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HYPERSURFACES OF INFINITE DIMENSIONAL BANACH SPACES, BERTINI THEOREMS AND EMBEDDINGS OF PROJECTIVE SPACES

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Abstract: Let V, E be infinite dimensional Banach spaces, $\mathbf{P}(V)$ the projective space of all one-dimensional linear subspaces of V, W a finite codimensional closed linear subspace of $\mathbf{P}(V)$ and $X \subset \mathbf{P}(V)$ a closed analytic subset of finite codimension such that $\mathbf{P}(W) \subset X$ and X is not a linear subspace of $\mathbf{P}(V)$. Here we show that X is singular at some point of $\mathbf{P}(W)$. We also prove that any closed embedding $j\colon \mathbf{P}(V) \to \mathbf{P}(E)$ with $j(\mathbf{P}(V))$ finite codimensional analytic subset of $\mathbf{P}(E)$ is a linear isomorphism onto a finite codimensional closed linear subspace of $\mathbf{P}(E)$.

1 – Introduction

For any locally convex and Hausdorff complex topological vector space V let $\mathbf{P}(V)$ be the projective space of all one-dimensional linear subspaces of V. In section 2 we will prove the following result.

Theorem 1. *Let V be an infinite dimensional complex Banach space, W a finite codimensional closed linear subspace of V and $X \subset \mathbf{P}(V)$ a finite codimensional closed analitic subset such that $M := \mathbf{P}(W) \subseteq X$. Assume that X is not a linear subspace of $\mathbf{P}(V)$. Then X is singular and its singular locus* $\mathrm{Sing}(X)$ *contains a closed finite codimensional analytic subset T of M.*

By [6], Th. III.3.1.1, $\mathrm{Sing}(X)$ is a closed analytic subset of $\mathbf{P}(V)$.

As a very easy corollary of Theorem 1 we will prove the following result.

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Proposition 1. *Let V be an infinite dimensional complex Banach space, W a finite codimensional closed linear subspace of V, $X \subset \mathbf{P}(V)$ a finite codimensional closed analytic subset of $\mathbf{P}(V)$ and Q a degree d hypersurface of $\mathbf{P}(W)$ such that $Q \subset X$. Then for every integer t such that $0 < t < d$ every degree t hypersurface Y of $\mathbf{P}(V)$ containing X is singular at at least one point of Q.*

A key point of our proof of Theorem 1 is the following Bertini type result which will be proved in section 2

Theorem 2. *Let V be an infinite dimensional complex Banach space, A a finite dimensional linear subspace of V and Y a closed analytic hypersurface of $\mathbf{P}(V)$. Then there exists a linear subspace B of V such that $A \subset B$, $\dim(B) = \dim(A) + 1$ and $\mathrm{Sing}(Y) \cap A = \mathrm{Sing}(Y \cap B) \cap A$.*

We believe that Theorem 2 has an independent interest, because it allows quite often to transfer properties which are known in the case of a finite-dimensional ambient projective space to the case of finite-codimensional closed submanifolds of $\mathbf{P}(V)$ with V any Banach space. We used this informal principle to guess the truth of Theorem 1 and then proved Theorem 2 to prove our guess.

In section we will prove the following classification of all finite codimensional embeddings of infinite dimensional projective spaces.

Theorem 3. *Let V and E be infinite dimensional complex Banach spaces. Let $j\colon \mathbf{P}(V) \to \mathbf{P}(E)$ be a closed embedding with $j(\mathbf{P}(V))$ finite codimensional closed analytic subset of $\mathbf{P}(E)$. Then j is a linear isomorphism onto a finite codimensional closed linear suspace of $\mathbf{P}(E)$.*

As far as we know this is the first uniqueness result for finite-codimensional embeddings. It shows that the assumption of finite-codimensionality is extremely strong and probably too restrictive.

2 – Proof of Theorems 1 and 2 and of Proposition 1

Proof of Theorem 2: Set $m := \dim(A)$. Let G_A be the closed analytic subset of the Grassmannian $G(m+1, V)$ of all $(m+1)$-dimensional linear subspaces of V parametrizing the $(m+1)$-dimensional linear subspaces containing A ([2], §2, or [6], p. 89). We have $G_A \cong G(1, V/A) = \mathbf{P}(V/A)$.

Set $Z := \mathbf{P}(A)\backslash(\mathbf{P}(A) \cap \operatorname{Sing}(Y))$. We may assume $Z \neq \emptyset$, otherwise the statement of Theorem 2 is vacuosly true. For every $P \in Z \cap Y$ let $T_P Y \subset \mathbf{P}(V)$ be the Zariski tangent space of Y at P. Since $P \in Z$, Y is smooth at P. Thus $T_P Y$ is a closed hyperplane of $\mathbf{P}(V)$. Set $G_A(P) := \{U \in G_A \colon U \subset T_P Y\}$. $G_A(P)$ is a closed analytic subset of G_A of codimension m. If $B \in G_A \backslash G_A(P)$, then $B \cap Y$ is smooth at P. Since $\dim(Z) = m-1$, there is $B \in G_A$ with $B \notin G_A(P)$ for all $P \in Z$, i.e. such that $\operatorname{Sing}(Y) \cap A = \operatorname{Sing}(Y \cap B) \cap A$. ∎

Lemma 1. *Fix positive integers m, k and d with $d \geq 2$ and $2k \geq m > k$. Let $Y \subset \mathbf{P}^m$ be a degree d hypersurface containing a dimension k linear subspace L of $\mathbf{P}^m$. Then* $\operatorname{Sing}(Y) \cap L \neq \emptyset$.

Proof: Take homogeneous coordinates $x_0, \ldots, x_m$ of $\mathbf{P}^m$ such that $L = \{x_{k+1} = \ldots = x_m = 0\}$. Let F be a degree d homogeneous equation of Y. Since $L \subset Y$, there are degree $d-1$ homogeneous polynomials G_i, $k+1 \leq i \leq m$, such that $F = \sum_{i=k+1}^m x_i\, G_i$. Since $d-1 > 0$, the polynomials G_i, $k+1 \leq i \leq m$, are not constant. Since $m-k \geq k$, the restriction to L of the $m-k$ homogeneous polynomial G_i, $k+1 \leq i \leq m$, must have at least one common zero, P. At P every partial derivative $\partial F/\partial x_i$, $0 \leq i \leq m$, vanishes. Hence Y is singular at P. ∎

Proof of Theorem 1: Taking a minimal closed linear subspace of $\mathbf{P}(V)$ containing X instead of $\mathbf{P}(V)$ we reduce to the case in which $X \neq \mathbf{P}(V)$ and X is not contained in any closed hyperplane of $\mathbf{P}(V)$. By [6], Th. III.2.3.1, X is the zero-locus of finitely many continuous homogeneous polynomials on V, i.e. the intersection of finitely many closed algebraic hypersurfaces of $\mathbf{P}(V)$. Let Y be any closed analytic hypersurface of $\mathbf{P}(V)$ containing X. By assumption we have $d := \deg(Y) > 1$.

(**a**) Here we will check that $\operatorname{Sing}(Y) \cap M \neq \emptyset$ and that $\operatorname{Sing}(Y) \cap M$ contains a finite codimensional closed analytic subset $T(Y)$ of M. Assume that this is not true. Then for an arbitrary integer n we may find a dimension n projective subspace E of M such that $E \cap \operatorname{Sing}(Y) = \emptyset$. Let a be the codimension of X in $\mathbf{P}(V)$. Take any integer $n \geq 2a+1$ and any E as above. Using Theorem 2 we obtain the existence of a dimension $n+a$ linear subspace N of $\mathbf{P}(V)$ such that $\operatorname{Sing}(Y \cap N) \cap E = \emptyset$, contradicting Lemma 1.

(**b**) Take finitely many closed analytic hypersurfaces $Y_1, \ldots, Y_x$ such that $X = Y_1 \cap \ldots \cap Y_x$. By part (a) and the infinite dimensionality of M we have $\operatorname{Sing}(Y_1) \cap \ldots \cap \operatorname{Sing}(Y_x) \cap M \neq \emptyset$ and that $\operatorname{Sing}(Y_1) \cap \ldots \cap \operatorname{Sing}(Y_x) \cap M$ contains a finite codimensional closed analytic subset of M. Since $\mathrm{Sing}(Y_1)\cap\ldots\cap\mathrm{Sing}(Y_x) \subseteq \mathrm{Sing}(X)$, we are done. ∎

Proof of Proposition 1: Since $t<d$, Y contains $\mathbf{P}(W)$. Hence by Theorem 1 Y is singular at each point of a non-empty closed analytic subset B of $\mathbf{P}(W)$ with finite codimension in $\mathbf{P}(W)$. Since $Q \cap B \neq \emptyset$, we are done. ∎

3 – Proof of Theorem 3

Proposition 2. *Let V be a locally convex and Hausdorff complex topological vector space, Y any Hausdorff complex analytic set and C any finite dimensional connected closed analytic subset of $\mathbf{P}(V)$. Assume that C is not a point. Then there is no holomorphic map $\phi\colon \mathbf{P}(V) \to Y$ such that $\phi|\phi^{-1}(Y\backslash\phi(C))\colon \phi^{-1}(Y\backslash\phi(C)) \to Y\backslash\phi(C)$ is a surjective biholomorphism, while $\phi(C)$ is a point, i.e. there is no contraction $\phi\colon \mathbf{P}(V) \to Y$ of C.*

Proof: The result is well-known if V is finite dimensional; it follows from the result quoted at the end of this proof. Hence we may assume V infinite dimensional. Assume the existence of such a contraction ϕ. Since $\phi(C)$ is finite, there is an open neighborhood Ω of $\phi(C)$ in Y such that the holomorphic functions on Ω induce an embedding of Ω as a closed analytic subset of an open subset of a complex topological vector space. Hence $U := \phi^{-1}(\Omega)$ is an open neighborhood of C in $\mathbf{P}(V)$ such that the holomorphic functions on U separates the points of $U\backslash C$. Since C is finite dimensional, the vector space $H^0(C,\mathcal{O}_C(1))$ is finite dimensional. Hence the linear span $\langle C\rangle$ of C in $\mathbf{P}(V)$ is finite dimensional. The holomorphic functions on $U\cap\langle C\rangle$ separate distict points of $U\cap\langle C\rangle\backslash C$. Since $U\cap\langle C\rangle$ is a neighborhood of C in the finite dimensional projective space $\langle C\rangle$ and C has positive dimension, this is well-known to be false (see [3] and references therein for stronger statements). ∎

Proof of Theorem 3: By [4], Th. 7.1, for every holomorphic line bundle L on $\mathbf{P}(V)$ there is a unique integer t such that $L \cong \mathcal{O}_{\mathbf{P}(V)}(t)$. Let d be the unique integer such that $j^*(\mathcal{O}_{\mathbf{P}(E)}(1)) \cong \mathcal{O}_{\mathbf{P}(V)}(d)$. The line bundle $\mathcal{O}_{\mathbf{P}(V)}(t)$ has no global section if $t<0$, it is trivial and with only the costants as global sections if $t=0$, while if $t>0$ its global sections are given by the degree t continuous homogeneous polynomials on V. Thus $d>0$. Every closed curve of $j(\mathbf{P}(V))$ has degree divisible by d. Since every finite codimensional closed analytic subset of $\mathbf{P}(E)$ contains a line ([1], Th. 1.1, or modify the proof of a similar statement given in [7], Lemma 1.4), we obtain $d = 1$. This implies that any line of $\mathbf{P}(V)$ is sent isomorphically onto a line of $\mathbf{P}(E)$. This implies that for any two points P, Q of $j(\mathbf{P}(V))$ such that $P \neq Q$ the line spanned by P and Q is contained in $j(\mathbf{P}(V))$. Thus $j(\mathbf{P}(V))$ is a linear subspace of $\mathbf{P}$, proving the result. ■

Remark 1. In the statement of Theorems 1 and 2 and of Proposition 1 we assumed that V is a Banach space and not a more general topological vector space only because in their proof we quoted [6], p. 89 and Th. III.2.3.1. In the statement of Theorem 3 we assumed that V is a Banach space only to quote [4], Th. 7.1. □

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