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MULTIPLICATION OPERATORS ON WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

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Abstract: Let V be a Nachbin family on the Hausdorff completely regular space X, E a locally convex space, $\mathcal{B}(E)$ the algebra of all continuous operators on E and $\psi : X \to \mathcal{B}(E)$ a map. We give necessary and sufficient conditions for the induced linear mapping $M_\psi \colon f \mapsto \psi(\ )(f(\ ))$ to be a multiplication operator on a subspace of the weighted space of E-valued continuous functions $CV(X,E)$. Next, we characterize the bounded multiplication operators and show that, at least whenever X is a $V_{\mathbb{R}}$-space, such an operator is precompact if and only if it is trivial.

1 – Introduction

Throughout this paper X will stand for a Hausdorff completely regular space and E for a Hausdorff locally convex space over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). We assume that the topology of E is given by a family $\mathbb{P}$ of seminorms. The space of all continuous E-valued functions on X will be denoted by $C(X,E)$, while $\mathcal{B}(E)$ denotes the algebra of all continuous linear operators on E. If $F \subset C(X,E)$ is a locally convex space (for a given topology), we will call a multiplication operator on F every *continuous* linear mapping M_ψ from F into *itself*, where $\psi : X \to \mathcal{B}(E)$ is a map and $M_\psi(f)(x) := \psi(x)(f(x))$ for every $f \in F$ and $x \in X$. Particularly interesting locally convex spaces contained in $C(X,E)$ are the so-called weighted spaces, namely $CV(X,E)$ and $CV_0(X,E)$, where V is a Nachbin family on X. These spaces were intensively investigated by many authors (e.g. [1], [2], [3], [5], [8], [10] and many others). The multiplication operators on the

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weighted spaces $CV(X,E)$ and $CV_0(X,E)$ were considered first by R.K. Singh and S.J. Manhas in [7] in the two particular settings: $\psi : X \to \mathbb{C}$ and $\psi : X \to E$, where $M_\psi(f)(x) := \psi(x)f(x)$, the multiplication being pointwise. In the latter case E is assumed to be a locally multiplicatively convex algebra. The authors gave a necessary and sufficient condition for M_ψ to be a multiplication operator on either $CV(X,E)$ and $CV_0(X,E)$. The same authors considered in [8] (and [9]) the general (i.e. operator-valued) case. They asserted (Theorem 2.1 of [8] corrected in [9]) that if X is a $k_{\mathbb{R}}$-space and ψ a continuous map from X into $\mathcal{B}(E)$, endowed with the topology of uniform convergence on the bounded subsets of E, then M_ψ is a multiplication operator on $CV(X,E)$ if and only if the following condition holds

$$\forall\, v \in V \ \ \forall\, P \in \mathbb{P}, \ \exists\, u \in V \ \ \exists\, Q \in \mathbb{P}: \ \ v(x)\, P(\psi_x(a)) \le u(x)\, Q(a), \ \ a \in E, \ \ x \in X \,.$$

In the present paper we first show by a counter-example (see Example 1. 1) that the assertion above of [8] is not true in the full generality. However, we show that, under the additional assumption of essentiality of $CV(X,E)$ the equivalence holds (see Theorem 5). At this point, notice that $CV(X,E)$ need not be essential even for relatively nice spaces X and Nachbin families V (see Example 1. 2).

Next, we characterize those maps ψ inducing multiplication operators on a subspace F of $CV(X,E)$, unifying in this way, the study for a large class of subspaces of $CV(X,E)$ containing, as special ones, $CV(X,E)$ itself and $CV_0(X,E)$. In particular, we get an extention of (the analogous of) the result of [8] to a large class of completely regular spaces X including the $k_{\mathbb{R}}$-spaces, the sequential spaces and the pseudocompact ones.

Finally, we characterize the maps ψ for which M_ψ is a bounded operator (in the sense of P. Uss [11]) on a subspace of $CV(X,E)$ and show that, at least when X is a $V_{\mathbb{R}}$-space without isolated points, M_ψ is compact only if it is trivial.

2 – Preliminaries

Henceforth, the space of all continuous and bounded (resp. continuous and vanishing at infinity, continuous with compact support) E-valued functions on X will be denoted by $C_b(X,E)$ (resp. $C_0(X,E)$, $\mathcal{K}(X,E)$). $B(X)$ and $B_0(X)$ denote respectively the spaces of all bounded $\mathbb{K}$-valued functions and all bounded $\mathbb{K}$-valued ones vanishing at infinity. A function $f : X \to \mathbb{K}$ is said to vanish at infinity if for every $\epsilon > 0$, there exists a compact set $K \subset X$ such that $|f(x)| < \epsilon$ whenever $x \in X \backslash K$. We will let V be a Nachbin family on X. This is a collection of non negative upper semicontinuous (u.s.c.) functions v on X such that for every $v_1, v_2 \in V$ and $\lambda > 0$, there exists $v \in V$ with $\max(\lambda v_1, \lambda v_2) \leq v$ and for every $x \in X$, $v(x) \neq 0$ for some $v \in V$. With V we associate the so-called weighted spaces :

$$CV(X,E) := \Big\{ f \in C(X,E) \colon \ vP(f) \in B(X), \ \forall P \in \mathbb{P}, \ \forall\, v \in V \Big\} ,$$

$$CV_0(X,E) := \Big\{ f \in CV(X,E) \colon \ vP(f) \in B_0(X), \ \forall P \in \mathbb{P}, \ \forall\, v \in V \Big\} ,$$

both equipped with the natural weighted topology $\tau_{V,\mathbb{P}}$ generated by the family $\mathbb{P}_V := \{P_v,\ P \in \mathbb{P},\ v \in V\}$ of seminorms; where

$$P_v(f) := \sup\Big\{ v(x)P(f(x)),\ x \in X \Big\}, \qquad f \in CV(X,E) .$$

For $F \subset CV(X,E)$, set $coz(F) := \{x \in X : \ f(x) \neq 0,$ for some $f \in F\}$ and $B_{P,v}(F) := B_{P,v} \cap F$ with $B_{P,v}$ the closed unit ball of the seminorm P_v in $CV(X,E)$. If $coz(F) = X$, then F is said to be essential. In the scalar case (i.e. $E = \mathbb{K}$), we will omit the symbols E and $\mathbb{P}$ from the notations and then write $CV(X)$ and $CV_0(X)$ instead of $CV(X,\mathbb{K})$ and $CV_0(X,\mathbb{K})$ respectively and τ_V instead of $\tau_{V,|\ |}$.

A subspace F of $CV(X,E)$ is said to be E-solid (resp. EV-solid) if for every $g \in C(X,E)$, the following condition is satisfied

$$(ES) \quad g \in F \iff \forall P \in \mathbb{P},\ \exists\, Q \in \mathbb{P},\ f \in F \colon\ P{\circ}g \leq Q{\circ}f \ \text{ pointwise on } coz(F)$$

(resp.
(EVS)
$g \in F \iff \forall P \in \mathbb{P},\ v \in V,\ \exists u \in V,\ Q \in \mathbb{P},\ f \in F \colon\ v.P{\circ}g \leq u.Q{\circ}f$ on $coz(F)$).

The classical solid spaces are nothing but the $\mathbb{K}$-solid ones. Moreover, it is easily seen that every EV-solid subset of $CV(X,E)$ is E-solid and that every E-solid F satisfies either $C_b(X)F \subset F$ and the condition,

$$(M) \qquad P(f(\ ))a \in F \quad \text{for all } \ P \in \mathbb{P}, \ \ a \in E \ \text{ and all } \ f \in F .$$

The spaces $CV(X,E)$, $CV_0(X,E)$ and $\mathcal{K}(X,E)$ are all EV-solid, while $CV(X,E) \cap C_b(X,E)$, $CV_0(X,E) \cap C_b(X,E)$, $CV(X,E) \cap C_0(X,E)$ and $CV_0(X,E) \cap C_0(X,E)$ are E-solid but need not be EV-solid. Actually, $C_0(\mathbb{R})$ and $C_b(\mathbb{R})$ are not $\mathbb{C}V$-solid for $V = \{\lambda e^{-\frac{1}{n}} :\ n \in \mathbb{N},\ \lambda > 0\}$.

The algebra of all continuous operators T from a locally convex space E into another F will be denoted by $\mathcal{B}(E,F)$. If $\mathcal{A}$ is a collection of subsets of E, then we will mean by $\mathcal{B}_{\mathcal{A}}(E,F)$ the subspace of $\mathcal{B}(E,F)$ consisting of those operators T which are bounded on the members of $\mathcal{A}$, together with the topology $\tau_{\mathcal{A}}$ of uniform convergence on the elements of $\mathcal{A}$. This topology is generated by the suprema of finitely many seminorms of the form $P_A(T) := \sup\{P(T(a)),\ a \in A\}$, A running over $\mathcal{A}$ and P over a family of seminorms defining the topology of F. If $\mathcal{A}$ consists of all the finite (resp. bounded) subsets of E, then we will write $\mathcal{B}_\beta(E)$ (resp. $\mathcal{B}_\sigma(E)$) for $\mathcal{B}_{\mathcal{A}}(E,E)$ and β (resp. σ) for $\tau_{\mathcal{A}}$.

3 – Multiplication operators on $CV(X,E)$

We start this section by giving an example in which $CV(X,E)$ is trivial and another where M_ψ is a multiplication operator on $CV(X,E)$ although the condition of [8] is not satisfied. This shows that the essentiality condition misses really in [8].

Example 1. **1.** Let X be the set of all rationals with the natural topology. This is of course a metrizable space. Consider on X the Nachbin family consisting of all non negative continuous functions. We claim that $CV(X,E)$ is reduced to $\{0\}$ for every E. Indeed, assume that, for a given E, $f(x) \neq 0$ for some $x \in X$ and some $f \in CV(X,E)$. Since E is Hausdorff, there exists some $P \in \mathbb{P}$ so that $P(f(x)) \neq 0$. With no loss of generality, we assume that $P(f(x)) = 1$. Then there exists $\epsilon > 0$ such that $P(f(t)) > \frac{1}{2}$ whenever $|t - x| < \epsilon$. For an irrational r with $|r - x| < \epsilon$, the function $t \mapsto \dfrac{1}{|t-r|}$ belongs to V and then must verify $\sup\{v(t)\,P(f(t)),\ t \in X\} < +\infty$. But this is clearly not true.

2. Set $X := [0,1] \cup Q_{[1,2]}$, where $Q_{[1,2]}$ denotes the set of all the rationals contained in $[1,2]$. Consider on X the Nachbin family consisting of all the maxima of finitely many continuous functions of the form $\lambda\, v_r(x) = \dfrac{\lambda}{|x-r|}$, r running over $[1,2]\backslash Q_{[1,2]}$ and λ over $\mathbb{R}^+\backslash\{0\}$. If $E = \mathbb{C}$, then $CV(X)$ is nothing but the Banach algebra $C[0,1]$ with the uniform norm. For a fixed irrational r_0 from $[1,2]$, take $\psi := v_{r_0}$. Then $M_\psi : f \mapsto \psi f$ is obviously a multiplication operator on $CV(X)$. However, the condition of [8] is not enjoyed by ψ since $\dfrac{1}{|x-r_0|^2}$ cannot be dominated by a weight from V. □

The following lemma shows that the corner stone in (the repaired) Theorem 2.1 of [8] is the continuity of $M_\psi(f)$ for every $f \in CV(X,E)$. It also shows what property of $CV(X,E)$ is involved either in the necessity or in the sufficiency.

Lemma 2. *Let $\psi : X \to \mathcal{B}(E)$ be a map and F a subspace of $CV(X,E)$. If F is a $C_b(X)$-module and satisfies the condition (M) and if M_ψ is a multiplication operator on F, then the following condition holds*

$$(1) \qquad \forall v \in V,\ \ \forall P \in \mathbb{P},\ \ \exists u \in V,\ \ \exists Q \in \mathbb{P}\colon \quad v(x)\, P(\psi_x(a)) \le u(x)\, Q(a)\,, \qquad a \in E,\ x \in coz(F)\ .$$

If in addition F is EV-solid and $M_\psi(F) \subset C(X,E)$, then the converse holds as well.

Proof: Assume that M_ψ is a multiplication operator on F. Then for every $v \in V$ and $P \in \mathbb{P}$, there exist $u \in V$ and $Q \in \mathbb{P}$ so that $P_v(M_\psi(f)) \le Q_u(f)$, f running over F. In particular, for every $x \in coz(F)$ and every $f \in F$,

$$(2) \qquad v(x)\, P(\psi_x(f(x))) \ \le\ \sup\Big\{u(t)\, Q(f(t)),\ t \in X\Big\}\ .$$

Choose $g \in F$ so that $g(x) \ne 0$. With no loss of generality, we may assume that $Q(g(x)) = 1$. Consider then $h_n \in C_b(X)$ such that $h_n(x) = 1$, $0 \le h_n \le 1$ and $h_n = 0$ outside of $U_n := \{t \in X\colon\ u(t) < u(x) + \dfrac{1}{n}$ and $Q(g(t)) < 1 + \dfrac{1}{n}\}$. Now, for every $a \in E$, put $f_n := h_n Q(g(\ ))a$. This is an element of F, for F is a $C_b(X)$-module and enjoies (M). Moreover, applying (2) to f_n, we get

$$v(x)\, P(\psi_x(a)) \ \le\ \Big(u(x) + \frac{1}{n}\Big)\Big(1 + \frac{1}{n}\Big) Q(a)$$

which gives (1) since n is arbitrary.

Conversely, assume that (1) is satisfied. Since $M_\psi(f)$ is continuous for every $f \in F$, we only have to show that $M_\psi(f)$ belongs to F and that M_ψ is continuous. Let $v \in V$ and $P \in \mathbb{P}$ be given. By (1), there exist $u \in V$ and $Q \in \mathbb{P}$ such that:

$$v(x)\, P(\psi_x(a)) \ \le\ u(x)\, Q(a)\,, \qquad \forall\, a \in E,\ \ x \in coz(F)\ .$$

In particular,

$$v(x)\, P(\psi_x(f(x)) \ \le\ u(x)\, Q(f(x))\,, \qquad \forall\, f \in F,\ \ x \in X\ .$$

Since F is EV-solid, $M_\psi(f)$ belongs to F and the passage to the supremum, first on the right hand side and then on the left hand one, yields $P_v(M_\psi(f)) \le Q_u(f)$ which shows the continuity of M_ψ. ∎

The first consequence of Lemma 2 is that if M_ψ is a multiplication operator on $CV(X,E)$, then so is it also on any EV-solid subspace F of $CV(X,E)$. However, the converse fails to hold in general even in the scalar case. Here is such an example.

Example 3. Set again $X := [0,1] \cup Q_{[1,2]}$ as above, $E := \mathbb{C}$ and $\psi = v_{\sqrt{2}}$. For the Nachbin family V consisting of all the positive constant functions on X, we have $CV_0(X) = C[0,1]$ with the uniform norm, while $CV(X)$ is the algebra of all continuous and bounded functions on X with the uniform norm. It is easy to see that M_ψ is a multiplication operator on $CV_0(X)$ but not on $CV(X)$. □

The following theorems yield conditions ensuring the continuity of $M_\psi(f)$ for every $f \in F$ so that we can apply Lemma 2.

Theorem 4. *Let F be an EV-solid subspace of $CV(X,E)$ and $\psi\colon X \to \mathcal{B}_\beta(E)$ be a continuous function. Suppose that, for every $x \in X$, there exists a neighbourhood Ω of x with $\psi(\Omega)$ equicontinuous on E. Then M_ψ is a multiplication operator on F if and only if* (1) *holds.*

Proof: By Lemma 2, we only have to show that $M_\psi(f)$ is continuous for every $f \in F$. Let $x_0 \in X$ and $f \in F$ be given. By assumption, there exists an open set Ω containing x_0 such that $\{\psi_x, x \in \Omega\}$ is equicontinuous on E. Then, for every $P \in \mathbb{P}$, there exist some $Q \in \mathbb{P}$ and some $M > 0$ so that

$$P(\psi_x(a)) \leq M\, Q(a)\,, \quad \forall\, x \in \Omega, \ \ \forall\, a \in E\ .$$

But f and ψ are continuous at x_0. Then, for arbitrary $\epsilon > 0$, there exists a neighbourhood Ω' of x_0 so that $Q(f(x) - f(x_0)) \leq \epsilon/(2M)$ and $P_{\{f(x_0)\}}(\psi_x - \psi_{x_0}) \leq \epsilon/2$ for every $x \in \Omega'$. Hence, for $x \in \Omega \cap \Omega'$, we have

$$\begin{aligned} P\Big(M_\psi(f)(x) - M_\psi(f)(x_0)\Big) &\leq P\Big(\psi_x(f(x) - f(x_0))\Big) + P\Big((\psi_x - \psi_{x_0})(f(x_0))\Big) \\ &\leq M\, Q\Big(f(x) - f(x_0)\Big) + P_{\{f(x_0)\}}(\psi_x - \psi_{x_0}) \\ &\leq M\, \epsilon/2\, M + \epsilon/2 \;=\; \epsilon\ . \end{aligned}$$

This shows the continuity of $M_\psi(f)$ at x_0. Since the latter is arbitrary in X, $M_\psi(f)$ is continuous on X. ■

Now, we provide an extension of the result of [8] to a wider class of completely regular spaces. To this aim, let γ be a property a net $(x_i)_{i\in I}$ may satisfy or not. We will call a γ-net any net enjoying the property γ. A function $f\colon X \to Y$ from X into a topological space Y will be said to be γ-continuous if, for every $x \in X$ and every γ-net $(x_i)_{i\in I}$ of X converging in X to x, $(f(x_i))_{i\in I}$ converges to $f(x)$. The space X is then called a $\gamma_{\mathbb{R}}$-space if every γ-continuous function from X into the real line (or equivalently into any completely regular space) is continuous on X. Here are some examples of such a property γ. Let us say that $(x_i)_{i\in I}$ is a s-, k-, c- or b-net if respectively $I = \mathbb{N}$, $\{x_i,\, i \in I\}$ is contained in a compact set, $\{x_i,\, i \in I\}$ is countable or $\{x_i,\, i \in I\}$ is bounding (i.e. every continuous scalar function on X is bounded on $\{x_i,\, i \in I\}$). In this way, the $k_{\mathbb{R}}$-spaces, in the present sense, are nothing but the classical ones, every sequential space is a $s_{\mathbb{R}}$-space and every pseudo-compact space is a $b_{\mathbb{R}}$-space. Moreover, every $s_{\mathbb{R}}$-space is a $k_{\mathbb{R}}$-space and every $k_{\mathbb{R}}$-space is a $b_{\mathbb{R}}$-space. Finally, if V is a Nachbin family on X, we will call a V-net any one $(x_i)_{i\in I}$ contained in $N_{v,1} := \{x \in X : v(x) \geq 1\}$ for some $v \in V$. In this way, we get the classical $V_{\mathbb{R}}$-spaces introduced in [1]. Now, if $\mathcal{A}$ consists of the γ-nets $(x_i)_{i\in I}$ converging in E, then we denote $\mathcal{B}_{\mathcal{A}}(E)$ by $\mathcal{B}_\gamma(E)$ and $\tau_{\mathcal{A}}$ by τ_γ. It is then clear that β is coarser than τ_s whenever the constant nets are γ-nets and that σ is finer than τ_b. Finally, one has $\tau_s \leq \tau_k \leq \tau_b$.

In the following, we will assume that the property γ is preserved by continuous functions. This is the case for $\gamma \in \{s, c, k, b\}$.

Theorem 5. *Let F be an EV-solid subspace of $CV(X,E)$, X a $\gamma_{\mathbb{R}}$-space for some property γ and $\psi : X \to \mathcal{B}_\gamma(E)$ a continuous map. Then M_ψ is a multiplication operator on F if and only if* (1) *holds.*

Proof: Here again, we have to show the continuity of $M_\psi(f)$ for every $f \in F$. Since X is a $\gamma_{\mathbb{R}}$-space, it suffices to show that $M_\psi(f)$ is γ-continuous. Let then $f \in F$ and $x_0 \in X$ be given. If $(x_i)_{i\in I}$ is a γ-net in X converging to x_0, then also $(f(x_i))_{i\in I}$ is a γ-net converging to $f(x_0)$. Since ψ is continuous, for every $P \in \mathbb{P}$ and $\epsilon > 0$, there exists a neighbourhood Ω of x_0 so that:

$$P_{\{f(x_i), i\in I\}}(\psi_y - \psi_{x_0}) < \frac{\epsilon}{2}\,, \qquad y \in \Omega\ . \tag{3}$$

But there exists $i_0 \in I$ with $x_i \in \Omega$ whenever $i_0 \leq i$. Hence (3) gives

$$\sup\Big\{P\Big((\psi_{x_i} - \psi_{x_0})(f(x_i))\Big),\ i \geq i_0\Big\} < \epsilon/2\ .$$

Moreover, since ψ_{x_0} is continuous, there are $Q \in \mathbb{P}$ and $M > 0$ such that:

$$P(\psi_{x_0}(a)) \leq M\, Q(a)\,, \qquad \forall\, a \in E\ .$$

Now, the convergence of $(f(x_i))_i$ to $f(x_0)$ yields some $i_1 \in I$ so that, whenever $i_1 \leq i$, $Q(f(x_i) - f(x_0)) \leq \epsilon/(2M)$. Hence, for i larger than both i_0 and i_1, we have

$$\begin{aligned} P\Big(M_\psi(f)(x_i) - M_\psi(f)(x_0)\Big) &\leq P\Big((\psi_{x_i} - \psi_{x_0})(f(x_i))\Big) + P\Big(\psi_{x_0}(f(x_i) - f(x_0))\Big) \\ &\leq P\Big((\psi_{x_i} - \psi_{x_0})(f(x_i))\Big) + M\, Q\Big(f(x_i) - f(x_0)\Big) \\ &\leq 2\,\frac{\epsilon}{2} \;=\; \epsilon\ . \end{aligned}$$

Whence the γ-continuity of $M_\psi(f)$ at x_0 and then on the whole of X since x_0 is arbitrary. ∎

As a consequence we get the following corollary including in particular Theorem 2.1 of [8].

Corollary 6. *Let F be a EV-solid subspace of $CV(X,E)$ and $\psi\colon X \to \mathcal{B}_\sigma(E)$ a continuous map. If X is a $b_{\mathbb{R}}$-space (in particular a $k_{\mathbb{R}}$-space, a sequential space or a pseudo-compact one), then M_ψ is a multiplication operator if and only if* (1) *holds.* ∎

In the theorems 4 and 5 we assume that ψ is continuous and obtain that $M_\psi(F) \subset C(X,E)$. We bring this section to an end by a kind of converse.

Proposition 7. *Let F be a subspace of $C(X,E)$ enjoying (M) and $\psi\colon X \to \mathcal{B}(E)$ a map. If $M_\psi(F) \subset C(X,E)$, then ψ is necessarily continuous on $coz(F)$ when $\mathcal{B}(E)$ is equipped with the topology β.*

Proof: Let $x \in coz(F)$, $Q \in \mathbb{P}$ and $f \in F$ be such that $Q(f(x)) = 1$. Set

$$\Omega \;:=\; \left\{ t \in X\colon\; |1 - Q(f(t))| < \frac{1}{2} \right\} .$$

This is an open set containing x and contained in $coz(F)$. For every $P \in \mathbb{P}$, $a \in E$ and $\epsilon > 0$, since $x \mapsto \psi_x(f(x))$ is continuous, there exists an open neighbourhood Ω' of x so that

$$P\Big(\psi_t\Big(Q(f(t))a\Big) - \psi_x\Big(Q(f(x))a\Big)\Big) \;<\; \frac{\epsilon}{4}\,, \qquad t \in \Omega' \;.$$

If $P(\psi_x(a)) = 0$, then $P\left(\psi_t(Q(f(t))a)\right) < \frac{\epsilon}{4}$ for every $t \in \Omega'$. If t is also in Ω, we get $P(\psi_t(a)) < \frac{\epsilon}{2}$ which shows that ψ is continuous at x. Now, if $P(\psi_x(a)) \neq 0$, then put

$$\Omega'' \;:=\; \left\{ t \in X\colon\; \left|\frac{1}{Q(f(t))} - 1\right| < \frac{\epsilon}{2\,P(\psi_x(a))} \right\} .$$

For $t \in \Omega \cap \Omega' \cap \Omega''$, we get:

$$\begin{aligned}
P_{\{a\}}(\psi_t - \psi_x) \;&=\; P\Big(\psi_t(a) - \psi_x(a)\Big) \;\leq \\
&\leq\; P\left(\frac{\psi_t(Q(f(t))a)}{Q(f(t))} - \frac{\psi_x(Q(f(x))a)}{Q(f(t))}\right) + P\left(\frac{\psi_x(Q(f(x))a)}{Q(f(t))} - \psi_x(a)\right) \\
&\leq\; \frac{1}{Q(f(t))}\, P\Big(\psi_t\Big(Q(f(t))a\Big) - \psi_x\Big(Q(f(x))a\Big)\Big) + \left|\frac{1}{Q(f(t))} - 1\right| P(\psi_x(a)) \\
&\leq\; 2\,\frac{\epsilon}{4} + \frac{\epsilon}{2\,P(\psi_x(a))}\; P(\psi_x(a)) \;=\; \epsilon \;.
\end{aligned}$$

This shows that ψ is β-continuous on $coz(F)$. ∎

4 – Compact multiplication operators

In a large class of locally convex spaces of continuous functions the precompact sets are equicontinuous. This is the case, as shown by K.D. Bierstedt in [1], for $CV(X)$ whenever X is a $V_{\mathbb{R}}$-space. K.D. Bierstedt's result was extended in [6] to the space $CV_p(X,E) := \{f \in CV(X,E) : (vf)(X)$ is precompact in E for every $v \in V\}$. In order to extend this result to $CV(X,E)$, let the subscript c in $\mathcal{B}_c(CV(X,E),E)$ stand for the topology of uniform convergence on precompact subsets of $CV(X,E)$. While δ_x denotes the evaluation $f \mapsto f(x)$ at the point x, Δ will be the evaluation map $x \mapsto \delta_x$ defined from X into $\mathcal{B}(CV(X,E),E)$.

Lemma 8. *The evaluation map Δ is continuous from X into $\mathcal{B}_c(CV(X,E),E)$ if and only if every precompact subset of $CV(X,E)$ is equicontinuous.*

Proof: Necessity: Assume that H is precompact in $CV(X,E)$ and let us show that H is equicontinuous on X. Fix $x_0 \in X$, $P \in \mathbb{P}$ and $\epsilon > 0$. Since Δ is continuous at x_0, there exists some open set Ω containing x_0 such that $\Delta(\Omega) \subset \delta_{x_0} + \epsilon\, H_P^o$. Here,

$$H_P^o := \left\{T \in \mathcal{B}_c(CV(X,E),E)\colon\ P_H(T) := \sup_{h\in H} P(T(h)) \,\le\, 1\right\} .$$

Thus $\sup_{h\in H} P(h(x)-h(x_0)) \le \epsilon$ for every $x \in \Omega$. This shows the equicontinuity of H at x_0. Since x_0 is arbitrary, H is equicontinuous on X.

Sufficiency: Let $x_0 \in X$ and U a neighbourhood of δ_{x_0} in $\mathcal{B}_c(CV(X,E),E)$ be given. There exist some $\epsilon > 0$, $P \in \mathbb{P}$ and some precompact set $H \subset CV(X,E)$ so that $\delta_{x_0} + \epsilon\, H_P^o \subset U$. Since H is equicontinuous, there exists some open set Ω containing x_0 with $\sup_{h\in H} P(h(x) - h(x_0)) < \epsilon$ for every $x \in \Omega$. Hence $\delta_x - \delta_{x_0} \in \epsilon\, H_P^o$ for every $x \in \Omega$. This gives $\Delta(\Omega) \subset U$ and then Δ is continuous at x_0. As x_0 is arbitrary, Δ is continuous on X. ∎

Proposition 9. *If X is a $V_{\mathbb{R}}$-space, then every precompact subset of $CV(X,E)$ is equicontinuous.*

Proof: In view of Lemma 7 and our assumption on X, it suffices to show that Δ is continuous on each $N_{v,1} := \{x \in X :\ v(x) \ge 1\}$. Let then $v \in V$ and $x \in N_{v,1}$ be given. If U is a neighbourhood of δ_x in $\mathcal{B}_c(CV(X,E),E)$, then there exist $P \in \mathbb{P}$, a precompact set $H \subset CV(X,E)$ and $\epsilon > 0$ such that $\delta_x + \epsilon\, H_P^o \subset U$. But there exist $h_i \in H$, $i \in \{1,2,...,n\}$, so as $H \subset \bigcup_{i=1}^{n} (h_i + \frac{\epsilon}{3}\, B_{P,v})$. Consider a neighbourhood Ω of x enjoying, for every $i=1,2...,n$ and $t\in\Omega$, $P(h_i(t)-h_i(x))<\frac{\epsilon}{3}$. Now, if $t\in\Omega\cap N_{v,1}$ and $h\in H$, then $h=h_i+f$ for some $i\in\{1,2,...,n\}$ and some $f\in\frac{\epsilon}{3}B_{v,P}$. Hence

$$\begin{aligned} P\Big(\delta_t(h)-\delta_x(h)\Big) &= P\Big(h(t)-h(x)\Big) \\ &\le P\Big(h_i(t)-h_i(x)\Big)+P\Big(f(t)-f(x)\Big) \\ &\le P\Big(h_i(t)-h_i(x)\Big)+\frac{P_v(f)}{v(t)}+\frac{P_v(f)}{v(t_0)} \\ &\le 3\,\frac{\epsilon}{3} \;=\; \epsilon\;. \end{aligned}$$

Since h is arbitrary in H, $P_H(\Delta(t)-\Delta(x))<\epsilon$ and thus Δ is continuous on $N_{v,1}$. ∎

Next, we show that the precompact (and then the compact) multiplication operators are often trivial. For this purpose, we need a further result. Let us first point out that, if H is a subset of $CV(X)$ such that $C_b^+(X)H\subset H$, then it is easy to show that $\dfrac{1}{v(x)}=\sup\{|f(x)|,\ f\in B_v\cap H\}$ for every $x\in coz(H)$; B_v standing for the closed unit ball of $|\ |_v$ in $CV(X)$. As a consequence, if G is a subset of $CV(X,E)$ such that $C_b^+(X)G\subset G$, then for every $x\in coz(G)$, there exists $P\in\mathbb{P}$ with $\dfrac{1}{v(x)}=\sup\{P(f(x)),\ f\in B_{P,v}\cap G\}$. Here and in the following we put $\frac{1}{0}=+\infty$. If in addition G satifies (M), we get the following

Lemma 10. *Let G be a subset of $CV(X,E)$ such that $C_b^+(X)G\subset G$ and G satisfies (M). Then for every $P\in\mathbb{P}$, $v\in V$ and $x\in coz(G)$, the equality $\dfrac{1}{v(x)}=\sup\{P(f(x)),\ f\in B_{P,v}\cap G\}$ holds.*

Proof: Let $x\in coz(G)$, $v\in V$ and $P\in\mathbb{P}$ be given. Then there exist $f\in G$ and $Q\in\mathbb{P}$ with $Q(f(x))=1$. Consider $a\in E$ so that $P(a)=1$. If $v(x)=0$, set $U_n:=\{t\in X\colon v(t)<\frac{1}{n}$ and $1-\frac{1}{n}<Q(f(t))<1+\frac{1}{n}\}$ and consider $h_n\in C_b(X)$ enjoying $0\le h_n\le n$, $h_n(x)=n$ and $\operatorname{supp}h_n\subset U_n$. The function $g_n:=\dfrac{n}{n+1}\,h_n\,Q(f(\ ))a$ belongs to G and

$$\begin{aligned} P_v(g_n) &= \sup\Big\{v(t)\,\frac{n}{n+1}\,h_n(t)\,Q(f(t))\,P(a),\ t\in X\Big\} \\ &= \sup\Big\{v(t)\,\frac{n}{n+1}\,h_n(t)\,Q(f(t))\,P(a),\ t\in U_n\Big\} \\ &\le \frac{1}{n}\,\frac{n}{n+1}\,n\Big(1+\frac{1}{n}\Big) \;=\; 1\;. \end{aligned}$$

Furthermore, $\sup\{P(g_n(x)),\ n\in\mathbb{N}\} = +\infty = \frac{1}{v(x)}$. Now, assume that $v(x)\neq 0$ and consider for $n > \frac{1}{v(x)}$ the open set:

$$U_n := \left\{ t\in X\colon \frac{v(x)}{v(x)+\frac{1}{n}} < Q(f(t)) < \frac{v(x)}{v(x)-\frac{1}{n}} \ \text{ and }\ v(t) < v(x)+\frac{1}{n} \right\}.$$

Choose then $h_n\in C_b(X)$ with $0\le h_n\le \dfrac{1}{v(x)+\frac{1}{n}}$, $h_n(x) = \dfrac{1}{v(x)+\frac{1}{n}}$ and $\operatorname{supp} h_n\subset U_n$. Then $g_n := \dfrac{v(x)-\frac{1}{n}}{v(x)}\, h_n\, Q(f(\ ))a$ belongs to G and

$$\begin{aligned} P_v(g_n) &= \sup\left\{ v(t)\, \frac{v(x)-\frac{1}{n}}{v(x)}\, h_n(t)\, Q(f(t))\, P(a),\ t\in U_n \right\} \\ &\le \left(v(x)+\frac{1}{n}\right) \frac{v(x)-\frac{1}{n}}{v(x)}\, \frac{1}{v(x)+\frac{1}{n}}\, \frac{v(x)}{v(x)-\frac{1}{n}} \;=\; 1 . \end{aligned}$$

Finally,

$$\sup_n P(g_n(x)) = \sup_n \left(\frac{v(x)-\frac{1}{n}}{v(x)}\right) \left(\frac{1}{v(x)+\frac{1}{n}}\right) = \frac{1}{v(x)} . \ \blacksquare$$

Recall that a linear mapping $T\colon F\subset CV(X,E)\to F$ is said to be bounded (resp. precompact, compact, equicontinuous) if it maps some 0-neighbourhood into a bounded (resp. precompact, compact, equicontinuous) subset of F.

Proposition 11. *Let $F\subset CV(X,E)$ be a $C_b(X)$-module and $\psi\colon X\to\mathcal{B}(E)$ a map such that M_ψ maps F into $C(X,E)$. If X has no isolated points and M_ψ is equicontinuous on F, then $M_\psi = 0$.*

Proof: Assume that M_ψ is equicontinuous and $M_\psi(f_0)\neq 0$ for some $f_0\in F$. Then there exists $x\in coz(F)$ with $\psi_x(f_0(x))\neq 0$. Since M_ψ is equicontinuous, there exist $P\in\mathbb{P}$ and $v\in V$ so that $M_\psi(B_{P,v}(F))$ is equicontinuous on X and in particular at x. With no loss of generality we assume that $f_0\in B_{P,v}$. Hence, for every $Q\in\mathbb{P}$ and $\epsilon>0$, there exists a neighbourhood Ω of x such that $Q[\psi_t(f(t))-\psi_x(f(x))]<\epsilon$ for every $t\in\Omega$ and $f\in B_{P,v}(F)$. Since x is not isolated, there exists some $t\in\Omega\cap coz(F)$ with $t\neq x$. Take then $g_t\in C_b(X)$ satisfying $g_t(x)=1$, $g_t(t)=0$ and $0\le g_t\le 1$. Then, $g_t f_0\in B_{P,v}(F)$ and then $Q[\psi_x(f_0(x))]<\epsilon$. Since ϵ and Q are arbitrary, $\psi_x(f_0(x))=0$. This is a contradiction. $\blacksquare$

Corollary 12. *Let $F \subset CV(X,E)$ be a $C_b(X)$-module and $\psi\colon X \to \mathcal{B}(E)$ a map such that M_ψ maps F into $C(X,E)$. If X is a $V_{\mathbb{R}}$-space without isolated points, then M_ψ is precompact if and only if $M_\psi = 0$.* ∎

Remark. An equicontinuous linear mapping need not be continuous. Actually, it may even be unbounded on some bounded set. For such an example, take $x_0 \in \beta\mathbb{R}\backslash\mathbb{R}$ and $T\colon (C_b(\mathbb{R}), \tau_c) \to (C(\mathbb{R}), \tau_c)$ with $T(f) := \tilde{f}(x_0)\,1$. Here, $\beta\mathbb{R}$ is the Stone–Čech compactification of $\mathbb{R}$, $\tilde{f}$ the Stone extension of f and τ_c the compact open topology. The map T is equicontinuous but not bounded on the bounded set $A := \{f_n,\, n \in \mathbb{N}\}$, where $f_n(x) := \min(|x|, n)$. □

However, we get

Proposition 13. *Let $\psi : X \to \mathcal{B}(E)$ and $F \subset CV(X,E)$ be such that $M_\psi(F) \subset C(X,E)$ and F satisfies (M). If M_ψ is a bounded multiplication operator on F, then there exist $P \in \mathbb{P}$ and $v \in V$ such that :*

(4) $\forall\, u \in V,\ Q \in \mathbb{P},\ \ \exists\, \lambda > 0\colon\ \ u(x)\,Q(\psi_x(a)) \le \lambda\, v(x)\, P(a),\ x \in coz(F),\ a{\in}E\ .$

If in addition F is EV-solid, then also the converse is true.

Proof: If M_ψ is bounded, then it is bounded on $B_{P,v}(F)$ for some $P \in \mathbb{P}$ and some $v \in V$. Then, for every $u \in V$ and $Q \in \mathbb{P}$, there exists $\lambda > 0$ so that $Q_u(M_\psi(f)) \le \lambda$ for every $f \in B_{P,v}(F)$. In particular, $u(x)\,Q[\psi_x(f(x))] \le \lambda$; x running over X. But for $f \in B_{P,v}(F)$ and $a \in B_P$, the function $P(f(\ ))a$ belongs to $B_{P,v}$ and by (M) to F. Hence $P(f(x))\,Q(\psi_x(a)) \le \lambda$, $x \in X$ and $f \in B_{P,v}(F)$. Using Lemma 4, we get

$$u(x)\,Q(\psi_x(a)) \le \lambda\, v(x)\,, \qquad x \in coz(F) \text{ and } a \in B_P\ .$$

Let $a \in E$ be arbitrary, if $P(a) = 0$, then also $P(na) = 0$ for every $n \in \mathbb{N}$ and then $u(x)\,Q(\psi_x(a)) = 0$ for every $x \in X$. Whence $u(x)\,Q(\psi_x(a)) \le \lambda\, v(x)\, P(a)$, for every $a \in E$. Assume now that $P \in \mathbb{P}$ and $v \in V$ enjoy (4). We claim that $M_\psi(B_{P,v}(F))$ is contained and bounded in F. Indeed, for every $u \in V$ and $Q \in \mathbb{P}$, there exists, by (4), $\lambda > 0$ so that $u(x)\,Q(\psi_x(a)) \le \lambda\, v(x)\, P(a),\ x \in coz(F)$ and $a \in E$. In particular $u(x)\,Q(\psi_x(f(x))) \le \lambda\, v(x)\, P(f(x))$, for every $f \in F$ and $x \in coz(F)$. In virtue of (EVS), $M_\psi(f) \in F$, and the latter inequality leads to $Q_u(M_\psi(f)) \le \lambda\, P_v(f)$ for every $f \in F$. This shows that M_ψ is bounded on $B_{P,v}(F)$. ∎

Now, we examine the cases $V = \mathcal{K}$, the set of all positive multiples of characteristic functions of the compact subsets of X, and $V = S$, the set of all non negative u.s.c. functions vanishing at infinity on X.

Proposition 14. *Let $\psi\colon X \to \mathcal{B}(E)$ be a map and F a subspace of $CV(X,E)$ satisfying (M) with $V \in \{\mathcal{K}, S\}$.*

1. *If M_ψ is a bounded multiplication operator on $(F, \tau_{\mathcal{K},\mathbb{P}})$, then the support of ψ is contained in $K \cup z(F)$ for some compact $K \subset X$. Here, $z(F) := X \setminus coz(F)$.*

2. *If M_ψ is a bounded multiplication operator on $(F, \tau_{S,\mathbb{P}})$, then ψ vanishes at infinity when $\mathcal{B}(E)$ is endowed with the topology β.*

Proof: **1.** Let $K \subset X$ be a compact set and $P \in \mathbb{P}$ such that, for every compact $H \subset X$ and every $Q \in \mathbb{P}$, there exists $\lambda > 0$ with

$$1_H(x)\, Q(\psi_x(a)) \le \lambda\, 1_K(x)\, P(a)\,, \qquad a \in E, \quad x \in coz(E)\ .$$

If $x \notin z(F) \cap K$, then taking a compact H containing x and not intersecting K, we get $\psi_x = 0$. This shows that $\operatorname{supp}\psi \subset z(F) \cap K$.

2. Since M_ψ is bounded, there exist $P \in \mathbb{P}$ and $v \in S$ so that for every $Q \in \mathbb{P}$, there exists $\lambda > 0$ with $\sqrt{v(x)}\, Q(\psi_x(a)) \le \lambda\, v(x)\, P(a)$, for every $x \in coz(F)$ and $a \in E$. This gives $Q_{\{a\}}(\psi_x) \le \lambda\, P(a)\sqrt{v(x)}$. Since $\sqrt{v}$ vanishes at infinity, $Q_{\{a\}}(\psi_x)$ also does. This shows that $\psi\colon X \to \mathcal{B}_\beta(E)$ vanishes at infinity. ∎

The following example shows that the converse in both 1. and 2. does not hold.

Example. Let $X := \widehat{\mathbb{N}}$ be the one point compactification of $\mathbb{N}$ and $E := C[0,1]$ the algebra of all continuous functions on $[0,1]$ equipped with the norm of $L^1[0,1]$. For every $n \in \mathbb{N}$, consider the function g_n defined on [0,1] by $g_n(x) = n^{\frac{2}{3}}(1 - nx)$ if $x \le \frac{1}{n}$ and $g_n(x) = 0$ otherwise. Then $g_n \in E$ and g_n tends to 0 as n tends to infinity. For every $g \in E$ and $x \in X$, set

$$\psi(x)(g) \;=\; \begin{cases} g_x g \;: & x \in \mathbb{N}, \\ \quad 0 \;: & x = \infty\ , \end{cases}$$

Since the multiplication of E is separately continuous, we get a continuous function ψ from X into $\mathcal{B}_\beta(E)$. Now, for every $m \in \mathbb{N}$, consider a continuous piecewise linear function h_m with $h_m(t) = m$ if $t \le \frac{1}{2m}$, $h_m(t) = 0$ if $t \ge \frac{1}{2m} + \alpha_m$, α_m being so chosen that $||h_m|| := \int_0^1 |h_m(t)|\, dt \le 1$. Next, set φ_m the constant function on X with value h_m. Then φ_m belongs to the unit ball $B_{|||\ |||}$ of the norm $|||\ |||$ of $CV(X,E)$, with $V = \{\lambda 1,\ \lambda > 0\}$. But,

$$|||M_\psi(\varphi_m)||| \;=\; \sup_n ||\psi_n(\varphi_m(n))|| \;=$$

$$
\begin{aligned}
&= \sup_n ||g_n\, h_m|| \\
&= \sup_n \int_0^1 |g_n(t)h_m(t)|\, dt \\
&\geq \int_0^{\frac{1}{2m}} m^{\frac{5}{3}}(1 - m\, x)\, dx \\
&\geq \frac{m^{\frac{2}{3}}}{4}.
\end{aligned}
$$

Hence $\sup_{\varphi \in B_{|||\ |||}} |||M_\psi(\varphi)||| \geq \sup_m |||M_\psi(\varphi_m)||| \geq \sup_m \frac{m^{\frac{2}{3}}}{4} = +\infty$. This shows that M_ψ is not bounded. □

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