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ON THE RIGIDITY OF HORIZONTAL SLICES

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Abstract: In this paper, we proved a rigidity theorem of the Hodge metric for concave horizontal slices and a local rigidity theorem for the monodromy representation.

I – Introduction

Let (X,ω) be a polarized simply connected Calabi–Yau manifold. That is, X is an n-dimensional compact Kähler manifold with zero first Chern class and $[\omega]\in H^2(X,\mathbb{Z})$ is a Kähler metric. By the famous theorem of Yau [12], there is a Kähler metric on X in the same cohomological class of $[\omega]$ such that its Ricci curvature is zero.

Let Θ be the holomorphic tangent bundle of X. In [9], Tian proved that the universal deformation space of the complex structure is smooth. The complex dimension of the universal deformation space is $\dim H^1(X,\Theta)$. In other words, there are no obstructions towards the deformation of the complex structure of Calabi–Yau manifold. A good reference for the proof is in [3].

Take $n=3$ for example. A natural question is that to what extent the Hodge structure, namely, the decomposition of $H^3(X,\mathbb{C})$, into the sum of $H^{p,q}$'s $(p+q=3)$, determines a Calabi–Yau threefold. Let's recall the concept of classifying space in [4], which is a generalization of classical period domain. In the case of Calabi–Yau threefold, the classifying space D is defined as the set of the

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filtrations of $H = H^3(X, \mathbb{C})$ by

$$0 \subset F^3 \subset F^2 \subset F^1 \subset H$$

with $\dim F^3 = 1$, $\dim F^2 = n = \dim H^1(X, \Theta)$, $\dim F^1 = 2n+1$, and $H^{p,q} = F^p \cap \overline{F}^q$, $H = F^p \oplus \overline{F^{4-p}}$ $(p+q=3)$ together with a quadratic form Q such that

1) $i\,Q(x, \overline{x}) < 0$ if $0 \neq x \in H^{3,0}$,

2) $i\,Q(x, \overline{x}) > 0$ if $0 \neq x \in H^{2,1}$,

where $i = \sqrt{-1}$.

There is a natural map from the universal deformation space into the classifying space. Intuitively, this is because D is just the set of all the possible "Hodge decompositions". Such a map is called a period map. In the case of Calabi–Yau, the map is a holomorphic immersion. Thus in that case, the infinitesimal Torelli theorem is valid [4].

It can be seen that D fibers over a symmetric space D_1. But such a symmetric space needs not to be Hermitian. Even D_1 is Hermitian symmetric, D still needs not fiber holomorphically over D_1. Although in that case, there is a complex structure on D such that D becomes homogeneous Kählerian [8].

Griffiths introduced the concept of horizontal distribution in [4]. He proved that the image of the universal deformation space, via the period map to the classifying space, is an integral submanifold of the horizontal distribution. A horizontal slice is an integral complex submanifold of the horizontal distribution. In this terminology, the universal deformation space is a horizontal slice of the classifying space. The horizontal distribution is a highly nonintegrable system.

Because of the above result of Griffiths, it is interesting to study horizontal slices of a classifying space. The local properties of horizontal slices have been studied in [6] and [5].

In [6], we introduced a new Kähler metric on a horizontal slice U. We call such a metric the Hodge metric. The main result in [6] is that (see also [5] for the case $n = 3$)

Theorem. *Let $U \to D$ be a horizontal slice. Then the restriction of the natural invariant Hermitian metric of D to U is actually Kählerian. We call such a Kähler metric the Hodge metric of U. The holomorphic bisectional curvature of the Hodge metric is nonpositive. The Ricci curvature of the Hodge metric is negative away from zero.*

In this paper, we study some global rigidity properties of horizontal slices. In order to do that, we observe that the universal deformation space U carries less global information than the moduli space $\mathcal{M}$, which is essentially the quotient of the universal deformation space by a discrete subgroup Γ of $\mathrm{Aut}(U)$. The group Γ is called the monodromy group of the moduli space. The volume of the space $\Gamma\backslash U$ is finite with respect to the Hodge metric. This is the consequence of the theorem of Viehweg [11], the theorem of Tian [10] and the above theorem.

For a horizontal slice U of D, if Γ is a discrete subgroup of U such that the volume of $\Gamma\backslash U$ is finite, then a general conjecture is that whether Γ completely determines the space $\Gamma\backslash U$. In the case where $n=2$, this is correct by the superrigidity theorem of Margulis [7]. In general, this is a very difficult problem.

We consider the following weaker rigidity problem: to what extent the complex structure of the moduli space determines the metrics on the moduli space and the monodromy representation? To this problem, we have the following result in this paper.

First, we proved that, if $\Gamma\backslash U$ is a complete concave manifold, then the complex structure of $\Gamma\backslash U$ completely determines the Hodge metric of $\Gamma\backslash U$. More precisely, we proved that if for some discrete group Γ of $\mathrm{Aut}(U)$, $\Gamma\backslash U$ is a concave complete complex manifold, then the Hodge metric defined on U is intrinsic. In other words, the Hodge metric doesn't depend on the choice of the holomorphic immersion $U \to D$ from which it becomes a horizontal slice.

Theorem 1.1. *If the moduli space $\Gamma\backslash U$ is a concave manifold. Then the Hodge metric is intrinsically defined.*

The second main result of this paper is the local rigidity of the monodromy representation. The result is in Theorem 5.1. We combine the superrigidity theorem of Margulis [7] together with some ideas of Frankel [2] in the proof of the theorem.

Theorem 1.2. (For definition of the notations, see §5) *Let $\Gamma\backslash U$ be of finite Hodge volume. Suppose further that $\mathcal{G}_0$ is semisimple and $\mathcal{G}_0/\mathcal{K}_0$ is a Hermitian symmetric space but is not a complex ball, where $\mathcal{K}_0$ is the maximum compact subgroup of $\mathcal{G}_0$. Then the representation $\Gamma \to \mathcal{G}$ is locally rigid.*

The motivation behind the above results is that in the case of K-3 surfaces, the moduli space is a local symmetric space of rank 2. But even in the case of Calabi–Yau threefold, little has been known about the moduli space. We wish to involve certain kinds of metrics (Weil–Petersson metric, Hodge metric, etc) in the study of the moduli space of Calabi–Yau manifolds. The metrics have applications in Mirror Symmetry of Calabi–Yau manifolds [13].

2 – Preliminaries

In this section, we give some definitions and notations which will be used throughout this paper. Unless otherwise stated, the materials in this section are from the book of Griffiths [4].

Let X be a compact Kähler manifold. A C^∞ form on X decomposes into (p,q)-components according to the number of dz's and $d\overline{z}$'s. Denoting the C^∞ n-forms and the $C^\infty(p,q)$ forms on X by $A^n(X)$ and $A^{p,q}(X)$ respectively, we have the decomposition

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X) .$$

The cohomology group is defined as

$$\begin{aligned} H^{p,q}(X) &= \Big\{ \textit{closed } (p,q)\textit{-forms} \Big\} \Big/ \Big\{ \textit{exact } (p,q)\textit{-forms} \Big\} \\ &= \Big\{ \phi \in A^{p,q}(X) \mid d\phi = 0 \Big\} \Big/ d\, A^{n-1}(X) \cap A^{p,q}(X) . \end{aligned}$$

The following theorem is well known:

Theorem (Hodge Decomposition Theorem)**.** *Let X be a compact Kähler manifold of dimension n. Then the n-th complex de Rham cohomology group of X can be written as a direct sum*

$$H^n_{DR}(X,\mathbb{Z}) \otimes \mathbb{C} = H^n_{DR}(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

such that $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

Remark 2.1. We can define a filtration of $H^n_{DR}(X,\mathbb{C})$ by

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 = H = H^n_{DR}(X,\mathbb{C})$$

such that

$$H^{p,q}(X) = F^p \cap \overline{F}^q .$$

So the set $\{H^{p,q}(X)\}$ and $\{F^p\}$ are equivalent in defining the Hodge decomposition. In the remaining of this paper, we will use both notations interchangeably. □

Definition 2.1. A Hodge structure of weight j, denoted by $\{H_Z, H^{p,q}\}$, is given by a lattice H_Z of finite rank together with a decomposition on its complexification $H = H_Z \otimes \mathbb{C}$

$$H = \bigoplus_{p+q=j} H^{p,q}$$

such that

$$H^{p,q} = \overline{H^{q,p}} \ . \ \square$$

A polarized algebraic manifold is a pair (X, ω) consisting of an algebraic manifold X together with a Kähler form $\omega \in H^2(X, \mathbb{Z})$. Let

$$L\colon\ H^j(X, \mathbb{C}) \to H^{j+2}(X, \mathbb{C})$$

be the multiplication by ω, we recall below two fundamental theorems of Lefschetz:

Theorem (Hard Lefschetz Theorem). *On a polarized algebraic manifold (X, ω) of dimension n,*

$$L^k\colon\ H^{n-k}(X, \mathbb{C}) \to H^{n+k}(X, \mathbb{C})$$

is an isomorphism for every positive integer $k \leq n$.

From the theorem above, we know that

$$L^{n-j}\colon\ H^j(X, \mathbb{C}) \to H^{2n-j}(X, \mathbb{C})$$

is an isomorphism for $j \geq 0$. The primitive cohomology $P^j(X, \mathbb{C})$ is defined to be the kernel of L^{n-j+1} on $H^j(X, \mathbb{C})$.

Theorem (Lefschetz Decomposition Theorem). *On a polarized algebraic manifold (X, ω), we have for any integer j the following decomposition*

$$H^j(X, \mathbb{C}) = \bigoplus_{k=0}^{[\frac{n}{2}]} L^k P^{j-2k}(X, \mathbb{C}) \ .$$

It follows that the primitive cohomology groups determine completely the full complex cohomology.

In this paper we are only interested in the cohomology group $H^n_{DR}(X, \mathbb{C})$. Define

$$H_Z = P^n(X, \mathbb{C}) \cap H^n(X, \mathbb{Z})$$

and

$$H^{p,q} = P^n(X, \mathbb{C}) \cap H^{p,q}(X) \ .$$

Suppose that Q is the quadric form on $H^n_{DR}(X,\mathbb{C})$ induced by the cup product of the cohomology group. Q can be represented by

$$Q(\phi,\psi) \;=\; (-1)^{n(n-1)/2}\int \phi\wedge\psi\;.$$

Q is a nondegenerated form, and is skewsymmetric if n is odd and is symmetric if n is even. It satisfies the two Hodge–Riemannian relations

1) $Q(H^{p,q},H^{p',q'})=0$ unless $p'=n-p,\; q'=n-q$;

2) $(\sqrt{-1})^{p-q}\,Q(\phi,\overline{\phi})>0$ for any nonzero element $\phi\in H^{p,q}$.

Let H_Z be a fixed lattice, n an integer, Q a bilinear form on H_Z, which is symmetric if n is even and skewsymmetric if n is odd. And let $\{h^{p,q}\}$ be a collection of integers such that $p+q=n$ and $\sum h^{p,q}=\operatorname{rank} H_Z$. Let $H=H_Z\otimes\mathbb{C}$.

Definition 2.2. A polarized Hodge structure of weight n, denoted by $\{H_Z,F^p,Q\}$, is given by a filtration of $H=H_Z\otimes\mathbb{C}$

$$0\subset F^n\subset F^{n-1}\subset\cdots\subset F^0\subset H$$

such that

$$H=\;F^p\oplus\overline{F}^{n-p+1}$$

together with a bilinear form

$$Q\colon\; H_Z\otimes H_Z\to\mathbb{Z}$$

which is skewsymmetric if n is odd and symmetric if n is even such that it satisfies the two Hodge–Riemannian relations:

1) $Q(F^p,F^{n-p+1})=0$ unless $p'=n-p,\; q'=n-q$;

2) $(\sqrt{-1})^{p-q}\,Q(\phi,\overline{\phi})>0$ if $\phi\in H^{p,q}$ and $\phi\neq 0$
where $H^{p,q}$ is defined by

$$H^{p,q}=F^p\cap\overline{F}^q\;. \;\square$$

Definition 2.3. With the notations as above, the classifying space D for the polarized Hodge structure is the set of all the filtration

$$0\subset F^n\subset\cdots\subset F^1\subset H\,,\qquad \dim F^p=f^p$$

with $f^p=h^{n,0}+\cdots+h^{n,n-p}$ on which Q satisfies the Hodge–Riemannian relations as above. $\square$

D is a complex homogeneous space. Moreover, D can be written as $D = G/V$ where G is a noncompact semisimple Lie group and V is its compact subgroup. In general, D is *not* a homogeneous Kähler manifold.

3 – The canonical map and the horizontal distribution

In this section we study some elementary properties of classifying space and horizontal slice.

Suppose $D = G/V$ is a classifying space. We fix a point of D, say p, which can be represented by the subvector spaces of H

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset H$$

or the set

$$\left\{H^{p,q} \mid p+q=n\right\}$$

described in the previous section. We define the subspaces of H:

$$H^+ = H^{n,0} + H^{n-2,2} + \cdots\ ,$$
$$H^- = H^{n-1,1} + H^{n-3,3} + \cdots\ .$$

Suppose K is the subgroup of G such that K leaves H^+ invariant. Then we have

Lemma 3.1. *The identity component K_0 of K is the maximal connected compact subgroup of G containing V. In particular, V itself is a compact subgroup.*

Proof: Recall that $V \subset G \subset \operatorname{Hom}(H_R, H_R)$ is a *real* subgroup, where $H_R = H_Z \otimes \mathbb{R}$. Without losing generality, we assume V fixes p. Then we have

$$VF^p \subset F^p$$

where $p = 1, ..., n$. This implies that

$$V\overline{F^q} \subset \overline{F^q}$$

for $q = 1, ..., n$. So

$$VH^{p,q} = V(F^p \cap \overline{F^q}) \subset VF^p \cap V\overline{F^q} = H^{p,q}\ .$$

Thus V leaves H^+ invariant and thus $V \subset K$.

In order to prove that K_0 is a compact subgroup, we fix some $H^+, H^- \subset H$. Note that if $0 \neq x \in H^+$, then from the second Hodge–Riemannian relation

$$(\sqrt{-1})^n Q(x, \overline{x}) > 0 \ .$$

So for any norm on H^+, there is a $c > 0$ such that

$$\frac{1}{c}\,\|x\|^2 \ \geq\ (\sqrt{-1})^n Q(x, \overline{x}) \ \geq\ c\,\|x\|^2 \ .$$

For the same reason, we have

$$\frac{1}{c}\,\|x\|^2 \ \geq\ -(\sqrt{-1})^n Q(x, \overline{x}) \ \geq\ c\,\|x\|^2$$

for $x \in H^-$.

Let $g \in K_0$. For any x, let $x = x^+ + x^-$ be the decomposition of x into H^+ and H^- parts. Then

$$\|gx^{\pm}\|^2 \ \leq\ \pm\frac{1}{c}(\sqrt{-1})^n Q(gx^{\pm}, \overline{gx^{\pm}}) \ =\ \pm\frac{1}{c}(\sqrt{-1})^n Q(x^{\pm}, \overline{x^{\pm}}) \ \leq\ \frac{1}{c^2}\,\|x^{\pm}\|^2 \ .$$

Thus

$$\|g\| \leq C \ .$$

So the norm of the element of K_0 is uniformly bounded. Consequently, K_0 is a compact subgroup.

Suppose that $K' \supset K_0$ is a compact connected subgroup. Suppose $\mathfrak{k}'$ is the Lie algebra of K', then if K_0 is not maximal, there is a $\xi \in \mathfrak{k}'$ such that $\xi \notin \mathfrak{f}_0$ for the Lie algebra $\mathfrak{f}_0$ of K_0.

Suppose $\xi = \xi_1 + \xi_2$ is the decomposition for which

$$\begin{aligned} &\xi_1\colon\ H^+ \to H^+,\ \ H^- \to H^- \ , \\ &\xi_2\colon\ H^+ \to H^-,\ \ H^- \to H^+ \ . \end{aligned}$$

Then we have

Lemma 3.2. $\xi_1, \xi_2 \in \mathfrak{g}_R$ *for the Lie algebra* $\mathfrak{g}_R$ *of* G.

Proof: First we observe that

$$\begin{aligned} Q(H^+, H^+) = Q(H^-, H^-) = 0\,, &\qquad n \text{ odd} \ , \\ Q(H^+, H^-) = Q(H^-, H^+) = 0\,, &\qquad n \text{ even} \ , \end{aligned}$$

by the type consideration. Since Q is invariant under the action of G by definition, we have

$$Q(\xi x, y) + Q(x, \xi y) = 0 \ .$$

Thus

$$Q(\xi_1 x, y) + Q(x, \xi_1 y) + Q(\xi_2 x, y) + Q(x, \xi_2 y) \ = \ 0 \ .$$

If n is odd then if $x \in H^+$, $y \in H^+$ or $x \in H^-$, $y \in H^-$ then

$$Q(\xi_1 x, y) + Q(x, \xi_1 y) = 0$$

so in this case

$$Q(\xi_2 x, y) + Q(x, \xi_2 y) = 0$$

and if $x \in H^+$, $y \in H^-$ or $x \in H^-$, $y \in H^+$ then we have

$$Q(\xi_2 x, y) + Q(x, \xi_2 y) = 0$$

automatically. Thus we concluded

$$Q(\xi_2 x, y) + Q(x, \xi_2 y) = 0$$

for any $x, y \in H$. So $\xi_2 \in \mathfrak{g}_R$ and thus $\xi_1 \in \mathfrak{g}_R$.

The same is true if n is even. ∎

We define the Weil operator

$$C\colon\ H^{p,q} \to H^{p,q}\,, \qquad C|_{H^{p,q}} = (\sqrt{-1})^{p-q} \ .$$

Then we have

$$C|_{H^+} = (\sqrt{-1})^n\,, \qquad C|_{H^-} = -(\sqrt{-1})^n \ .$$

Let

$$Q_1(x, y) = Q(Cx, \overline{y}) \ .$$

Then we have

Lemma 3.3. *Q_1 is an Hermitian inner product.*

Proof: Let

$$x = x_1 + x_2$$

be the decomposition of x such that $x_1 \in H^+$ and $x_2 \in H^-$.

If n is odd, then $\overline{x_2} \in H^+$. So $Q(x_1, \overline{x_2}) = 0$; if n is even, then $\overline{x_2} \in H^-$. So $Q(x_1, \overline{x_2}) = 0$.

If $x \neq 0$ we have

$$\begin{aligned} Q_1(x,x) &= Q_1(x_1,x_1) + Q_1(x_2,x_2) + Q_1(x_1,x_2) + Q_1(x_2,x_1) \\ &= Q_1(x_1,x_1) + Q_1(x_2,x_2) \\ &= (\sqrt{-1})^n \left(Q(x_1,\overline{x_1}) - Q(x_2,\overline{x_2}) \right) \ > \ 0 \ . \end{aligned}$$

Thus $Q_1(\cdot,\cdot)$ is a Hermitian product on H. Furthermore, it defines an inner product on $H_R = H_Z \otimes \mathbb{R}$. ∎

Now back to the proof of Lemma 3.1, we have

$$Q_1(\xi_2 x, y) = Q(C\xi_2 x, \overline{y}) = -Q(\xi_2 Cx, \overline{y}) = Q(Cx, \xi_2 \overline{y}) = Q_1(x, \xi_2 y) \ .$$

Thus ξ_2 is a Hermitian metrics under the metric Q_1. Since K' is a compact group, there is a constant C such that

$$\|\exp(t\,\xi_2)\| \leq C < +\infty$$

for all $t \in \mathbb{R}$ which implies $\xi_2 = 0$. ∎

Lemma 3.4. *Let*

$$D_1 = \left\{ H^{n,0} + H^{n-2,2} + \cdots \mid \{H^{p,q}\} \in D \right\} \ .$$

Then the group G acts on D_1 transitively with the stable subgroup K_0, and D_1 is a symmetric space.

Proof: For $x, y \in D_1$, let $H_x^{p,q}, H_y^{p,q}$ be the corresponding points in D. Since D is homogeneous, we have a $g \in G$ such that

$$g\{H_x^{p,q}\} \ = \ H_y^{p,q} \ .$$

So $gx = y$. This proves that G acts on D_1 transitively. By definition, K_0 fixes the H^+ of the fixed point $p \in D$. By Lemma 3.1, D_1 is a symmetric space. ∎

Definition 3.1. We call the map p

$$p\colon\, G/V \to G/K_0 \,, \qquad \{H^{p,q}\} \mapsto H^{n,0} + H^{n-2,2} + \cdots$$

the natural projection of the classifying space. □

There are universal holomorphic bundles $\underline{F}^n, ..., \underline{F}^1, \underline{H}$ over D, namely we assign any point p of D the linear space

$$0 \subset F^n \subset \cdots \subset F^1 \subset H$$

or in other words, assign every point of D the space $H = H_Z \otimes \mathbb{C}$, with the Hodge decomposition

$$H = \sum H^{p,q} .$$

It is well known that the holomorphic tangent bundle $T(D)$ can be realized by

$$T(D) \subset \bigoplus \mathrm{Hom}(F^p, H/F^p) = \bigoplus_{r>0} \mathrm{Hom}(H^{p,q}, H^{p-r,q+r})$$

such that the following compatible condition holds

$$\begin{array}{ccc} F^p & \longrightarrow & F^{p-1} \\ \downarrow & & \downarrow \\ H/F^p & \longrightarrow & H/F^{p-1} \end{array}$$

We define a subbundle $T_h(D)$ called the horizontal bundle of D, by

$$T_h(D) = \left\{ \xi \in T(D) \mid \xi F^p \subset F^{p-1} \right\} .$$

$T_h(D)$ is called the horizontal distribution of D. The properties of the horizontal bundle or the horizontal distribution play an important role in the theory of moduli space.

Let $\mathfrak{g}_R$ be the Lie algebra of G. Suppose

$$\mathfrak{g}_R = \mathfrak{k}_0 + \mathfrak{p}_0$$

is the Cartan decomposition of $\mathfrak{g}_R$ into the compact and noncompact part.

Lemma 3.5. *If we identify $T_0(G)$ with the Lie algebra $\mathfrak{g}_R$. Then*

$$E \subset \mathfrak{p}_0$$

where E is the fiber of $T_h(D)$ at the original point.

Proof: Suppose

$$\{0 \subset f^n \subset f^{n-1} \subset \cdots f^1 \subset H\} \quad \text{or} \quad \{h^{p,q}\}$$

is the set of subspace representing the point eV of $D = G/V$. Suppose $X \in E$.

Then $X \in E$ if

$$X\colon\ f^k \to f^{k-1}\ .$$

Let $X = X_1 + X_2$ be the Cartan decomposition with $X_1 \in \mathfrak{k}_0$, and $X_2 \in \mathfrak{p}_0$. Let

$$\begin{aligned} h^+ &= h^{n,0} + h^{n-2,2} + \cdots \\ h^- &= h^{n-1,1} + h^{n-3,3} + \cdots \end{aligned}$$

be the subspaces of H.

By definition $X_1 \in \mathfrak{k}_0$, we see that

$$X_1\colon\ \ h^+ \to h^+,\quad h^- \to h^-\ .$$

Since X maps f^k to f^{k-1}, so does X_1. So X_1 must leave f^k invariant because X_1 sends h^+ to h^+, and h^- to h^-.

From the above argument we see that $X_1 \in \mathfrak{v}$, the Lie algebra of V. Thus the action X on the classifying space is the same as X_2. But $X_2 \in \mathfrak{p}_0$. This completes the proof. ∎

On the other hand, $\forall\, h \in V$, $X \in E$, we have $Ad(h)X \in E$. So there is a representation

$$\rho\colon\ V \to \mathrm{Aut}(E),\quad h \mapsto Ad(h)\ .$$

Suppose T' is the homogeneous bundle

$$T' = G \times_V E$$

whose local section can be represented as C^∞ functions

$$f\colon\ G \to E$$

which is V equivariant

$$f(ga) = Ad(a^{-1})\, f(g)$$

for $a \in V,\ g \in G$. Our next lemma is

Lemma 3.6.

$$T' = T_h(D)\ .$$

Proof: What we are going to prove is that both vector bundles will be coincided as subbundles of $T(D)$.

Suppose $\xi \in T'_{gV}$ for $g \in G$ where T'_{gV} is the fiber of T' at gV. Then ξ can be represented as

$$\xi = (g, \xi_1) \quad \text{for} \ \ \xi_1 \in E \ .$$

So the 1-jet in the ξ direction is $(g + \varepsilon\, g\, \xi_1)\, V$ for ε small. Such a point is

$$(g + \varepsilon\, g\, \xi_1)\, \{f^p\} = (1 + \varepsilon\, g\, \xi_1\, g^{-1})\, \{F^p\}$$

where $\{F^p\} = g\,\{f^p\}$.

Suppose $\xi_2 = g\, \xi_1\, g^{-1}$, then

$$\xi_2\, F^p \subset F^{p-1} \ .$$

Thus $\xi_2 \in (T_h)_{gV}(D)$ and

$$T'_{gV} \subset (T_h)_{gV}(D) \ .$$

Thus

$$T' \subset T_h(D)$$

and T' is the subbundle of $T_h(D)$. But since they coincides at the origin, they are equal. ∎

Corollary 3.1. *Suppose $T_v(D)$ is the distribution of the tangent vectors of the fibers of the natural projection*

$$p\colon\ D \to G/K$$

then

$$T_v(D) \cap T_h(D) = \{0\} \ .$$

Proof:

$$T_v(D) = G \times_V \mathfrak{v}_1$$

where $\mathfrak{k}_0 = \mathfrak{v} + \mathfrak{v}_1$ and $\mathfrak{v}_1$ is the orthonormal complement of the Lie algebra $\mathfrak{v}$ of V. ∎

Definition 3.2. Let U be a complex manifold. If $U \subset D$ is a complex submanifold such that $T(U) \subset T_h(D)|_U$. Then we say that U is a horizontal slice. If

$$f\colon\ U \to D$$

is an immersion and $f(U)$ is a horizontal slice, then we say that (U, f) or U is a horizontal slice. In a word, a horizontal slice U of D is a complex integral submanifold of the distribution $T_h(D)$. □

Because to become a horizontal slice is a local property, we make the following definition:

Definition 3.3. Suppose Γ is a discrete subgroup of U and suppose $\Gamma \subset G$ for $D = G/V$. Then if $U \to D$ is a horizontal slice, we also say that $\Gamma\backslash U$ is a horizontal slice. □

Corollary 3.2. *If $f\colon U \to D$ is a horizontal slice, then*

$$p\colon U \subset D \to G/K_0$$

is an immersion, where $p\colon D \to G/K_0$ is the natural projection in Definition 3.1. ■

4 – A metric rigidity theorem

In this section, we prove that, for concave horizontal slices, the Hodge metric is intrinsically defined. That is, the Hodge metric does not depend on the immersion to the classifying space.

To be precise, suppose $\Gamma\backslash U \to \Gamma\backslash D$ is a horizontal slice. Then we can define the Hodge metric on $\Gamma\backslash U$. But as a complex manifold, the horizontal immersion $\Gamma\backslash U \to \Gamma\backslash D$ may not be unique. If a metric defined on $\Gamma\backslash U$ is independent of the choice of the immersion, we say such a metric is defined intrinsically.

For the moduli space of a Calabi–Yau threefold, the Hodge metric is defined intrinsically by the main result in [5]. It is interesting to ask if the property is true for general horizontal slices.

Definition 4.1. The classifying space D, as a homogeneous complex manifold, has a natural invariant Kähler form ω_H. In general, $d\omega_H \neq 0$. However, if $U \to D$ is a horizontal slice, then $d\omega_H = 0$ (cf. [6]). The metric $\omega_H|_U$ is called the Hodge metric. □

Definition 4.2. We say a complex manifold M is *concave*, if there is an exhaustion function φ on M such that the Hessian of φ has at least two negative eigenvalues at each point outside some compact set. □

Any pluriharmonic function on a concave manifold is a constant.

Suppose $f_i\colon U \to D,\ i = 1,2$ are two horizontal slices. Suppose we have $\Gamma \in \operatorname{Aut}(U), \Gamma_0 \in \operatorname{Aut} D$ and we have the group homomorphism

$$\rho\colon \Gamma \to \Gamma_0 \ ,$$

such that

$$f_i(\gamma x) = \rho(\gamma)\, f_i(x)\,, \qquad i = 1,2, \quad \gamma \in \Gamma, \quad x \in U \ ,$$

where the action $\rho(\gamma)$ on D is the left translation.

The main results of this section are the following two theorems:

Theorem 4.1. *With the notations as above, suppose that $\Gamma\backslash U$ has no non-constant pluriharmonic functions. Then there is an isometry $f\colon\ f_1(U) \to f_2(U)$ such that $f \circ f_1 = f_2$.*

Proof of Theorem 4.1: Let $D_1 = G/K_0$ be the symmetric space defined in Lemma 3.4. We denote $\tilde{f}_1\colon U \to G/K_0$ and $\tilde{f}_2\colon U \to G/K_0$ to be the two natural projections, that is $\tilde{f}_i = p \circ f_i$ where p is defined in Definition 3.1. By Corollary 3.2, both maps are immersions. Let

$$g\colon U \to \mathbb{R}\,, \qquad g(x) = d(f_1(x), f_2(x))$$

where $d(\cdot,\cdot)$ is the distance function of G/K_0. Thus since G/K_0 is a Cartan–Hardamad manifold, $g(x)$ is smooth if $g(x) \neq 0$.

Let $p \in U$ and $X \in T_pU$. Let $X_1 = (\tilde{f}_1)_{*p}X,\ X_2 = (\tilde{f}_2)_{*p}X$. Let σ be the geodesic ray starting at p with vector X. i.e.

$$\begin{cases} \sigma''(t) = 0, \\ \sigma(0) = p, \quad \sigma'(0) = X \ . \end{cases}$$

Suppose the smooth function $\sigma(s,t)$ is defined as follows: for fixed s, $\sigma(s,t)$ is the geodesic in G/K connecting $\tilde{f}_1(\sigma(s))$ and $\tilde{f}_2(\sigma(s))$. Furthermore, we assume that $\sigma(0,t)$ is normal. i.e. t is the arc length. Define

$$\tilde{X}(s) \;=\; \frac{d}{ds}\bigg|_{s=0} \sigma(s,t)$$

be the Jacobi field of the variation. In particular

$$\begin{cases} \tilde{X}(0) = X_1, \\ \tilde{X}(l) = X_2 \ , \end{cases}$$

where $l = g(x)$. Suppose T is the tangent vector of $\sigma(0,t)$, we have the second variation formula

$$\begin{aligned} XX(g)|_p &= \langle \nabla_{X_2} X_2, T\rangle - \langle \nabla_{X_1} X_1, T\rangle \\ &\quad + \int_0^l |\nabla_T \tilde{X}|^2 - R(T,\tilde{X},T,\tilde{X}) - (T\langle \tilde{X}, T\rangle)^2 \end{aligned}$$

where ∇ is the connection operator on G/K_0 and $R(\cdot,\cdot,\cdot,\cdot)$ is the curvature tensor.

We also have the first variation formula

$$Xg = \langle (\tilde{f}_2)_* X, T\rangle - \langle (\tilde{f}_1)_* X, T\rangle \; .$$

By [6, Theorem 1.1], we know f_i $(i = 1,2)$ are pluriharmonic. That is, we have the following

$$\nabla_{(\tilde{f}_i)_* X} (\tilde{f}_i)_* \, X + \nabla_{(\tilde{f}_i)_* JX} (\tilde{f}_i)_* \, JX + (\tilde{f}_i)_* \, J[X, JX] = 0$$

for $i = 1, 2$.

Define

$$D(X,X) = XXg + (JX)\,(JX)g + J[X,JX]\,g \; .$$

Using the fact that J is ∇-parallel, we see

$$\begin{aligned} D(X,X)g &= \int_0^l |\tilde{X}'|^2 - R(T,\tilde{X},T,\tilde{X}) - (T\langle \tilde{X}, T\rangle)^2 \\ &\quad + \int_0^l |\widetilde{JX}'|^2 - R(T,\widetilde{JX},T,\widetilde{JX}) - (T\langle \widetilde{JX}, T\rangle)^2 \end{aligned}$$

where $\widetilde{JX}$ is the Jacobi connecting $\tilde{f}_1(J\sigma(t))$ and $\tilde{f}_2(J\sigma(t))$.

Claim: If $g(x) \neq 0$, then Hessian of g at x is semipositive.

Proof: Let $(\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n})$ be the holomorphic normal frame at $p \in U$. In order to prove g is plurisubharmonic, it suffices to prove that $\frac{\partial^2 g}{\partial z_i \, \partial \overline{z_i}} \geq 0$. But

$$4\,\frac{\partial^2 g}{\partial z_i \, \partial \overline{z_i}} = \frac{\partial^2 g}{\partial x_i^2} + \frac{\partial^2 g}{\partial y_i^2} = D\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\Big)\, g \; .$$

Let $X = \frac{\partial}{\partial x_i}$ in the second variation formula. Since the curvature of the symmetric space is nonpositive,

$$D\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\Big)\, g \geq \int_0^l |\tilde{X}'|^2 - (T\langle \tilde{X}, T\rangle)^2 + |\widetilde{JX}'|^2 - (T\langle \widetilde{JX}, T\rangle)^2$$

because the curvature operator is nonpositive. On the other hand

$$
\begin{aligned}
&|X'|^2 - (T\langle \tilde{X}, T\rangle)^2 \;=\; |\tilde{X}' - \langle \tilde{X}', T\rangle\, T|^2 \;\ge\; 0\;, \\
&|JX'|^2 - (T\langle \widetilde{JX}, T\rangle)^2 \;=\; |\widetilde{JX}' - \langle \widetilde{JX}', T\rangle\, T|^2 \;\ge\; 0\;.
\end{aligned}
\tag{4.1}
$$

Thus g is plurisubharmonic if $g(x) \neq 0$.

$g^2(x)$ is a smooth function on U. It is easy to see that g^2 is a plurisubharmonic function. But g^2 is also Γ-invariant so it descends to a function on $\Gamma\backslash U$. Thus g^2 and g must be constant.

Since g is a constant, by Equation (4.1) and the second variational formula, we have

$$
\begin{cases}
\tilde{X}' - \langle \tilde{X}', T\rangle\, T \;=\; 0, \\
R(T, \tilde{X}, T, \tilde{X}) = 0\;.
\end{cases}
$$

Moreover, by the first variational formula,

$$\langle \tilde{X}, T\rangle(0) \;=\; \langle \tilde{X}, T\rangle(l)\;.$$

Since $\tilde{X}$ is a Jacobi field, $\tilde{X}'' \equiv 0$. Furthermore, by the above equations, we have $\tilde{X}' \equiv 0$.

This proves that there is an isometry

$$\tilde{f} \quad \tilde{f}_1(U) \to \tilde{f}_2(U)\,, \qquad \tilde{f}_1(x) \mapsto \tilde{f}_2(x)$$

which sends $\tilde{f}_1(x)$ to $\tilde{f}_2(x)$ and thus we have $\tilde{f} \circ \tilde{f}_1 = \tilde{f}_2$.

The theorem follows from the fact that $f_i(U)$ and $\tilde{f}_i(U)$ are isometric for $i = 1, 2$. ∎

5 – Local rigidity of the group representation

In this section we study the monodromy group representation on a horizontal slice.

We assume that U is a horizontal slice. Let $\Gamma \subset \mathrm{Aut}(U)$ be a discrete group. Suppose $\Gamma\backslash U$ is of finite volume with respect to the Hodge metric.

For the sake of simplicity, we assume that Γ is also the subgroup of the left translation of $D = G/V$, the classifying space. There is a natural map $\Gamma\backslash U \to \Gamma\backslash G/K_0$ where G/K_0 is the symmetric space of $D = G/V$ as in Definition 3.1. Let

$$\mathcal{G} \;=\; \Big\{ a \in G \mid a \in \mathrm{Aut}(U) \Big\}\;.$$

Let $\mathcal{G}_0$ be the identity component of $\mathcal{G}$.

The main theorem of this section is

Theorem 5.1. *Let $\Gamma\backslash U$ be of finite Hodge volume. Suppose further that $\mathcal{G}_0$ is semisimple and $\mathcal{G}_0/\mathcal{K}_0$ is a Hermitian symmetric space but is not a complex ball, where $\mathcal{K}_0$ is the maximum compact subgroup of $\mathcal{G}_0$. Then the representation $\Gamma \to \mathcal{G}$ is locally rigid.*

By local rigidity we mean that if $\rho_t : \Gamma \to \mathcal{G}$ is a continuous set of representations for $t \in (-\epsilon, \epsilon)$, then there is an a_t for any $|t| < \epsilon$ such that $\rho_t = Ad(a_t)\,\rho_0$.

Before proving the rigidity theorem, we make the following assumption. We postpone the proof of the assumption to the end of this section.

Assumption 5.1. *Let $\mathcal{K}_0$ be a maximal compact subgroup of $\mathcal{G}_0$. Suppose $\Gamma_1 = \Gamma \cap \mathcal{G}_0$. We assume that $\Gamma_1 \backslash \mathcal{G}_0/\mathcal{K}_0$ has finite volume with respect to the standard Hermitian metric on $\mathcal{G}_0/\mathcal{K}_0$. In this case, we will call Γ_1 has finite covolume.*

We prove a series of lemmas.

Let

$$\mathcal{G}_1 = \Gamma + \mathcal{G}_0$$

be the group generated by Γ and $\mathcal{G}_0$ in G.

Let

$$\Gamma_1 = \Gamma \cap \mathcal{G}_0 \ .$$

Lemma 5.1. *Let $\pi : U \to \Gamma\backslash U$ be the projection. Then for any $x \in U$, the projection of the $\mathcal{G}_0$ orbit $\pi(\mathcal{G}_0 x)$ is a closed, locally connected, properly embedded smooth submanifold of $\Gamma\backslash U$.*

Proof (cf. [2])**:** $\mathcal{G}_0 x$ is a closed properly embedded, locally connected smooth submanifold of U, we claim:

Claim: $\pi^{-1}(\pi(\mathcal{G}_0 x)) = \mathcal{G}_1 x$.

Proof: We know that $\mathcal{G} \subset N(\mathcal{G}_0)$, the normalizer of $\mathcal{G}_0$ in $\mathcal{G}$. So $\forall\, \xi \in \mathcal{G}_0$, $b \in \Gamma$, there is a $\eta \in \mathcal{G}_0$ such that $b\,\xi = \eta\, b$. Thus $\forall\, g \in \mathcal{G}_1$, $g = g_1 g_2$ where $g_1 \in \Gamma$ and $g_2 \in \mathcal{G}_0$. So

$$\pi(gx) = \pi(g_1 g_2\, x) = \pi(g_2 x) \in \pi(\mathcal{G}_0 x) \ .$$

Thus $gx \in \pi^{-1}\pi(\mathcal{G}_0 x)$.

On the other hand, if $y \in \pi^{-1}\pi(\mathcal{G}_0 x)$, then $\pi(y) \in \pi(\mathcal{G}_0 x)$, thus by definition, $y \in \mathcal{G}_1 x$.

Since $\mathcal{G}_1 x$ is a properly embedded, locally connected smooth submanifold of U and $\mathcal{G}_1 x$ is Γ invariant. The lemma is proved by observing $\pi(\mathcal{G}_1 x) = \pi(\mathcal{G}_0 x)$. ∎

In order to prove Theorem 5.1, we use the following famous theorem of Margulis [7] about the superrigidity of symmetric spaces:

Theorem (Margulis)**.** *Suppose that $\mathcal{G}_0$ is defined as above. If Γ_1 is of finite covolume, then for any homomorphism*

$$\varphi\colon\ \Gamma_1 \to \Gamma_1$$

there is a unique extension

$$\tilde{\varphi}\colon\ \mathcal{G}_0 \to \mathcal{G}_0$$

of group homomorphism.

The following lemma is a straightforward consequence of the above theorem of Margulis.

Lemma 5.2. *If $x \in \mathcal{G}_0$ such that*

$$xy = yx$$

for all $y \in \Gamma_1$, then $x = e$.

Proof: Let $\varphi\colon\ \Gamma_1 \to \Gamma_1$ by $y \to xyx^{-1}$. Then φ has an extension $\tilde{\varphi}\colon\ \mathcal{G}_0 \to \mathcal{G}_0$. This extension is unique. So we must have $\tilde{\varphi}(y) = xyx^{-1} = y$. Since $\mathcal{G}_0$ is semisimple, we have $x = e$. ∎

Lemma 5.3. *Let $\Gamma_1 = \Gamma \cap \mathcal{G}_0$, then*

$$\mathrm{Out}(\Gamma_1)/\,\mathrm{Inn}(\Gamma_1)$$

is a finite group.

Here $\mathrm{Out}(\Gamma_1)$ denotes the group of isomorphisms of Γ_1 and $\mathrm{Inn}(\Gamma_1)$ denotes the group of conjugations of Γ_1.

Proof: Let $\varphi\colon \Gamma_1 \to \Gamma_1$ be an element in $\mathrm{Out}(\Gamma_1)$. Since Γ_1 has finite covolume, we know there is a unique extension $\tilde{\varphi}\colon\ \mathcal{G}_0 \to \mathcal{G}_0$.

Thus $\tilde{\varphi} \in \mathrm{Out}(\mathcal{G}_0)$. Since $\mathrm{Out}(\mathcal{G}_0) = \mathrm{Inn}(\mathcal{G}_0)$ because $\mathcal{G}_0$ is semisimple, there is a $b \in \mathcal{G}_0$ such that $\tilde{\varphi}(x) = bxb^{-1}$. Define

$$\tilde{\varphi}\colon \mathcal{G}_0/\mathcal{K}_0 \to \mathcal{G}_0/\mathcal{K}_0\,, \qquad a\mathcal{K}_0 \to ba\mathcal{K}_0\ .$$

It is a Γ-equivariant holomorphic map. The lemma then follows from the following proposition. ∎

Proposition 5.1. *Suppose* $\Gamma\backslash G/K$ *is of finite volume, then* $\mathrm{Aut}(\Gamma\backslash G/K)$ *is a finite group.*

Proof: Since $\Gamma\backslash G/K$ is a Hermitian symmetric space, we know $\mathrm{Aut}(\Gamma\backslash G/K)$ is the same as $\mathrm{Iso}(\Gamma\backslash G/K)$.

Suppose $\mathrm{Iso}(\Gamma\backslash G/K)$ is not finite. Then we have a sequence of isometries $f_1, f_2, \ldots$. Let $p \in \Gamma\backslash G/K$ be a fixed point and let V be a normal coordinate neighborhood of p. The we know that $\{f_i(p)\}$ must be bounded, otherwise there is a subsequence of f_i such that $f_i(U)$ will be mutually disjoint. This will contradict to the fact that $\Gamma\backslash G/K$ has finite volume, because

$$\mathrm{vol}(\Gamma\backslash G/K)\ \geq\ \sum \mathrm{vol}(f_i(U))\ =\ +\infty\ .$$

A contradiction. Let $q = \lim f_i(p)$. For any $x \in \Gamma\backslash G/K$, if i is large enough such that $d(f_i(p), q) < 1$, then

$$d(f_i(x), q)\ \leq\ d(f_i(x), f_i(p)) + 1\ =\ d(x,p) + 1\ .$$

By Ascoli theorem, there is a subsequence of f_i such that f_i converges to an $f \in \mathrm{Iso}(\Gamma\backslash G/K)$. Thus $\mathrm{Iso}(\Gamma\backslash G/K)$ is not discrete. So there is a holomorphic vector field X on $\Gamma\backslash G/K$.

Suppose $X = X^i \frac{\partial}{\partial z^i}$ in local coordinate, and $\|X\|^2 = G_{i\bar{j}} X^i \overline{X^j}$. Suppose the local coordinate is normal, then

$$\partial_k \overline{\partial}_l \|X\|^2 = R_{i\bar{j}k\bar{l}} X^i \overline{X}^j + \partial_k X^i \overline{\partial_l X^i} \tag{5.1}$$

where $R_{i\bar{j}k\bar{l}}$ is the curvature tensor of the symmetric space. Thus in particular $\partial\overline{\partial}\|X\|^2 \geq 0$.

By the theorem of [1], $\Gamma\backslash G/K$ is a concave manifold. Thus $\|X\|^2$ is a constant. On the other hand, from equation (5.1), we have

$$\Delta\|X\|^2 =\ \mathrm{Ric}(X) + |\nabla X|^2\ .$$

So $\mathrm{Ric}(X) = 0$ and thus $X \equiv 0$. This is a contradiction. ∎

From the above proposition, there is an integer n such that $a^n = e$ for all

$$a \in \operatorname{Out}(\Gamma_1)/\operatorname{Inn}(\Gamma_1)\ .$$

Let $\tilde{\Gamma}$ be the subgroup of $\mathcal{G}_1$ generated by Γ_1 and a^n where $a \in \Gamma$. Then we have an exact sequence

$$1 \to \Gamma_1 \to \tilde{\Gamma} \to \tilde{B} \to 1 \tag{5.2}$$

where $\tilde{B}$ is the quotient $\tilde{\Gamma}/\Gamma_1$. For any $\overline{b} \in \tilde{\Gamma}/\Gamma_1$ with $b \in \tilde{\Gamma}$, we have $b\,\Gamma_1 b^{-1} \subset \Gamma_1$. So $b \in \operatorname{Out}(\Gamma_1)$. But by the definition of $\tilde{\Gamma}$, b is a trivial element in $\operatorname{Out}(\Gamma_1)/\operatorname{Inn}(\Gamma_1)$. So there is a $c \in \Gamma_1$ such that bc is commutative to Γ_1. So there is a homomorphism

$$\eta\colon\ B \to \tilde{\Gamma}\,, \qquad \overline{b} \mapsto b\,c\ .$$

We can thus define a homomorphism

$$\xi\colon\ \Gamma_1 \times \tilde{\Gamma}/\Gamma_1 \to \tilde{\Gamma}$$

such that

$$\xi(a,b) = a\,\eta(b)$$

which is an isomorphism. In other words, the exact sequence (5.2) splits.

Lemma 5.4. *Let* $\tilde{\mathcal{G}}_1 = \tilde{\Gamma} + \mathcal{G}_0$. *Then*

$$\tilde{\mathcal{G}}_1 = \mathcal{G}_0 \times \tilde{B}\ .$$

Proof: We have

$$\tilde{\Gamma} = \Gamma_1 \times \tilde{B}\ .$$

Define

$$\varphi\colon\ \mathcal{G}_0 \times \tilde{B} \to \tilde{\mathcal{G}}_1\,, \qquad \varphi(a,b) = ab\ .$$

Then φ is an isomorphism. ∎

Thus we know a family of representation of $\tilde{\Gamma}$ splits to the representation to the discrete group $\tilde{B}$ and Lie group $\mathcal{G}_0$ respectively.

Lemma 5.5. *If the representation* $\tilde{\Gamma} \to \tilde{\mathcal{G}}_1$ *is locally rigid, then the representation* $\Gamma \to \tilde{\mathcal{G}}$ *is also locally rigid.*

Proof: Let $\varphi_t : \Gamma \to \tilde{\mathcal{G}}$ be a local family of representations, $t \in (-\epsilon, \epsilon)$. Then we see that φ_t restricts to a trivial family of representations on $\tilde{\Gamma}$. That is, there are $a_t \in \tilde{\mathcal{G}}_1 \subset \tilde{\mathcal{G}}$ with $a_0 = e$ such that $\varphi_t(x) = a_t \, \varphi_0(x) \, a_t^{-1}$ for $x \in \tilde{\Gamma}$. Let $\xi_t = Ad(a_t^{-1}) \, \varphi_t$. Then we know $\xi_t(x) = \varphi_0(x)$ for all $x \in \tilde{\Gamma}$. Now if $x \in \Gamma$, then $x^n \in \tilde{\Gamma}$. So we have $(\xi_t(x))^n = (\varphi_0(x))^n$ and $\xi_0(x) = \varphi_0(x)$. Thus $\xi_t(x) = \varphi_0(x)$. ∎

In the rest of this section, we prove Assumption 5.1.

Lemma 5.6. *$\mathcal{G}_1$ is a closed subgroup of G.*

Proof: We know that $\mathcal{G}_1 \subset \mathcal{G}$. Let $x_m \in \mathcal{G}_1$ such that $x_m \to x$ for $x \in G$. Then $x \in \mathrm{Aut}(\mathcal{M})$ so $x \in \mathcal{G}$. Thus for sufficient large m, x_m and x are in the same component. In particular, we have $x \in \mathcal{G}_1$. ∎

Lemma 5.7. *Let $p \in U$, we have*

$$\inf_{q \in \mathcal{G}_1 \setminus \mathcal{G}_0} d(qp, \mathcal{G}_0 p) \; > \; 0 \; .$$

Proof: Suppose the assertion is not true, then we have $\{q_m\} \in \mathcal{G}_1$ and $g_m \in \mathcal{G}_0$ such that

$$d(q_m \, p, \; g_m \, p) \to 0 \, , \qquad m \to +\infty$$

or

$$d(g_m^{-1} q_m \, p, \; p) \to 0 \, , \qquad m \to +\infty \; .$$

It is easy to check that $\mathcal{G}_0 p$ is a homogeneous manifold, with compact stable group.

Thus, there are $k_m \in \mathcal{K}_0$, a compact subgroup of $\mathcal{G}_0$ such that

$$g_m^{-1} q_m \, k_m \to e \; .$$

So by passing a subsequence if necessary, we know

$$g_m^{-1} q_m \to \; g \in \mathcal{K}_0 \subset \mathcal{G}_0 \; .$$

This contradicts the fact that $\mathcal{G}_0$ is open.

Let $x, y \in U$. Let

$$L_1 = \mathcal{G}_0 x \, , \qquad L_2 = \mathcal{G}_0 y$$

be the two $\mathcal{G}_0$ orbits. We can define

$$f(p) = d(p, L_2)$$

be the distance of a point $p \in L_1$ to L_2. ∎

Lemma 5.8. *$f(p)$ is a constant.*

Proof: Let $p, q \in L_1$. Then there is a $g \in \mathcal{G}_0$ such that $q = gp$. We have

$$d(q, L_2) \leq d(q, \xi) = d(gp, \xi) = d(p, g^{-1}\xi) \ .$$

This proves

$$d(q, L_2) \leq d(p, L_2) \ .$$

On the other hand, we also have

$$d(q, L_2) \leq d(p, L_2) \ .$$

Thus $d(q, L_2) = d(p, L_2)$ and $f(p)$ is a constant. ∎

Define the distance between two orbits L_1, L_2 by $d(L_1, L_2) = f(p)$. If $L_1 \neq L_2$, $d(L_1, L_2) > 0$.

Let $\overline{a} \in \mathcal{G}_1/\mathcal{G}_0$. Then $\overline{a}L$ defines another orbit. So we have a map

$$\mathcal{G}_1/\mathcal{G}_0 \to \mathbb{R} , \quad \overline{a} \mapsto d(\overline{a}L, L) \ .$$

We know that $d(\overline{a}L, L) > 0$ for $\overline{a} \neq 0$. Furthermore we have

Lemma 5.9.

$$\varepsilon \ = \ \inf_{\overline{a} \neq 0} d(\overline{a}L, L) \ > \ 0 \ .$$

Proof: This is a consequence of the previous two lemmas. ∎

For any orbit $\mathcal{G}_0 p$, $\Gamma \backslash \mathcal{G}_0 p$ is a closed, properly embedded submanifolds (Lemma 5.1). We fix one of them, say L.

Let

$$W = \left\{ x \in U \mid d(x, L) < \frac{\varepsilon}{100} \right\}$$

where ε is defined in the previous lemma. Then for any $a \in \mathcal{G}_1 \backslash \mathcal{G}_0$, $aU \cap U = \emptyset$. In particular

$$\Gamma \backslash W = \Gamma_1 \backslash W$$

in $\Gamma \backslash U$.

Now that

$$\mathrm{vol}(\Gamma \backslash U) \ \geq \ \mathrm{vol}(\Gamma \backslash W) \ = \ \mathrm{vol}(\Gamma_1 \backslash W) \ .$$

For any $p \in \Gamma_1 \backslash U$, there is a unique $q \in L$ such that

$$d(p, q) = d(p, L) \ .$$

Now we can prove the following proposition which implies the assumption:

Proposition 5.2. *If* $\mathrm{vol}(\Gamma\backslash U) < +\infty$, *then*

$$\mathrm{vol}(\Gamma\backslash L) < +\infty\ .$$

Proof: Let $f(p) = d(p, L)$. Then by the coarea formula

$$\mathrm{vol}(\Gamma_1\backslash U)\ = \int_0^\varepsilon \left(\int_{f=c} \frac{1}{|\nabla f|}\right) dc\ .$$

But $|\nabla f| \le 1$. So

$$\mathrm{vol}(\Gamma_1\backslash U)\ \ge \int_0^\epsilon \mathrm{vol}(f=c)\, dc$$

so at least there is a c s.t.

$$\mathrm{vol}(f=c)\ <\ \infty\ .$$

Note that $\dim\{f=c\} = \dim U - 1$. The proposition then follows from the induction.

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