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# OPTIMAL CONTROL AND "STRANGE TERM" FOR A STOKES PROBLEM IN PERFORATED DOMAINS

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**Abstract:** We study a problem of optimal control for Stokes equations in perforated domains with Dirichlet conditions on the boundary of holes. We consider different sizes of holes.

## 1 – Introduction

The aim of this paper is to study an optimal control problem for Stokes equations in perforated domains with Dirichlet conditions on the boundary of holes.

Let $\Omega$ be a bounded connected open set in $\mathbb{R}^n$ $(n_2)$ with Lipschitz boundary $\partial\Omega$. Let $\varepsilon$ be a sequence of positive real numbers which tends to zero. We cover the set $\Omega$ with a regular mesh of size $2\varepsilon$, each cell is a cube $P_i^\varepsilon,\ i=1,...,N(\varepsilon)$, similar to $[-\varepsilon,\varepsilon]^n$. We make a hole $T_i^\varepsilon$ at the center of each cube $P_i^\varepsilon$, included in $\Omega$. We define the holes as follows: each hole $T_i^\varepsilon$ is equal to $a_\varepsilon\,T$ where $T$ is a given closed set independent of $\varepsilon$, and $a_\varepsilon$ is the size of the hole $(0<a_\varepsilon<\varepsilon)$. Then the perforated domain $\Omega_\varepsilon$ is defined by $\Omega_\varepsilon=\Omega\backslash\bigcup T_i^\varepsilon$. There are different possible sizes of the holes which can be considered ("critical", smaller and larger holes). So we define a ratio $\sigma_\varepsilon$ between the current size of the holes and the critical one:

$$\sigma_\varepsilon=(\varepsilon^n/a_\varepsilon^{n-2})^{1/2}\ \text{ for } n\geq 3\,,\qquad \sigma_\varepsilon=\varepsilon\left(\log(a_\varepsilon/\varepsilon)\right)^{1/2}\ \text{ for } n=2\ . \tag{1.1}$$

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If the limit of $\sigma_\varepsilon$ as $\varepsilon$ tends to zero, is positive and finite then the size of the holes is called critical. If the $\lim_{\varepsilon\to 0} \sigma_\varepsilon = +\infty$, the size of holes is smaller and if $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$, the holes are larger (cf. Cioranescu and Murat [2] and Allaire [1]).

Throughout all the sequel, we use the convention of summation over repeated indices.

We denote by $\tilde{}$ the extension by zero onto the holes.

Let $B = (b_{ij})$ be a symmetric matrix such that

$$\alpha_m\, \xi_i\, \xi_i \le b_{ij}(x)\, \xi_i\, \xi_j \le \alpha_M\, \xi_i\, \xi_i \quad \text{a.e. in } \Omega \text{ and } b_{ij} \in L^\infty(\Omega)\ , \tag{1.2}$$

where $\alpha_m$ and $\alpha_M$ are constants such that $\alpha_M > \alpha_m > 0$.

For $\varepsilon > 0$ fixed, we define the optimal control problem as follows.

Let $\mathcal{U}_{ad}^\varepsilon \subset L^2(\Omega_\varepsilon)^n$ be a closed convex set. Let $f \in L^2(\Omega)^n$ be a given function and let $N > 0$ be a given constant. For $\theta_\varepsilon \in \mathcal{U}_{ad}^\varepsilon$, we define the state equation of the Stokes problem by

$$\begin{cases} \nabla p_\varepsilon - \Delta u_\varepsilon \;=\; f + \theta_\varepsilon & \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon \;=\; 0 & \text{in } \Omega_\varepsilon, \\ u_\varepsilon \;=\; 0 & \text{on } \partial\Omega_\varepsilon\ . \end{cases} \tag{1.3}$$

where $u_\varepsilon$, $p_\varepsilon$ are respectively the velocity, the pressure of the fluid and $\theta_\varepsilon$ is the control.

The cost functional is then given by

$$J_\varepsilon(\theta_\varepsilon) \;=\; \frac{1}{2}\int_{\Omega_\varepsilon} B\, \nabla u_\varepsilon\, \nabla u_\varepsilon\; dx \;+\; \frac{N}{2}\int_{\Omega_\varepsilon} \theta_\varepsilon^2\; dx\ . \tag{1.4}$$

The second integral corresponds to the cost of the control whereas the first one corresponds to the energy of the fluid. The matrix $B$ is used in order to generalize the usual energy (we obtain this energy when the matrix $B$ is equal to identity).

The optimal control $\theta_\varepsilon^\star$ is the function in $\mathcal{U}_{ad}^\varepsilon$ which minimizes $J_\varepsilon(\theta_\varepsilon)$ for $\theta_\varepsilon \in \mathcal{U}_{ad}^\varepsilon$, i.e.

$$\theta_\varepsilon^\star \in \mathcal{U}_{ad}^\varepsilon \quad \text{and} \quad J_\varepsilon(\theta_\varepsilon^\star) = \min_{\theta_\varepsilon \in \mathcal{U}_{ad}^\varepsilon} J_\varepsilon(\theta_\varepsilon)\ . \tag{1.5}$$

This problem admits a unique optimal solution $\theta_\varepsilon^\star$ (see Lions [6]).

The problem (1.3)–(1.5) can be reduced to a system of equations by introducing the adjoint state $(v_\varepsilon, p'_\varepsilon)$ of $(u_\varepsilon, p_\varepsilon)$. Thus we get

$$\begin{cases} \nabla p'_\varepsilon + \Delta v_\varepsilon \;=\; \operatorname{div}(B\, \nabla u_\varepsilon) & \text{in } \Omega_\varepsilon, \\ \operatorname{div} v_\varepsilon \;=\; 0 & \text{in } \Omega_\varepsilon, \\ v_\varepsilon \;=\; 0 & \text{on } \partial\Omega_\varepsilon\ . \end{cases} \tag{1.6}$$

where $(v, p'_\varepsilon) \in (H^1(\Omega)^n \times L_0^2(\Omega))$.

The optimal control $\theta_\varepsilon^\star$ is characterized by the variational inequality

$$(1.7)\qquad \theta_\varepsilon^\star \in \mathcal{U}_{ad}^\varepsilon \quad \text{and} \quad \int_{\Omega_\varepsilon} (v_\varepsilon + N\,\theta_\varepsilon^\star)\,(\theta_\varepsilon - \theta_\varepsilon^\star)\;dx \;\geq\; 0 \quad \forall\,\theta_\varepsilon \in \mathcal{U}_{ad}^\varepsilon\,.$$

Our aim is to study the limiting behaviour of the optimal control $\theta_\varepsilon^\star$ as $\varepsilon \to 0$.

In fact, it can be shown that (up to a subsequence) $\widetilde{\theta_\varepsilon^\star} \rightharpoonup \theta_0^\star$ weakly in $L^2(\Omega)^n$. Our objective is to characterize $\theta_0^*$ as the optimal control of a similar problem set in the non-perforated domain $\Omega$.

The type of optimal control problem which we consider, was studied by Kesavan and Vanninathan [5], Kesavan and Saint Jean Paulin [3] in non-perforated domains and by Kesavan and Saint Jean Paulin [4] in perforated domains. They studied in [4] the Laplace problem with Neumann conditions on the boundary. Also Rajesh [7] considered the optimal control problem for the Dirichlet problem in perforated domains and he obtained a "strange term" in the limit.

This paper is organized as follows. In Section 2, we recall some hypotheses (H1)–(H6) in perforated domains concerning the holes (see Allaire [1]) and the main results of the homogenization of Stokes equations. In Section 3, we consider the critical case and we homogenize the adjoint problem and establish convergence results of energies which appear in the cost functional. In Section 4, we obtain the limiting optimal control problem. In Section 5, we study the optimal problem for smaller sizes of holes (for which $\lim\limits_{\varepsilon\to 0} \sigma_\varepsilon = +\infty$).

**Notation.** Throughout this paper, $C$ denotes various real positive constants independent of $\varepsilon$. The duality products between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and between $(H_0^1(\Omega))^n$ and $(H^{-1}(\Omega))^n$, are each denoted by $\langle\;,\;\rangle$. □

We denote by $(e_k)_{1\leq k\leq n}$ the canonical basis of $\mathbb{R}^n$.

**Definition 1.1.** We define the set $L_0^2(\Omega)$ by

$$(1.8)\qquad L_0^2(\Omega) \;=\; \left\{ f \in L^2(\Omega) \;\middle|\; \int_\Omega f(x)\,dx = 0 \right\}\,. \;\square$$

## 2 – Hypotheses on the perforations and preliminary results

We make on the holes the same assumptions as Allaire [1], so there exist functions $(\omega_k^\varepsilon, r_k^\varepsilon, \mu_k)$ and a linear mapping $R_\varepsilon$ such that

(H1) $\omega_k^\varepsilon \in H^1(\Omega)^n,\ r_k^\varepsilon \in L^2(\Omega)$,

(H2) $\operatorname{div} \omega_k^\varepsilon = 0$ in $\Omega$ and $\omega_k^\varepsilon = 0$ in $T_i^\varepsilon$,

(H3) $\omega_k^\varepsilon \rightharpoonup e_k$ weakly in $H^1(\Omega)^n$ and $r_k^\varepsilon \rightharpoonup 0$ weakly in $L_0^2(\Omega)$,

(H4) $\mu_k \in W^{-1,\infty}(\Omega)^n$,

(H5) $\forall\, v_\varepsilon$ and $\forall\, v$ such that $v_\varepsilon \rightharpoonup v$ weakly in $H^1(\Omega)^n$, $v_\varepsilon = 0$ in $T_i^\varepsilon$ and $\forall\, \phi \in \mathcal{D}(\Omega)$,

$$\Big\langle \nabla r_k^\varepsilon - \Delta \omega_k^\varepsilon\,,\ \phi\, v_\varepsilon \Big\rangle \to \langle \mu_k, \phi v\rangle\ ,$$

$$\text{(H6)} \qquad \begin{cases} R_\varepsilon \in \mathcal{L}(H_0^1(\Omega)^n, H_0^1(\Omega_\varepsilon)^n), \\ \text{If } u \in H_0^1(\Omega_\varepsilon)^n \text{ then } R_\varepsilon\, \widetilde{u} = u \text{ in } \Omega_\varepsilon, \\ \text{If } \operatorname{div} u = 0 \text{ in } \Omega \text{ then } \operatorname{div}(R_\varepsilon u) = 0 \text{ in } \Omega_\varepsilon, \\ ||R_\varepsilon u||_{H_0^1(\Omega_\varepsilon)^n} \le c\, ||u||_{H_0^1(\Omega)^n}\ . \end{cases}$$

**Example 2.1.** The assumptions (H1)–(H6) are satisfied in the particular case where each hole $T_i^\varepsilon$ is a ball of radius $a_\varepsilon$ where $a_\varepsilon = C_0\, \varepsilon^{n/n-2}$ for $n \ge 3$ and $a_\varepsilon = e^{-C_0/\varepsilon^2}$ for $n = 2$ with $C_0 > 0$ and in a such geometry we can compute explicitly the functions $\omega_k^\varepsilon$, $r_k^\varepsilon$ and $\mu_k$ which satisfy (H1)- -(H6) (see [1]). In this case, the diameter of the holes is such that $a_\varepsilon << \varepsilon$.

Note also that, the case where the diameter of the holes $a_\varepsilon$ is of the same order as $\varepsilon$ corresponds to the classical homogenization. □

Assumptions (H1)–(H6) hold throughout the paper.

We define the matrix $M \in (W^{-1,\infty}(\Omega))^{n\times n}$ by (see [1])

$$M e_k = \mu_k\ . \tag{2.1}$$

This matrix is symmetric and under the above assumptions, we have the following result which is due to Allaire [1].

The extension $\widetilde{u_\varepsilon}$ of the velocity $u_\varepsilon$ and the extension $P^\varepsilon p_\varepsilon$ of the pressure $p_\varepsilon$ (defined by Allaire [1]) satisfy

**Theorem 2.2** (Allaire [1]). *Depending on the size of the holes, there are three different limit flow regimes for the solution of (1.3):*

**(i)** *If* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = +\infty$ *then* $(\widetilde{u_\varepsilon}, P^\varepsilon p_\varepsilon)$ *converges strongly to* $(u,p)$ *in* $H^1_0(\Omega)^n \times L^2_0(\Omega)$, *where* $(u,p)$ *is the unique solution of the Stokes problem*

$$\left\{\begin{aligned} \nabla p - \Delta u &= f+\theta \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}\right. \tag{2.2}$$

**(ii)** *If* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = \sigma > 0$ *then there exist a measure* $\mu^k$ *and a matrix* $M$ *such that* $M e_k = \mu^k$ *such that* $(\widetilde{u_\varepsilon}, P^\varepsilon p_\varepsilon)$ *converges weakly to* $(u,p)$ *in* $H^1_0(\Omega)^n \times L^2_0(\Omega)$, *where* $(u,p)$ *is the unique solution of the Brinkman-type law*

$$\left\{\begin{aligned} \nabla p - \Delta u + Mu &= f+\theta \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned}\right. \tag{2.3}$$

**Remark 2.3.** Under hypotheses similar to (H1)–(H6) (with a scaling depending of $\sigma_\varepsilon$), if $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$ then there exist a matrix $M_0$ such that $(\widetilde{u_\varepsilon}/\sigma^2_\varepsilon, P^\varepsilon p_\varepsilon)$ converges strongly to $(u,p)$ in $L^2(\Omega)^n \times L^2_0(\Omega)$, where $(u,p)$ is the unique solution of Darcy's law

$$\left\{\begin{aligned} u = M_0^{-1}(f - \nabla p + \theta) &\quad \text{in } \Omega,\\ \operatorname{div} u = 0 &\quad \text{in } \Omega,\\ u\,.\,n = 0 &\quad \text{on } \partial\Omega\,. \end{aligned}\right. \tag{2.4}$$

with $n$ the exterior normal vector to $\Omega$ (see Allaire [1] for more details concerning these hypotheses and the matrix $M_0$). □

## 3 – Homogenization and convergence of some energies

In this section and in Section 4, we assume that

$$\lim_{\varepsilon\to 0} \sigma_\varepsilon \;=\; \sigma > 0\,. \tag{3.1}$$

Following the approach of Kesavan and Saint Jean Paulin [4], we introduce the adjoint state variable and pass to the limit in the resulting system.

Assuming (3.1), there exists a sequence $(\omega^\varepsilon_k, r^\varepsilon_k)$ satisfying (H1)–(H6). We show that there exists $n$ distributions $\mu^k_B$ $(k = 1,...,n)$ and a matrix $M_B$

defined below by (3.8) such that, given any $f \in L^2(\Omega)^n$, if $(u_\varepsilon, p_\varepsilon)$ solves the Stokes problem (1.3), then (up to a subsequence), we have the following convergence of energies.

$$\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon \nabla u_\varepsilon\,dx \;\to\; \int_\Omega B\,\nabla u\,\nabla u\;dx + \langle M_B\,u,\,u\rangle \tag{3.2}$$

$$B\,\nabla u_\varepsilon \nabla u_\varepsilon\,dx \;\to\; B\,\nabla u\,\nabla u + {}^t(M_B\,u)\,u \quad \text{in } \mathcal{D}'(\Omega)\;, \tag{3.3}$$

where $(u, p)$ solves the problem (2.3).

This type of results was shown by Rajesh [7] for the Dirichlet problem for the Laplace operator.

We introduce some auxiliary test functions which are used to homogenize the adjoint problem (1.6).

**Lemma 3.1.** *Assume (3.1) and let $(\psi_k^\varepsilon, s_k^\varepsilon) \in H_0^1(\Omega_\varepsilon)^n \times L_0^2(\Omega_\varepsilon)$ be the solution of the auxiliary system*

$$\begin{cases} \nabla s_k^\varepsilon + \Delta\psi_k^\varepsilon \;=\; -\operatorname{div}({}^tB\,\nabla\omega_k^\varepsilon) & \text{in } \Omega_\varepsilon, \\ \operatorname{div}\psi_k^\varepsilon \;=\; 0 & \text{in } \Omega_\varepsilon, \\ \psi_k^\varepsilon \;=\; 0 & \text{on } \partial\Omega_\varepsilon\;. \end{cases} \tag{3.4}$$

*Then there exist $\psi_k$ and $s_k$ such that (for a subsequence)*

$$\widetilde{\psi_k^\varepsilon} \rightharpoonup \psi_k \quad \text{weakly in } H_0^1(\Omega)^n\;, \tag{3.5}$$

$$P_\varepsilon(s_k^\varepsilon) \rightharpoonup s_k \quad \text{weakly in } L_0^2(\Omega)\;. \tag{3.6}$$

**Proof:** Multiplying the first equation of (3.4) by $\psi_k^\varepsilon$, integrating by parts and taking into account the boundedness of $\omega_k^\varepsilon$ in $H^1(\Omega)^n$, we have the announced result. ∎

**Definition 3.2.** Let us define the distributions $\mu_B^k \in \mathcal{D}'(\Omega)$, $k = 1, ..., n$ by

$$\mu_B^k \;=\; -M\,\psi_k + (\nabla s_k + \Delta\psi_k)\;, \tag{3.7}$$

and the matrix $M_B \in (W^{-1,\infty})^{n\times n}$ by

$$M_B\,e_k \;=\; \mu_B^k\;. \;\square \tag{3.8}$$

**Proposition 3.3.** *Let $f \in L^2(\Omega)$. Define $M$ by (2.1) and $M_B$ by (3.8). Assume that (3.1) holds and that $\theta_\varepsilon$ is such that $\widetilde{\theta_\varepsilon}$ is bounded in $L^2(\Omega)^n$. Let $(u_\varepsilon, p_\varepsilon)$ and $(v_\varepsilon, p'_\varepsilon)$ in $(H_0^1(\Omega_\varepsilon)^n \times L_0^2(\Omega_\varepsilon))^2$ be the solution of the system*

$$\left\{\begin{aligned} \nabla p_\varepsilon - \Delta u_\varepsilon &= f + \theta_\varepsilon && \text{in } \Omega_\varepsilon, \\ \nabla p'_\varepsilon + \Delta v_\varepsilon &= \operatorname{div}(B\,\nabla u_\varepsilon) && \text{in } \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon = \operatorname{div} v_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ u_\varepsilon = v_\varepsilon &= 0 && \text{on } \partial\Omega_\varepsilon\ . \end{aligned}\right. \tag{3.9}$$

*Then, up to subsequences*

$$\left\{\begin{aligned} &\widetilde{\theta_\varepsilon} \rightharpoonup \theta && \text{weakly in } L^2(\Omega)^n, \\ &\widetilde{u_\varepsilon} \rightharpoonup u && \text{weakly in } H_0^1(\Omega)^n, \\ &\widetilde{v_\varepsilon} \rightharpoonup v && \text{weakly in } H_0^1(\Omega)^n \end{aligned}\right. \tag{3.10}$$

*and*

$$\left\{\begin{aligned} &P^\varepsilon p_\varepsilon \rightharpoonup p && \text{weakly in } L_0^2(\Omega), \\ &P^\varepsilon p'_\varepsilon \rightharpoonup p' && \text{weakly in } L_0^2(\Omega)\ , \end{aligned}\right. \tag{3.11}$$

*where the limits $(u,p)$ and $(v,p')$ are solution of the Brinkman type system*

$$\left\{\begin{aligned} \nabla p - \Delta u + Mu &= f + \theta && \text{in } \Omega, \\ \nabla p' + \Delta v - Mv &= \operatorname{div}(B\,\nabla u) - {}^t M_B\, u && \text{in } \Omega, \\ \operatorname{div} u = \operatorname{div} v &= 0 && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega\ . \end{aligned}\right. \tag{3.12}$$

**Proof:**

**Step 1**: *A priori estimates*

Since $\widetilde{\theta_\varepsilon}$ is bounded in $L^2(\Omega)$, it is clear that $\widetilde{u_\varepsilon}$ and $\widetilde{v_\varepsilon}$ are uniformly bounded in $H_0^1(\Omega)^n$ and, also $\{P^\varepsilon p_\varepsilon\}$ and $\{P^\varepsilon p'_\varepsilon\}$ are uniformly bounded in $L_0^2(\Omega)$.

Hence we can extract a subsequence (again indexed by $\varepsilon$ for convenience) such that (3.10) and (3.11) holds.

The homogenization of the state equation (1.3) is known (see Theorem 2.1 (ii)).

**Step 2**: *Energy method*

To pass to the limit in the second equation in (3.9), we use the test functions $(\omega_k^\varepsilon, r_k^\varepsilon)$ defined in (H1)–(H6) and the auxiliary functions $(\psi_k^\varepsilon, s_k^\varepsilon)$ defined by (3.4).

Let $\phi \in \mathcal{D}(\Omega)$. Multiplying the second equation in (3.9) by $\phi\omega_k^\varepsilon$ and integrating by parts and using assumption (H2), we get

$$\begin{aligned}\int_{\Omega_\varepsilon} p'_\varepsilon \, \nabla\phi \; \omega_k^\varepsilon \; dx \; &= -\int_{\Omega_\varepsilon} z_\varepsilon \, \nabla\phi \; \omega_k^\varepsilon \; dx - \int_{\Omega_\varepsilon} \nabla v_\varepsilon \, \nabla\omega_k^\varepsilon \; \phi \; dx \\ &\quad + \int_{\Omega_\varepsilon} B \, \nabla u_\varepsilon \, \nabla\omega_k^\varepsilon \; \phi \; dx \;, \end{aligned} \tag{3.14}$$

where

$$z_\varepsilon = \nabla v_\varepsilon - B \, \nabla u_\varepsilon \; . \tag{3.15}$$

Similarly, multiplying the first equation in (3.9) by $\phi \, \psi_k^\varepsilon$, integrating by parts and taking into account the definition of $\psi_k^\varepsilon$ (see equation (3.4)), we obtain

$$\begin{aligned}&\int_{\Omega_\varepsilon} (f + \theta_\varepsilon) \, \phi \; \psi_k^\varepsilon \; dx + \int_{\Omega_\varepsilon} p_\varepsilon \, \nabla\phi \; \psi_k^\varepsilon \; dx \; = \\ &\qquad = \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla\phi \; \psi_k^\varepsilon \; dx - \int_{\Omega_\varepsilon} u_\varepsilon \, \nabla\phi \; s_k^\varepsilon \; dx - \int_{\Omega_\varepsilon} u_\varepsilon \, \nabla\phi \, \nabla\psi_k^\varepsilon \; dx \\ &\qquad\quad - \int_{\Omega_\varepsilon} ({}^t B \, \nabla\omega_k^\varepsilon) \, \nabla u_\varepsilon \, \phi \; dx - \int_{\Omega_\varepsilon} ({}^t B \, \nabla\omega_k^\varepsilon) \, u_\varepsilon \, \nabla\phi \; dx \; . \end{aligned} \tag{3.16}$$

Adding (3.14) and (3.16) and transforming all the integrals over $\Omega_\varepsilon$ into integrals over $\Omega$, we get

$$\begin{aligned}&\int_{\Omega} (f + \widetilde{\theta}_\varepsilon) \, \phi \; \widetilde{\psi_k^\varepsilon} \; dx + \int_{\Omega} P^\varepsilon \, p_\varepsilon \, \nabla\phi \; \widetilde{\psi_k^\varepsilon} \; dx + \int_{\Omega} P^\varepsilon \, p'_\varepsilon \, \nabla\phi \; \omega_k^\varepsilon \; dx \; = \\ &\qquad = -\int_{\Omega} \widetilde{z}_\varepsilon \, \nabla\phi \; \omega_k^\varepsilon \; dx - \int_{\Omega} \nabla\widetilde{v}_\varepsilon \, \nabla\omega_k^\varepsilon \; \phi \; dx + \int_{\Omega} \nabla\widetilde{u_\varepsilon} \, \nabla\phi \; \widetilde{\psi_k^\varepsilon} \; dx \\ &\qquad\quad - \int_{\Omega} b_k^\varepsilon \, \widetilde{u_\varepsilon} \, \nabla\phi \; dx - \int_{\Omega} \widetilde{u_\varepsilon} \, \nabla\phi \; P^\varepsilon \, s_k^\varepsilon \; dx \;, \end{aligned} \tag{3.17}$$

where

$$b_k^\varepsilon \; = \; {}^t B \, \nabla\omega_k^\varepsilon + \nabla\widetilde{\psi_k^\varepsilon} \; . \tag{3.18}$$

Since $\operatorname{div} v_\varepsilon = 0$ in $\Omega_\varepsilon$, we get

$$\int_{\Omega} r_k^\varepsilon \, \phi \, \operatorname{div} \widetilde{v}_\varepsilon \; dx \; = \; 0 \; . \tag{3.19}$$

Adding (3.17) and (3.19) and integrating by parts, we get

$$
\begin{aligned}
&\int_\Omega (f+\widetilde{\theta}_\varepsilon)\,\phi\,\widetilde{\psi_k^\varepsilon}\,dx + \int_\Omega P^\varepsilon\,p_\varepsilon\,\nabla\phi\,\widetilde{\psi_k^\varepsilon}\,dx + \int_\Omega P^\varepsilon\,p'_\varepsilon\,\nabla\phi\,\omega_k^\varepsilon\,dx \;= \\
&\qquad = -\int_\Omega \widetilde{z}_\varepsilon\,\nabla\phi\,\omega_k^\varepsilon\,dx + \left\langle \Delta\omega_k^\varepsilon - \nabla r_k^\varepsilon,\, \phi\,\widetilde{v}_\varepsilon \right\rangle + \int_\Omega \widetilde{v}_\varepsilon\,\nabla\omega_k^\varepsilon\,\nabla\phi\,dx \\
&\qquad\quad - \int_\Omega r_k^\varepsilon\,\nabla\phi\,\widetilde{v}_\varepsilon\,dx + \int_\Omega \nabla\widetilde{u_\varepsilon}\,\nabla\phi\,\widetilde{\psi_k^\varepsilon}\,dx \\
&\qquad\quad - \int_\Omega b_k^\varepsilon\,\widetilde{u_\varepsilon}\,\nabla\phi\,dx - \int_\Omega \widetilde{u_\varepsilon}\,\nabla\phi\,P^\varepsilon s_k^\varepsilon\,dx\,.
\end{aligned}
\tag{3.20}
$$

**Step 3**: *Passing to the limit*

We now pass to the limit in (3.20) as $\varepsilon$ tends to 0. In order to do so, we need some preliminary results.

Using (H3), we have

$$\nabla\omega_k^\varepsilon \rightharpoonup 0 \quad \text{weakly in } \; L^2(\Omega)^{n\times n}\,. \tag{3.21}$$

By the definition (3.18) and using the convergences (3.5) and (3.21), we can extract a subsequence such that

$$b_k^\varepsilon \rightharpoonup \nabla\psi_k \quad \text{weakly in } \; L^2(\Omega)^{n\times n}\,. \tag{3.22}$$

Also by the definition (3.15) and using the convergence (3.10), we get (up to subsequences)

$$\widetilde{z}_\varepsilon \rightharpoonup z \,=\, \nabla v - B\,\nabla u \quad \text{weakly in } \; L^2(\Omega)^{n\times n}\,. \tag{3.23}$$

Now passing to the limit in (3.20), taking into account the convergences in (H3), (H5), (3.5), (3.6), (3.10), (3.11) and (3.21)–(3.23), we get, (up to subsequences)

$$
\begin{aligned}
&\int_\Omega (f+\theta)\;\phi\;\psi_k\,dx + \int_\Omega p\,\nabla\phi\;\psi_k\,dx + \int_\Omega p'\,\nabla\phi\;e_k\,dx \;= \\
&\qquad = -\int_\Omega z\,\nabla\phi\;e_k\,dx - \langle \mu_k,\,\phi\,v\rangle + \int_\Omega \nabla u\,\nabla\phi\;\psi_k\,dx \\
&\qquad\quad - \int_\Omega \nabla\psi_k\;u\,\nabla\phi\,dx - \int_\Omega u\,\nabla\phi\;s_k\,dx\,.
\end{aligned}
\tag{3.24}
$$

Therefore, integrating by parts the right-hand side of (3.24) and using Theorem 2.1 (ii), we have

$$
\begin{aligned}
&\int_\Omega M u\,\phi\,\psi_k\,dx - \int_\Omega \nabla p'\,\phi\,e_k\,dx \;= \\
&\qquad = \int_\Omega (\operatorname{div} z)\,\phi\,e_k\,dx - \langle \mu_k,\phi v\rangle + \int_\Omega \Delta\psi_k\,u\,\phi\,dx + \int_\Omega \nabla s_k\,u\,\phi\,dx\,.
\end{aligned}
\tag{3.25}
$$

Since the above relation holds for all $\phi \in \mathcal{D}(\Omega)$ and since $M$ is symmetric, we have

$$\nabla p' + \operatorname{div} z - Mv \;=\; -\,{}^t M_B\, u \tag{3.26}$$

i.e. $(u,p)$ and $(v,p')$ satisfy (3.12).

Since $M$ is symmetric and positive definite, the solutions $(u,p)$ and $(v,p')$ of (3.12) are unique, and therefore, it follows that the whole sequences $(u_\varepsilon, P^\varepsilon p_\varepsilon)$ and $(v_\varepsilon, P^\varepsilon p'_\varepsilon)$ converge. This completes the proof of the proposition. ∎

Now, we treat the convergence of the energies $\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx$. This type of convergence have been studied by Rajesh [7] for the Dirichlet problem. He has shown in [7] that "a strange term" for the energy appears in the limit using ideas of [2]. Similarly, we show a same type of result i.e. a strange term in the limiting energy for Stokes problem following ideas of [1] and [7].

**Theorem 3.4.** *Let $f \in L^2(\Omega)^n$ and $(u_\varepsilon, p_\varepsilon)$ be the solution of the Stokes problem (1.3). Let $M_B$ given by (3.8). Then*

$$\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx \;\to\; \int_\Omega B\,\nabla u\,\nabla u\;dx \;+\; \langle M_B\, u,\, u\rangle \tag{3.27}$$

*and*

$$B\,\nabla\widetilde{u_\varepsilon}\;\nabla\widetilde{u_\varepsilon} \;\to\; B\,\nabla u\;\nabla u \;+\; {}^t(M_B\, u)\, u \quad \text{in } \mathcal{D}'(\Omega)\;. \tag{3.28}$$

**Proof:** Using the fact that $(u_\varepsilon, p_\varepsilon)$ and $(v_\varepsilon, p'_\varepsilon)$ are solution of (3.9), we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx \;&=\; -\int_{\Omega_\varepsilon} (\nabla v_\varepsilon - B\,\nabla u_\varepsilon)\,\nabla u_\varepsilon\;dx \;+\; \int_{\Omega_\varepsilon} \nabla v_\varepsilon\,\nabla u_\varepsilon\;dx\\
&=\; -\int_{\Omega_\varepsilon} \nabla p'_\varepsilon\, u_\varepsilon\;dx \;+\; \int_{\Omega_\varepsilon} \nabla v_\varepsilon\,\nabla u_\varepsilon\;dx\\
&=\; \int_{\Omega_\varepsilon} \nabla v_\varepsilon\,\nabla u_\varepsilon\;dx\\
&=\; \int_{\Omega_\varepsilon} v_\varepsilon (f+\theta_\varepsilon)\;dx \;=\; \int_\Omega \widetilde{v_\varepsilon}(f+\widetilde{\theta_\varepsilon})\;dx\;.
\end{aligned} \tag{3.29}$$

Therefore, integrating by parts and using the homogenization results of Proposition 3.3 and the fact that $M$ is symmetric, we obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx &= \int_\Omega v(f+\theta)\;dx\\
&= \int_{\Omega_\varepsilon} v(\nabla p-\Delta u+Mu)\\
&= -\int_\Omega \Delta v\;u\;dx\;+\int_\Omega Mv\;u\;dx\\
&= \langle -\Delta v+Mv,\,u\rangle\\
&= \Big\langle \nabla p'-\operatorname{div}(B\,\nabla u)+{}^{t}M_B\,u,\,u\Big\rangle\\
&= \int_\Omega B\,\nabla u\,\nabla u\;dx\;+\;\langle M_B\,u,\,u\rangle\;,
\end{aligned}
\tag{3.30}
$$

which proves (3.27).

Let $\phi\in\mathcal{D}(\Omega)$. Set $z_\varepsilon$ defined by (3.15), integrating by parts and using the problem (3.9), we have

$$
\begin{aligned}
\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;\phi\;dx &= \int_{\Omega_\varepsilon} \nabla v_\varepsilon \nabla u_\varepsilon\;\phi\;dx\;-\int_{\Omega_\varepsilon} z_\varepsilon\,\nabla u_\varepsilon\;\phi\;dx\\
&= \int_{\Omega_\varepsilon} v_\varepsilon(f+\theta-\nabla p_\varepsilon)\,\phi\;dx\;-\int_{\Omega_\varepsilon} v_\varepsilon\,\nabla u_\varepsilon\,\nabla\phi\;dx\\
&\quad -\int_{\Omega_\varepsilon}\nabla p'_\varepsilon\,u_\varepsilon\,\phi\;dx\;+\int_{\Omega_\varepsilon} z_\varepsilon\,u_\varepsilon\,\nabla\phi\;dx\;.
\end{aligned}
\tag{3.31}
$$

Using the same arguments as in the proof of Proposition 3.3 and using system (3.12), we derive

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;\phi\;dx\;=\\
&= \int_\Omega v(f+\theta-\nabla p)\,\phi\;dx-\int_\Omega v\,\nabla u\,\nabla\phi\;dx-\int_\Omega\nabla p'\,u\,\phi\;dx+\int_\Omega z\,u\,\nabla\phi\;dx\\
&= \langle Mu,\,v\,\phi\rangle\;+\int_\Omega \nabla u\,\nabla v\;\phi\;dx\;-\int_\Omega z\,\nabla u\;\phi\;dx\;+\;\langle {}^{t}M_B\,u-Mv,\,\phi\,u\rangle\;.
\end{aligned}
\tag{3.32}
$$

Therefore, using the fact that $M$ is symmetric, we have

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;\phi\;dx &= \int_\Omega (\nabla v-z)\,\nabla u\;\phi\;dx\;+\;\langle {}^{t}M_B\,u,\,\phi\,u\rangle\\
&= \int_\Omega B\,\nabla u\,\nabla u\;\phi\;dx\;+\;\langle {}^{t}(M_B\,u)\,u,\,\phi\rangle\;.
\end{aligned}
\tag{3.33}
$$

This holds for all $\phi\in\mathcal{D}(\Omega)$. This proves (3.28) and completes the proof. ∎

Now we give some properties concerning the functions $(\mu_B^k)_{1\le k\le n}$.

**Theorem 3.5.** *Let $\mu_B^k$ be as defined in (3.7). Then*

$$\mu_B^k\, e\, i \;=\; \lim_{\varepsilon\to 0} B\,\nabla\omega_i^\varepsilon\,\nabla\omega_k^\varepsilon \quad \textit{in} \;\; \mathcal{D}'(\Omega)\;. \tag{3.34}$$

**Proof:** Let $\phi \in \mathcal{D}(\Omega)$. Using the problem (3.4), the expression (3.19) and integrating by parts, we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} B\,\nabla\omega_i^\varepsilon\,\nabla\omega_k^\varepsilon\;\phi\;dx \;&=\; \int_{\Omega_\varepsilon} (\nabla\psi_k^\varepsilon + {}^tB\,\nabla\omega_k^\varepsilon)\,\nabla\omega_i^\varepsilon\;\phi\;dx \;-\int_{\Omega_\varepsilon}\nabla\psi_k^\varepsilon\,\nabla\omega_i^\varepsilon\;\phi\;dx\\
&=\; -\int_{\Omega_\varepsilon} s_\varepsilon^k\,\omega_i^\varepsilon\,\nabla\phi\;dx \;-\int_{\Omega_\varepsilon}(\nabla\psi_k^\varepsilon + {}^tB\,\nabla\omega_k^\varepsilon)\;\omega_i^\varepsilon\,\nabla\phi\;dx\\
&\qquad + \int_{\Omega_\varepsilon}\psi_k^\varepsilon\,\Delta\omega_i^\varepsilon\;\phi\;dx + \int_{\Omega_\varepsilon}\psi_k^\varepsilon\,\nabla\omega_i^\varepsilon\,\nabla\phi\;dx\\
&=\; -\int_{\Omega_\varepsilon} s_k^\varepsilon\,\omega_i^\varepsilon\,\nabla\phi\;dx \;-\int_{\Omega_\varepsilon}(\nabla\psi_k^\varepsilon + {}^tB\,\nabla\omega_k^\varepsilon)\;\omega_i^\varepsilon\,\nabla\phi\;dx\\
&\qquad - \Big\langle \nabla r_i^\varepsilon - \Delta\omega_i^\varepsilon,\; \widetilde{\psi_k^\varepsilon}\,\phi\Big\rangle - \int_{\Omega_\varepsilon} r_i^\varepsilon\,\psi_k^\varepsilon\,\nabla\phi\;dx\;.
\end{aligned}\tag{3.35}$$

Passing to the limit (using the convergences (H3), (H5), (3.5) and (3.6)), we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} B\,\nabla\omega_i^\varepsilon\,\nabla\omega_k^\varepsilon\;\phi\;dx \;&=\; -\int_\Omega s_k\,e_i\,\nabla\phi\;dx \;-\int_\Omega \nabla\psi_k\;e_i\,\nabla\phi\;dx \;-\;\langle\mu_i,\,\psi_k\,\phi\rangle\\
&=\; \int_\Omega \nabla s_k\;e_i\,\phi\;dx + \int_\Omega \Delta\psi_k\,e_i\,\phi\;dx \;-\;\langle\mu_i,\,\psi_k\,\phi\rangle\\
&=\; \Big\langle \nabla s_k + \Delta\psi_k - M\psi_k,\;\phi\,e_i\Big\rangle\\
&=\; \langle\mu_B^k,\,\phi\,e_i\rangle\\
&=\; \langle\mu_B^k\,e_i,\,\phi\rangle\;.
\end{aligned}\tag{3.36}$$

This proves (3.34). ∎

**Corollary 3.6.** *If $B$ is symmetric positive definite, then $\mu_B^k$ is a positive measure and $M_B$ is symmetric.*

**Proof:** This is a consequence of Theorem 3.5. ∎

In the next section, we return to the control problem we started with.

## 4 – Optimal control

We denote by $\chi_\varepsilon$ the characteristic function of $\Omega_\varepsilon$. We now consider the optimal control problem (1.3)–(1.5) where the convex set $\mathcal{U}^\varepsilon_{ad} \subset L^2(\Omega_\varepsilon)$ is one of the following ones (see [3] and [4]).

$$\mathcal{U}^\varepsilon_{ad} = L^2(\Omega_\varepsilon)^n , \tag{4.1}$$

$$\mathcal{U}^\varepsilon_{ad} = \Big\{\theta \in L^2(\Omega_\varepsilon)^n \mid \widetilde{\theta} \geq \chi_\varepsilon\,\psi \ \text{ a.e. in } \Omega\Big\} , \tag{4.2}$$

$$\mathcal{U}^\varepsilon_{ad} = \Big\{\theta \in L^2(\Omega_\varepsilon)^n \mid \chi_\varepsilon\,\psi_1 \leq \widetilde{\theta} \leq \chi_\varepsilon\,\psi_2 \ \text{ a.e. in } \Omega\Big\} , \tag{4.3}$$

where $\psi$, $\psi_1$ and $\psi_2$ are given functions in $L^2(\Omega)^n$.

Now, since $\theta^\star_\varepsilon$ is optimal we have

$$\frac{N}{2}\int_{\Omega_\varepsilon} (\theta^\star_\varepsilon)^2\,dx \,\leq\, J_\varepsilon(\theta^\star_\varepsilon) \,\leq\, J_\varepsilon(\Theta_\varepsilon) \qquad \forall\,\Theta_\varepsilon \in \mathcal{U}^\varepsilon_{ad} . \tag{4.4}$$

This relation holds in particular with the following choice of $\Theta_\varepsilon$

$$\Theta_\varepsilon = \begin{cases} \chi_\varepsilon & \text{in the case of (4.1),} \\ \chi_\varepsilon\psi & \text{in the case of (4.2),} \\ \chi_\varepsilon\psi_2 & \text{in the case of (4.3) .} \end{cases} \tag{4.5}$$

In each of the three cases above, we have

**Lemma 4.1.** *The optimal control satisfies (up to a subsequence)*

$$\widetilde{\theta^\star_\varepsilon} \rightharpoonup \theta^\star_0 \quad \textit{weakly in } \ L^2(\Omega)^n . \tag{4.6}$$

**Proof:** Using (4.5), we have that $J_\varepsilon(\Theta_\varepsilon)$ is bounded in $L^2(\Omega_\varepsilon)^n$, so we derive from (4.4) the announced result. ∎

**Lemma 4.2.** *The characteristic function $\chi_\varepsilon$ of $\Omega_\varepsilon$ satisfies*

$$\chi_\varepsilon \rightharpoonup 1 \quad \textit{weakly} \star \textit{ in } \ L^\infty(\Omega) . \tag{4.7}$$

**Proof:** We have, up to a subsequence

$$\chi_\varepsilon \rightharpoonup \chi_0 \quad \text{weakly} \star \text{ in } \ L^\infty(\Omega) .$$

Since $\chi_\varepsilon\,\omega^\varepsilon_k = \omega^\varepsilon_k$, thus passing to the limit and by uniqueness, we obtain $\chi_0 = 1$. ∎

We proceed to characterize the limiting optimal control problem. We define the set $\mathcal{U}_{ad} \subset L^2(\Omega)$ as

$$\mathcal{U}_{ad} = L^2(\Omega)^n \,, \tag{4.8}$$

$$\mathcal{U}_{ad} = \Big\{\theta \in L^2(\Omega)^n \mid \theta \geq \psi \;\text{ a.e. in } \Omega\Big\} \,, \tag{4.9}$$

$$\mathcal{U}_{ad} = \Big\{\theta \in L^2(\Omega)^n \mid \psi_1 \leq \theta \leq \psi_2 \;\text{ a.e. in } \Omega\Big\} \,, \tag{4.10}$$

corresponding to the cases (4.1), (4.2) and (4.3) respectivly. We have the following convergence result of optimal control.

**Theorem 4.3.** *Let $M_B$ given by (3.8). For $\theta \in \mathcal{U}_{ad}$, let $(u,p) \in H_0^1(\Omega)^n \times L_0^2(\Omega)$ be the solution of (2.2). Let $J_0$ be the cost functional defined by*

$$J_0(\theta) = \frac{1}{2}\int_\Omega B\,\nabla u\,\nabla u\;dx + \frac{1}{2}\,\langle M_B\,u,\,u\rangle + \frac{N}{2}\int_\Omega \theta^2\,dx \,. \tag{4.11}$$

*Then $\theta_0^\star$ satisfy the condition of optimality*

$$\theta_0^\star \in \mathcal{U}_{ad} \quad \text{and} \quad J_0(\theta_0^\star) = \min_{\theta\in\mathcal{U}_{ad}} J_0(\theta) \,. \tag{4.12}$$

*Further we have the convergence of the minimal costs, i.e.*

$$\lim_{\varepsilon\to 0} J_\varepsilon(\theta_\varepsilon^\star) = J_0(\theta_0^\star) \,. \tag{4.13}$$

**Proof:**

**Step 1**: It is clear from the definition of $\mathcal{U}_{ad}$, that if $\theta \in \mathcal{U}_{ad}$ then $\chi_\varepsilon\theta \in \mathcal{U}_{ad}^\varepsilon$. Further, since $\widetilde{\theta_\varepsilon^\star} \rightharpoonup \theta_0^\star$ weakly in $L^2(\Omega)^n$ and $\mathcal{U}_{ad}$ is a closed convex set, we have $\theta_0^\star \in \mathcal{U}_{ad}$.

**Step 2**: Let $(u_\varepsilon^\star, p_\varepsilon^\star)$ be the solution of the state equation (1.3) corresponding to $\theta_\varepsilon = \theta_\varepsilon^\star$. Using the convergence (4.6) of Lemma 4.1, we get

$$\begin{cases} \widetilde{u_\varepsilon^\star} \rightharpoonup u^\star & \text{weakly in } H_0^1(\Omega)^n, \\ P^\varepsilon p_\varepsilon^\star \rightharpoonup p^\star & \text{weakly in } L_0^2(\Omega) \,, \end{cases} \tag{4.14}$$

where $(u^\star, p^\star)$ is solution of (2.2) with $\theta = \theta^\star$ in the right-hand side.

**Step 3**: Let $(w_\varepsilon, q_\varepsilon) \in H_0^1(\Omega_\varepsilon)^n \times L_0^2(\Omega_\varepsilon)$ be the solution of the state equation (1.3) with the control $\chi_\varepsilon\theta$, $\theta \in \mathcal{U}_{ad}$, that is

$$\begin{cases} \nabla q_\varepsilon - \Delta w_\varepsilon = f + \chi_\varepsilon\,\theta & \text{in } \Omega_\varepsilon, \\ \operatorname{div} w_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ w_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \,. \end{cases} \tag{4.15}$$

Since $\chi_\varepsilon \theta \rightharpoonup \theta$ weakly in $L^2(\Omega)^n$, it follows that $\widetilde{w_\varepsilon} \rightharpoonup w$ weakly in $H_0^1(\Omega)^n$ and $P^\varepsilon(q_\varepsilon) \rightharpoonup q$ weakly in $L_0^2(\Omega)$ where $(w, q)$ satisfy the following Brinkmann-type problem

$$\tag{4.16} \begin{cases} \nabla q - \Delta w + Mw = f + \theta & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \,, \end{cases}$$

(see Proposition 3.3). Further, using Theorem 3.4 for $\theta$ fixed, we have

$$\tag{4.17} \int_{\Omega_\varepsilon} B\,\nabla w_\varepsilon\,\nabla w_\varepsilon\;dx \;\rightarrow\; \int_\Omega B\,\nabla w\,\nabla w\;dx \;+\; \langle M_B\,w,\,w\rangle\;.$$

Thus

$$\tag{4.18} J_\varepsilon(\chi_\varepsilon\theta) \rightarrow J_0(\theta)\;.$$

Once again, using Theorem 3.4 but now for $\theta_\varepsilon = \theta_\varepsilon^\star$, we get

$$\tag{4.19} \int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon^\star\,\nabla u_\varepsilon^\star\;dx \;\rightarrow\; \int_\Omega B\,\nabla u^\star\,\nabla u^\star\;dx \;+\; \langle M_B\,u^\star,\,u^\star\rangle\;.$$

**Step 4**: Passing to the limit in the inequality

$$\tag{4.20} J_\varepsilon(\chi_\varepsilon\theta) \geq J_\varepsilon(\theta_\varepsilon^\star)\;.$$

and using Lemma 4.2, we get

$$\tag{4.21} J_0(\theta) \;\geq\; \frac{1}{2}\int_\Omega B\,\nabla u^\star\,\nabla u^\star\;dx \;+\; \limsup_{\varepsilon\to 0}\;\frac{N}{2}\int_{\Omega_\varepsilon}(\theta_\varepsilon^\star)^2\;dx \;+\; \frac{1}{2}\,\langle M_B\,u^\star,\,u^\star\rangle\;.$$

Thus taking $\theta = \theta^\star$ in the above inequality, we have

$$\tag{4.22} \limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(\theta_\varepsilon^\star)^2\;dx \;\leq\; \int_\Omega(\theta^\star)^2\;dx\;.$$

Moreover since $\widetilde{\theta_\varepsilon^\star} \rightharpoonup \theta_0^\star$ weakly in $L^2(\Omega)$, we get

$$\tag{4.23} \liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(\theta_\varepsilon^\star)^2\;dx \;\geq\; \int_\Omega(\theta^\star)^2\;dx\;.$$

Thus using (4.22) and (4.23), we derive

$$\tag{4.24} \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(\theta_\varepsilon^\star)^2\;dx \;=\; \int_\Omega(\theta^\star)^2\;dx\;.$$

We deduce (4.13). Now (4.12) follows from (4.21) and (4.24). ∎

## 5 – Case of smaller holes

We now assume that the size of the holes is smaller than the critical size, i.e.

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon = +\infty , \tag{5.1}$$

in other words,

$$a_\varepsilon << \varepsilon^{n/n-2} \text{ for } n \geq 3 , \quad a_\varepsilon = \exp^{-1/C_\varepsilon} \quad \text{and} \quad C_\varepsilon << \varepsilon^2 \text{ for } n = 2 . \tag{5.2}$$

Since the size of the holes satisfies (5.1), the hypothese (H3) is replaced by (see [1])

$$\omega_k^\varepsilon \to e_k \text{ strongly in } H^1(\Omega)^n \quad \text{and} \quad r_k^\varepsilon \to 0 \text{ strongly in } L_0^2(\Omega) . \tag{5.3}$$

**Remark 5.1.** Hypothese (5.3) is stronger than Hypothese (H3). □

We have the following result

**Proposition 5.2.** *Let the size of the holes satisfy (5.1). Assume that (H1), (H2), (H4)–(H6) and (5.3) hold. Let $(u_\varepsilon, p_\varepsilon)$ and $(v_\varepsilon, p'_\varepsilon)$ be the unique solution of (3.9). Then up to subsequences*

$$\begin{cases} \widetilde{\theta_\varepsilon} \rightharpoonup \theta & \text{weakly in } L^2(\Omega)^n, \\ \widetilde{u_\varepsilon} \to u & \text{strongly in } H_0^1(\Omega)^n, \\ \widetilde{v_\varepsilon} \to v & \text{strongly in } H_0^1(\Omega)^n \end{cases} \tag{5.4}$$

*and*

$$\begin{cases} P^\varepsilon(p_\varepsilon) \to p & \text{strongly in } L_0^2(\Omega), \\ P^\varepsilon(p'_\varepsilon) \to p' & \text{strongly in } L_0^2(\Omega) , \end{cases} \tag{5.5}$$

*where $(u, p)$ and $(v, p')$ are solution of the Stokes problem*

$$\begin{cases} \nabla p - \Delta u = f + \theta & \text{in } \Omega, \\ \nabla p' + \Delta v = \operatorname{div}(B\, \nabla u) & \text{in } \Omega, \\ \operatorname{div} u = \operatorname{div} v = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega . \end{cases} \tag{5.6}$$

**Proof:** To prove this result, we use the same arguments as in the proof of Proposition 3.3.

**Step 1**: The fact that $M = 0$ was established by Allaire [1] and also the following convergence result

$$\left\{\begin{array}{ll} \widetilde{u_\varepsilon} \to u & \text{strongly in } H_0^1(\Omega)^n, \\ P^\varepsilon(p_\varepsilon) \to p & \text{strongly in } L_0^2(\Omega) . \end{array}\right. \tag{5.7}$$

Following Allaire [1], we show now that $\mu_B^k$, defined by (3.7), is equal to zero. Since Hypotheses (H1), (H2), (H4)–(H6) and (5.3) are satisfied, all the results of Proposition 3.3 hold. But from (5.3) and Theorem 3.6, we deduce that $\mu_B^k = 0$ and hence $M_B = 0$. This proves that $(u, p)$ and $(v, p')$ satisfy the Stokes equations (5.6).

We complete the proof by showing the strong convergence of $\widetilde{v_\varepsilon}$ in $H_0^1(\Omega)^n$ and of $P^\varepsilon(p'_\varepsilon)$ in $L_0^2(\Omega)$.

**Step 2**: Using the convergences (3.10) and (3.11) of $\widetilde{v_\varepsilon}$ and $P^\varepsilon(p'_\varepsilon)$ respectively and defining $v$ by (5.6), we have the following convergence (using classical arguments)

$$\left\{\begin{array}{ll} \widetilde{v_\varepsilon} \to v & \text{strongly in } H_0^1(\Omega)^n, \\ P^\varepsilon(p'_\varepsilon) \to p' & \text{strongly in } L_0^2(\Omega) . \end{array}\right. \tag{5.8}$$

This ends the proof. ∎

We now give a convergence result of the optimal control. Let $\mathcal{U}_{ad}^\varepsilon \subset L^2(\Omega_\varepsilon)^n$ given by (4.1)–(4.3) and $\mathcal{U}_{ad} \subset L^2(\Omega)^n$ by (4.8)–(4.10). We have the following result

**Theorem 5.4.** *Let $\theta_\varepsilon^\star$ be the optimal control for the Stokes problem (1.3) and let the cost functional be given by (1.4). Then*

$$\widetilde{\theta_\varepsilon^\star} \rightharpoonup \theta_0^\star \quad \textit{weakly in } L^2(\Omega)^n \tag{5.9}$$

*and $\theta_0^\star$ is the optimal control for the problem*

$$\left\{\begin{aligned} \nabla p - \Delta u &= f + \theta && \textit{in } \Omega, \\ \operatorname{div} u &= 0 && \textit{in } \Omega, \\ u &= 0 && \textit{on } \partial\Omega , \end{aligned}\right. \tag{5.10}$$

*with the following cost functional*

$$J_0(\theta) = \frac{1}{2}\int_\Omega B\,\nabla u\,\nabla u\;dx + \frac{N}{2}\int_\Omega \theta^2\,dx . \tag{5.11}$$

**Proof:** By Lemma 4.1, we have (5.9). Now using Theorem 4.3 with the matrices $M$ and $M_B$ equal to zero, we have immediately the results. This completes the proof. ∎

**Remark 5.5.** In the case where $\sigma_\varepsilon \to 0$ (i.e. when the holes are larger), we have that $\widetilde{u_\varepsilon} \to 0$ strongly in $H^1(\Omega)^n$, hence $\nabla\widetilde{u_\varepsilon} \to 0$ strongly in $L^2(\Omega)^{n\times n}$. Then it is obvious that we have the following convergence of energy

$$\int_{\Omega_\varepsilon} B\,\nabla u_\varepsilon\,\nabla u_\varepsilon\;dx \;=\; \int_\Omega B\,\nabla\widetilde{u_\varepsilon}\,\nabla\widetilde{u_\varepsilon}\;dx \;\to\; 0\;.$$

Unfortunately, we could not succeed to conclude concerning the optimal control problem in this case. □

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