_{ae}\, J_{eb,c} = 0 \ . \tag{1.6}$$

Definition 1.1. $(\bar{M}^{2m}, g, J)$ is called a *nearly Kähler* (or *Tachibana*) *manifold* if the covariant differential $\bar{\nabla} J$ of J satisfies $(\bar{\nabla}_X J)(X) = 0$ for any tangent vector X. ▫

Remark 1.1. Equation $(\bar{\nabla}_X J)(X) = 0$ for any tangent vector X is equivalent to

$$J_{ab,c} + J_{ac,b} = 0 \ , \tag{1.7}$$

for all a, b and c. ▫

Let $(\bar{M}^{2m}, g, J)$ be a nearly Kähler manifold. Let M^m be a totally real submanifold, or a Lagrangian submanifold, of $\bar{M}^{2m}$. Then it follows that the image of the tangent space $T_x M^m$ under the mapping of the almost complex structure is the normal space $N_x M^m$, at every point $x \in M^n$. From now on, we agree on the following index ranges:

$$1 \le a, b, c, ... \le 2m \,, \quad 1 \le i, j, k, ... \le m, \quad \text{and} \quad i^* = m + i \ \text{ for } 1 \le i \le m \ .$$

Choose $\{e_1, e_2, ..., e_m;\ e_{1^*}, e_{2^*}, ..., e_{m^*}\}$ to be a local orthonormal frame field of the tangent bundle $T\bar{M}^{2m}$ such that e_i lies in TM^m and $e_{i^*} = Je_i$ lies in NM^m, for all $1 \le i \le m$. We call such a kind of frame an *adapted frame field* on $\bar{M}^{2m}$.

Let $\{\omega_1, \omega_2, ..., \omega_m;\ \omega_{1^*}, \omega_{2^*}, ..., \omega_{m^*}\}$ be the associated coframe field. Denote (ω_{ab}) to be the associated Levi–Civita connection form. Then (J_{ab}) can be expressed as

$$J_{ab} = \begin{cases} \delta_{ij}, & a = i,\ b = j^* \ ; \\ -\delta_{ij}, & a = i^*,\ b = j \ ; \\ 0, & \text{otherwise} \ . \end{cases} \tag{1.8}$$

From (1.5) and (1.8), we infer

$$-\bar{\nabla} J_{i^*j^*} = \bar{\nabla} J_{ij} = \omega_{i^*j} + \omega_{ij^*}\,, \qquad \bar{\nabla} J_{ij^*} = \bar{\nabla} J_{i^*j} = \omega_{i^*j^*} - \omega_{ij}\ . \tag{1.9}$$

Restricting (1.1) to M^n, we get $\omega_{k^*} = 0$ for all k. The structure equations of M^n are

$$\begin{cases} d\omega_i = \omega_{ij} \wedge \omega_j\ ; \\ d\omega_{ij} = \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}\ ; \\ d\omega_{i^*j^*} = \omega_{i^*k^*} \wedge \omega_{k^*j^*} + \Omega_{i^*j^*}\ . \end{cases} \qquad \begin{matrix} \Omega_{ij} = \omega_{ik^*} \wedge \omega_{k^*j} + \bar{\Omega}_{ij}\ ; \\ \Omega_{i^*j^*} = \omega_{i^*k} \wedge \omega_{kj^*} + \bar{\Omega}_{i^*j^*}\ ; \end{matrix} \tag{1.10}$$

From (1.1) we have the following relations:

$$\omega_{ij^*} \wedge \omega_i = 0\,, \qquad d\omega_{ij^*} = \omega_{ik} \wedge \omega_{kj^*} + \omega_{il^*} \wedge \omega_{l^*j^*} + \bar{\Omega}_{ij^*}\ .$$

Denote the curvature form $(\bar{\Omega}_{ab})$ by

$$\bar{\Omega}_{ab} = \frac{1}{2}\, \bar{R}_{abcd}\, \omega_c \wedge \omega_d\ . \tag{1.11}$$

By Cartan's lemma, we have

$$\omega_{ij^*} = h^{j^*}_{ik}\, \omega_k\,, \qquad h^{j^*}_{ik} = h^{j^*}_{ki}\,, \qquad h^{l^*}_{ijk} = h^{l^*}_{ikj} + \bar{R}_{l^*ijk}\ , \tag{1.12}$$

where $(h^{l^*}_{ijk})$ is the covariant differential of $(h^{l^*}_{ij})$ defined by

$$\nabla h^{l^*}_{ij} = h^{l^*}_{ijk}\, \omega_k = dh^{l^*}_{ij} + h^{l^*}_{kj}\, \omega_{ki} + h^{l^*}_{ik}\, \omega_{kj} + h^{s^*}_{ij}\, \omega_{s^*l^*}\ . \tag{1.13}$$

The second fundamental form σ and its covariant differential $\nabla\sigma$ are defined by

$$\sigma = h^{k^*}_{ij}\, \omega_i\, \omega_j\, e_{k^*}\,, \qquad \nabla\sigma = (\nabla h^{k^*}_{ij})\, \omega_i\, \omega_j\, e_{k^*}\ . \tag{1.14}$$

From the first equation of (1.9) and (1.12) we derive

$$J_{ij,k} = h^{j^*}_{ik} - h^{i^*}_{jk}\ . \tag{1.15}$$

It follows from (1.7) and (1.15) that

$$0 = J_{ij,k} + J_{ik,j} = h^{j^*}_{ik} + h^{k^*}_{ij} - 2\, h^{i^*}_{kj}\ , \tag{1.16}$$

$$0 = J_{jk,i} + J_{ji,k} = h^{k^*}_{ji} + h^{i^*}_{jk} - 2\, h^{j^*}_{ik}\ . \tag{1.17}$$

From (1.16) and (1.17), we derive

$$h^{j^*}_{ik} = h^{i^*}_{jk} \quad \text{or} \quad J_{ij,k} = h^{j^*}_{ik} - h^{i^*}_{jk} = 0\ . \tag{1.18}$$

It is known that the shape operator $A_{e_{j^*}}$ of M^m with respect to e_{j^*} can be expressed as $A_{e_{j^*}}(e_i) = h^{j^*}_{ik}\, e_k$. From (1.18) we get the following

Proposition 1.1. *Let M^m be a totally real submanifold of a nearly Kähler manifold $\bar{M}^{2m}$. Denote by A_ξ the shape operator of M^m with respect to a normal vector field ξ of M^m. Then*

$$A_{JX}(Y) = A_{JY}(X) \ , \tag{1.19}$$

for any two vector fields X and Y tangent to M^m.

Remark 1.2. When $\bar{M}^{2m}$ is chosen to be the 6-dimensional unit sphere S^6 and M^n is a 3-dimensional totally submanifold of S^6, Ejiri [3] has proved (1.19) in different way. □

As an direct application of Proposition 1.1, we have the following

Proposition 1.2. *Let M^m be a totally real submanifold of a nearly Kähler manifold $\bar{M}^{2m}$. Let $\{e_1, e_2, ..., e_m\}$ be a local orthonormal frame field on M^m. Denote $A_{i^*} = (h^{i^*}_{jk})$ to be the coefficient matrix of $A_{e_{i^*}}$ for every i. Then*

$$\mathrm{Tr}\Big(\sum_i A^2_{i^*}\Big)^2 = \sum_{i,j} (\mathrm{Tr}\, A_{i^*} A_{j^*})^2 \ . \tag{1.20}$$

Proof:

$$\mathrm{Tr}\Big(\sum_i A^2_{i^*}\Big)^2 = \sum_{i,j} h^{i^*}_{sk}\, h^{i^*}_{kl}\, h^{j^*}_{lr}\, h^{j^*}_{rs} = \sum_{l,s} h^{s^*}_{ik}\, h^{l^*}_{ki}\, h^{l^*}_{jr}\, h^{s^*}_{rj} = \sum_{l,s} (\mathrm{Tr}\, A_{l^*} A_{s^*})^2 \ . \ \blacksquare$$

We define a skew-symmetric tensor field G of type (1,2) by

$$G(X, Y) = (\bar{\nabla}_X J)\, Y \ , \tag{1.21}$$

for any vector fields X, Y on $\bar{M}^{2m}$. Then $G(e_a, e_b) = J_{ab,c}\, e_c$ for any a and b. Moreover it follows from (1.18) that

$$G(e_i, e_j) = J_{ij,k^*}\, e_{k^*} \in NM^m \ , \tag{1.22}$$

for any i and j, which implies the following

Proposition 1.3. *Let M^m be a totally real submanifold of a nearly Kähler manifold $\bar{M}^{2m}$. Define a skew-symmetric tensor field G of type $(1,2)$ by $G(X,Y) = (\bar{\nabla}_X J)\,Y$. Then $G(X,Y)$ is normal to M for any two vector fields X and Y tangent to M^m.*

Let us consider the behavior of $(h^{l^*}_{ijk})$. From (1.9), (1.13) and (1.20) we infer

$$\begin{aligned}
\nabla h^{l^*}_{ij} &= h^{l^*}_{ijk}\,\omega_k \\
&= dh^{l^*}_{ij} + h^{l^*}_{kj}\,\omega_{ki} + h^{l^*}_{ik}\,\omega_{kj} + h^{k^*}_{ij}\,\omega_{k^*l^*} \\
&= dh^{i^*}_{lj} + h^{k^*}_{lj}\,\omega_{ki} + h^{i^*}_{lk}\,\omega_{kj} + h^{i^*}_{kj}\,\omega_{k^*l^*} \\
&= dh^{i^*}_{lj} + h^{i^*}_{lk}\,\omega_{kj} + h^{k^*}_{lj}\,(\omega_{k^*i^*} - \bar{\nabla} J_{k^*i}) + h^{i^*}_{kj}(\bar{\nabla} J_{k^*l} + \omega_{kl}) \\
&= (dh^{i^*}_{lj} + h^{i^*}_{kj}\,\omega_{kl} + h^{i^*}_{lk}\,\omega_{kj} + h^{k^*}_{lj}\,\omega_{k^*i^*}) - h^{s^*}_{lj}\,J_{s^*i,k}\,\omega_k + h^{i^*}_{sj}\,J_{s^*l,k}\,\omega_k \\
&= \nabla h^{i^*}_{lj} + h^{i^*}_{sj}\,J_{s^*l,k}\,\omega_k - h^{s^*}_{lj}\,J_{s^*i,k}\,\omega_k\ .
\end{aligned}$$

After sorting the above equality, we get

$$h^{l^*}_{ijk} - h^{i^*}_{ljk} = h^{s^*}_{ij}\,J_{s^*l,k} - h^{s^*}_{lj}\,J_{s^*i,k}\ . \tag{1.23}$$

It follows from (1.14), (1.22) and (1.23) that

$$\begin{aligned}
g\Big(\nabla\sigma(e_k,e_j,e_i),\, e_{l^*}\Big) &= g\Big(\nabla\sigma(e_k,e_j,e_l),\, e_{i^*}\Big) + g\Big(\sigma(e_j,e_i),\, G(e_l,e_k)\Big) \\
&\quad - g\Big(\sigma(e_j,e_l),\, G(e_i,e_k)\Big)\ .
\end{aligned} \tag{1.24}$$

From (1.24) we have, for any $X = X_k\,e_k,\ Y = Y_j\,e_j,\ Z = Z_i\,e_i$ and $W = W_l\,e_l$,

$$\begin{aligned}
g\Big(\nabla\sigma(X,Y,Z),\, JW\Big) &= g\Big(\nabla\sigma(X,Y,W),\, JZ\Big) + g\Big(\sigma(Y,Z),\, G(W,X)\Big) \\
&\quad - g\Big(\sigma(Y,W),\, G(Z,X)\Big)\ .
\end{aligned}$$

Therefore we get the following

Proposition 1.4. *Let M^m be a Lagrangian submanifold of a nearly Kähler manifold $(\bar{M}^{2m}, g, J)$. Let σ be the second fundamental form of M^m. Define a skew-symmetric tensor field G of type $(1,2)$ by $G(X,Y) = (\bar{\nabla}_X J)\,Y$ for any vector fields X and Y tangent to M^m, where $\bar{\nabla}$ is the Levi–Civita connection with respect to g. Then*

$$\begin{aligned}
g\Big(\nabla\sigma(X,Y,Z),\, JW\Big) &= g\Big(\nabla\sigma(X,Y,W),\, JZ\Big) + g\Big(\sigma(Y,Z),\, G(W,X)\Big) \\
&\quad - g\Big(\sigma(Y,W),\, G(Z,X)\Big)\ ,
\end{aligned} \tag{1.25}$$

for any vector fields X, Y, Z and W tangent to M^m.

In the next section, we shall give an application to Proposition 1.4.

2 – A pinching problem on totally real submanifolds in S^6

In this section, we proceed to show an application of Propositions 1.1 and 1.4 to the theory of submanifold. Suppose that $\bar{M}^{2m}$ is a nearly Kähler manifold of constant curvature 1. Then the curvature tensor of $\bar{M}^{2m}$ can be represented by

$$\bar{R}_{abcd} = \delta_{ad}\,\delta_{bc} - \delta_{ac}\,\delta_{bd} \,. \tag{2.1}$$

In this case, we have from (1.12) that

$$h^{l^*}_{ijk} - h^{l^*}_{ikj} = \bar{R}_{l^*ijk} = 0 \,. \tag{2.2}$$

It is known that the Laplacian of h^α_{ij} is

$$\begin{aligned}\Delta h^{l^*}_{ij} = h^{l^*}_{ijkk} = {} & (\mathrm{Tr}\,A_{l^*})_{,ij} + (A_{r^*}A_{l^*}A_{r^*})_{ij} - (\mathrm{Tr}\,A_{l^*}A_{r^*})\,h^{r^*}_{ij} \\ & + (\mathrm{Tr}\,A_{r^*})\,(A_{l^*}A_{r^*})_{ij} - (A_{l^*}A_{r^*}A_{r^*})_{ij} + (A_{r^*}A_{l^*}A_{r^*})_{ij} \\ & - (A_{r^*}A_{r^*}A_{l^*})_{ij} + n\,h^{l^*}_{ij} - (\mathrm{Tr}\,A_{l^*})\,\delta_{ij} \,.\end{aligned} \tag{2.3}$$

Multiplying $h^{l^*}_{ij}$ on both-sides of (2.3) and taking sum with i, j and l, we derive

$$\begin{aligned}\sum_{i,j,l} h^{l^*}_{ij}\,\Delta h^{l^*}_{ij} = {} & h^{l^*}_{ij}(n\,H_{l^*})_{,ij} + (\mathrm{Tr}\,A_{i^*})\ \mathrm{Tr}(A_{j^*}A_{j^*}A_{i^*}) + n\,S - (\mathrm{Tr}\,A_{i^*})^2 \\ & - \sum_{i,j}\Big\{N(A_{i^*}A_{j^*} - A_{j^*}A_{i^*}) + (S_{i^*j^*})^2\Big\} \,,\end{aligned} \tag{2.4}$$

where we denote $S = \sum_{i,j,k}(h^{k^*}_{ij})^2$ the square of the length of the second fundamental form of M^n, $S_{i^*j^*} = \mathrm{Tr}(A_{i^*}A_{j^*})$ and $N(A_{i^*}A_{j^*} - A_{j^*}A_{i^*}) = -\,\mathrm{Tr}(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2$. Assume that M^m is minimal in $\bar{M}^{2m}$. Then $\mathrm{Tr}\,A_{i^*} = 0$ for all i. And (2.4) becomes

$$\sum_{i,j,k} h^{k^*}_{ij}\,\Delta h^{k^*}_{ij} = n\,S - \sum_{i,j}\Big\{N(A_{i^*}A_{j^*} - A_{j^*}A_{i^*}) + (S_{i^*j^*})^2\Big\} \,. \tag{2.5}$$

The Laplacian of S satisfies

$$\frac{1}{2}\,\Delta S = \sum_{i,j,k,l}(h^{l^*}_{ijk})^2 + \sum_{i,j,k} h^{k^*}_{ij}\,\Delta h^{k^*}_{ij} \,. \tag{2.6}$$

From now on, we assume $\bar{M}$ to be the 6-dimensional unit sphere with the standard induced metric from 7-dimensional Euclidean space $\mathbb{R}^7$. It is known that we can identify $\mathbb{R}^7$ with the set of all purely imaginary Cayley numbers.

Then S^6 can be equipped with an almost complex structure by the cross product of Cayley numbers. Moreover S^6 is nearly Kählerian but non-Kählerian. This enables us to study the pinching problem on the square of the length of the second fundamental form of 3-dimensional totally real submanifolds in S^6 in new viewpoint.

The main technique is to give a lower estimate to the right-hand side of (2.6). Up to now, the best estimation on the second part of the right-hand side of (2.5) was given by A.M. Li and J.M. Li [7].

Lemma 2.1 (Li's [7])**.** *Let $A_1, A_2, ..., A_p$ be symmetric $n\times n$-degree matrices, where $p \geq 2$. Denote $S_{\alpha\beta} = \mathrm{Tr}(A_\alpha^T A_\beta)$ and $S = S_{11} + S_{22} + \cdots + S_{pp}$. Then*

$$\sum_{\alpha,\beta}\Big\{N(A_\alpha A_\beta - A_\beta A_\alpha) + (S_{\alpha\beta})^2\Big\} \leq \frac{3}{2} S^2 , \tag{2.7}$$

where the equality holds if and only if one of the following conditions holds:

(**1**) $A_1 = A_2 = \cdots = A_p = 0$;

(**2**) *Only two of A_α's are different from zero. If we assume $A_1 \neq 0$, $A_2 \neq 0$ and $A_3 = \cdots = A_p = 0$, then $S_{11} = S_{22}$. Furthermore there exists an orthogonal $n\times n$-degree matrix U such that*

$$UA_1 U^T = \sqrt{\frac{S}{4}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \qquad UA_2 U^T = \sqrt{\frac{S}{4}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

In the rest of this section, we shall give a lower estimate to $|\nabla\sigma|^2 = \sum_{i,j,k,l}(h^{l^*}_{ijk})^2$.

It is easy to see that $JG(e_i, e_j)$ lies in T_xM^3 and is perpendicular to e_i and e_j for any $i \neq j$. So we can choose e_1, e_2 and e_3 such that (cf. Ejiri [3])

$$JG(e_1, e_2) = e_3 , \quad JG(e_2, e_3) = e_1 , \quad JG(e_3, e_1) = e_2 . \tag{2.8}$$

It follows from (2.8) that

$$J_{12,3^*} = J_{23,1^*} = J_{31,2^*} = -1; \quad \text{and} \quad J_{ij,k^*} = 0, \ \text{otherwise} . \tag{2.9}$$

Fixing e_1 and choosing e_2 and e_3 suitably, we can assume $h^{1^*}_{23} = 0$. Denote

$$h^{1^*}_{12} = x_1, \quad h^{1^*}_{13} = x_2, \quad h^{1^*}_{22} = x_3, \quad h^{1^*}_{33} = x_4, \quad h^{2^*}_{23} = x_5, \quad h^{2^*}_{33} = x_6 . \tag{2.10}$$

Then it follows from (1.18) that A_{i^*}'s can be expressed as

$$A_{1^*} = \begin{pmatrix} -x_3 - x_4 & x_1 & x_2 \\ x_1 & x_3 & 0 \\ x_2 & 0 & x_4 \end{pmatrix},$$

$$A_{2^*} = \begin{pmatrix} x_1 & x_3 & 0 \\ x_3 & -x_1 - x_6 & x_5 \\ 0 & x_5 & x_6 \end{pmatrix},$$

$$A_{3^*} = \begin{pmatrix} x_2 & 0 & x_4 \\ 0 & x_5 & x_6 \\ x_4 & x_6 & -x_2 - x_5 \end{pmatrix}.$$

And the square of the length of σ, say S, is expressed as

$$S \ = \ 4\,(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) \ + \ 2\,(x_3x_4 + x_1x_6 + x_2x_5)\ . \tag{2.12}$$

Let us consider the representations of $\{h_{ijk}^{l^*}\}$. Put

$$\begin{aligned} &h_{123}^{1^*} = y_1\,, \quad h_{122}^{1^*} = y_2\,, \quad h_{133}^{1^*} = y_3\,, \quad h_{112}^{1^*} = y_4\ , \\ &h_{233}^{1^*} = y_5\,, \quad h_{113}^{1^*} = y_6\,, \quad h_{223}^{1^*} = y_7\ . \end{aligned} \tag{2.13}$$

Since M^3 is minimal implies $h_{11k}^{l^*} + h_{22k}^{l^*} + h_{33k}^{l^*} = 0$ for any k, we have

$$h_{111}^{1^*} = -y_2 - y_3\,, \quad h_{222}^{1^*} = -y_4 - y_5\,, \quad h_{333}^{1^*} = -y_6 - y_7\ . \tag{2.14}$$

So we have

$$\begin{aligned} \sum_{i,j,k}(h_{ijk}^{1^*})^2 \ &= \ \sum_i (h_{iii}^{1^*})^2 \ + \ 3\sum_{i\neq j}(h_{iij}^{1^*})^2 \ + \ 6\,(h_{123}^{1^*})^2 \ = \\ &= \ 6\,y_1^2 \ + \ 4\,(y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2) \ + \ 2\,(y_2y_3 + y_4y_5 + y_6y_7)\ . \end{aligned} \tag{2.15}$$

On the other hand, it follows from (1.23) that

$$\begin{aligned} &h_{111}^{2^*} = \ y_4 + x_2\,, \quad h_{112}^{2^*} = \ y_2\,, \quad h_{113}^{2^*} = \ y_1 + x_4\ , \\ &h_{122}^{2^*} = -y_4 - y_5 + x_5\,, \quad h_{123}^{2^*} = \ y_7 + x_6\,, \quad h_{133}^{2^*} = \ y_5 - x_2 - x_5\ . \end{aligned} \tag{2.16}$$

Putting

$$h_{233}^{2^*} = y_8\,, \quad h_{223}^{2^*} = y_9\ , \tag{2.17}$$

then we have

$$h_{222}^{2^*} = -y_2 - y_8\,, \quad h_{333}^{2^*} = -x_4 - y_1 - y_9\ . \tag{2.18}$$

Therefore we infer from (2.16), (2.17) and (2.18),

$$\sum_{i,j,k}(h^{2^*}_{ijk})^2 \;=\; \sum_{i}(h^{2^*}_{iii})^2 \;+\; 3\sum_{i\neq j}(h^{2^*}_{iij})^2 \;+\; 6\,(h^{2^*}_{123})^2 \;= \tag{2.19}$$
$$\begin{aligned}
&= \;4\,y_1^2+4\,y_2^2+4\,y_4^2+6\,y_5^2+6\,y_7^2+4\,y_8^2+4\,y_9^2+2\,y_1y_9+2\,y_2y_8+6\,y_4y_5\\
&\quad+8\,x_4y_1+(2\,x_2-6\,x_5)\,y_4-(6\,x_2+12\,x_5)\,y_5+12\,x_6y_7+2\,x_4y_9\\
&\quad+4\,x_2^2+4\,x_4^2+6\,x_5^2+6\,x_6^2+6\,x_2x_5\;.
\end{aligned}$$

By the same procedure, we obtain

$$\begin{aligned}
&h^{3^*}_{111}= y_6-x_1\,, \quad h^{3^*}_{112}= y_1-x_3\,, \quad h^{3^*}_{113}= y_3\;,\\
&h^{3^*}_{122}= y_7+x_1+x_6\,, \quad h^{3^*}_{123}= y_5-x_5\,, \quad h^{3^*}_{133}= -x_6-y_6-y_7\;,\\
&h^{3^*}_{222}= y_9+x_3\,, \quad h^{3^*}_{223}= y_8\,, \quad h^{3^*}_{233}= -y_1-y_9\,, \quad h^{3^*}_{333}= -y_3-y_8\;.
\end{aligned} \tag{2.20}$$

Therefore we infer from (2.20) that

$$\sum_{i,j,k}(h^{3^*}_{ijk})^2 \;=\; \sum_{i}(h^{3^*}_{iii})^2 \;+\; 3\sum_{i\neq j}(h^{3^*}_{iij})^2 \;+\; 6\,(h^{3^*}_{123})^2 \;= \tag{2.21}$$
$$\begin{aligned}
&= \;6\,y_1^2+4\,y_3^2+6\,y_5^2+4\,y_6^2+6\,y_7^2+4\,y_8^2+4\,y_9^2+2\,y_3y_8+6\,y_1y_9+6\,y_6y_7\\
&\quad-2\,x_1y_6+2\,x_3y_9+6\,x_1y_7+6\,x_6y_6-6\,x_3y_1+12\,x_6y_7-12\,x_5y_5\\
&\quad+4\,x_1^2+4\,x_3^2+6\,x_5^2+6\,x_6^2+6\,x_1x_6\;.
\end{aligned}$$

Denote $F=|\nabla\sigma|^2$. Then

$$F \;=\; \sum_{i,j,k,l}(h^{l^*}_{ijk})^2 \;=\; \sum_{i,j,k}(h^{1^*}_{ijk})^2+\sum_{i,j,k}(h^{2^*}_{ijk})^2+\sum_{i,j,k}(h^{3^*}_{ijk})^2\;.$$

It follows from (2.15), (2.19) and (2.21) that

$$\begin{aligned}
F \;=\; &16\,y_1^2+8\,y_2^2+8\,y_3^2+8\,y_4^2+16\,y_5^2+8\,y_6^2+16\,y_7^2+8\,y_8^2+8\,y_9^2\\
&+2\,y_2y_3+2\,y_2y_8+2\,y_3y_8+8\,y_1y_9+8\,y_4y_5+8\,y_6y_7\\
&+(8\,x_4-6\,x_3)\,y_1+(2\,x_2-6\,x_5)\,y_4-(6\,x_2+24\,x_5)\,y_5\\
&+(6\,x_6-2\,x_1)\,y_6+(6\,x_1+24\,x_6)\,y_7+(2\,x_4+2\,x_3)\,y_9\\
&+4\,x_1^2+4\,x_2^2+4\,x_3^2+4\,x_4^2+12\,x_5^2+12\,x_6^2+6\,x_1x_6+6\,x_2x_5\;.
\end{aligned} \tag{2.22}$$

It is not difficult to check that the only critical point of F with respect to $(y_1, y_2, ..., y_9)$ is $P_0 = (y_1^0, y_2^0, ..., y_9^0)$, where

$$(2.23)\qquad \begin{aligned} &y_1^0 = \frac{1}{4}(x_3 - x_4)\,, \quad y_2^0 = 0\,, \quad y_3^0 = 0\,, \\ &y_4^0 = -\frac{1}{4}x_2\,, \quad y_5^0 = \frac{1}{4}(x_2 + 3\,x_5)\,, \quad y_6^0 = \frac{1}{4}x_1\,, \\ &y_7^0 = -\frac{1}{4}(x_1 + 3\,x_6)\,, \quad y_8^0 = 0\,, \quad y_9^0 = -\frac{1}{4}x_3\,. \end{aligned}$$

Furthermore we can see that P_0 is the minimum point of F. Substituting (2.23) into (2.22) and recalling (2.12), we obtain the minimum value of F:

$$(2.24)\qquad \begin{aligned} F_{min} &= 3\,x_1^2 + 3\,x_2^2 + 3\,x_3^2 + 3\,x_4^2 + 3\,x_5^2 + 3\,x_6^2 \\ &\quad + \frac{3}{2}x_5x_2 + \frac{3}{2}x_6x_1 + \frac{3}{2}x_3x_4 \\ &= \frac{3}{4}S\,. \end{aligned}$$

Therefore we get the following lower estimate to $|\nabla\sigma|^2$:

$$(2.25)\qquad |\nabla\sigma|^2 = \sum_{i,j,k,l} (h_{ijk}^{l^*})^2 \geq \frac{3}{4}S\,.$$

It follows from (2.5), (2.6), (2.7) and (2.25) that

$$(3.26)\qquad \Delta S \geq 3\,S\left(\frac{5}{2} - S\right).$$

Inequality (3.26) implies the following

Proposition 2.1. *Let M^3 be a closed 3-dimensional totally real submanifold of S^6. Denote by S the square of the length of the second fundamental form of M^3. If $S < 5/2$, then M^3 is totally geodesic.*

ACKNOWLEDGEMENTS – The author is very grateful to Professor A. Machado for his kind help and good advice. He would like to express his heartfelt gratitude to faculty and staffs of CMAF/UL for their hospitality during his stay and study in Portugal. And he wish to thank referee for his (her) earnest recommendation.

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