PORTUGALIAE MATHEMATICA
Vol. 61 Fasc. 2 – 2004
Nova Série

EXISTENCE RESULTS FOR FIRST AND SECOND ORDER NONCONVEX SWEEPING PROCESS WITH DELAY *

M. Bounkhel and M. Yarou

Abstract: In this paper, we prove several existence results for functional differential inclusions governed by nonconvex sweeping process of first and second order with perturbations depending on all the variables and with delay.

1 – Introduction

In this paper, we present some existence results for functional differential inclusions governed by nonconvex sweeping process of first and second order

$$\text{(FOSPD)} \qquad \begin{cases} \dot u(t) \in -N_{C(t)}(u(t)) + F_1(t,u_t) & \text{a.e. on } [0,T]\,; \\ u(t) \in C(t), & \text{for all } t \in [0,T]\,; \\ u(s) = \mathcal{T}(0)u(s) = \varphi(s), & \text{for all } s \in [-\tau,0]\,; \end{cases}$$

and

$$\text{(SOSPD)} \qquad \begin{cases} \ddot u(t) \in -N_{K(u(t))}(\dot u(t)) + F_2(t,u_t,\dot u_t) & \text{a.e. on } [0,T]\,; \\ \dot u(t) \in K(u(t)), & \text{for all } t \in [0,T]\,; \\ \mathcal{T}(0)\dot u = \varphi, & \text{on } [-\tau,0]\,; \end{cases}$$

where $\tau, T > 0$, $C : [0,T] \rightrightarrows H$ (resp. $K : H \rightrightarrows H$) is a set-valued mapping taking values in a Hilbert space H, and $F_1 : I\times\mathcal{C}_0 \rightrightarrows H$ (resp. $F_2 : I\times\mathcal{C}_0\times\mathcal{C}_0 \rightrightarrows H$) is a set-valued mapping with convex compact values. Here $N(C(t);u(t))$ (resp.

Received: December 15, 2002; *Revised*: June 21, 2003.
AMS Subject Classification: 49J52, 46N10, 58C20.
Keywords: uniformly prox-regular set; nonconvex sweeping processes; nonconvex differential inclusions.
* This Work was supported for the first author by the research center project No. Math/1422/22.

$N(K(u(t)); \dot{u}(t)))$ denotes the Clarke normal cone to $C(t)$ (resp. $K(u(t))))$ at $u(t)$ (resp. $\dot{u}(t)$) and $\mathcal{C}_0 := \mathcal{C}([-\tau, 0], H)$ is the Banach space of all continuous mappings from $[-\tau, 0]$ to H equipped with the norm of uniform convergence. For every $t \in I$, the function u_t is given by $u_t(s) = \mathcal{T}(t)u(s) = u(t+s)$, for all $s \in [-\tau, 0]$. Such problems have been studied in several papers (see for example [13, 15, 19]). In [13], some topological properties of solutions set for (FOSPD) problem in the convex case are established, and in [15], the compactness of the solutions set is obtained in the nonconvex case when $H = \mathbb{R}^d$, using important properties of uniformly r-prox regular sets developed recently in [8, 9, 21]. The (SOSPD) has been considered in [19] with a perturbation not depending of the third variable, $F(t, \mathcal{T}(t)u)$, continuous and with compact values.

For more details on functional differential inclusions for nonconvex sweeping process and related subjects, see [1, 2, 3, 4, 6, 7, 9, 14, 17, 18, 20, 22, 23, 24].

Our main purpose in this paper is to prove existence results for (FOSPD) and (SOSPD) when C has uniformly r-prox regular values and H is a separable Hilbert space. The paper is organized as follows. In Section 2, we recall some definitions and fundamental results needed in the sequel of the paper. Section 3 is devoted to prove (Theorem 3.1) the existence of approximate solutions for the (FOSPD) under the boundedness of F. Under two different assumptions on F we prove the existence of absolutely continuous solutions of (FOSPD) by proving the convergence of the approximate solutions established in Theorem 3.1. The last section is concerned with the existence of solutions for (SOSPD). Our approach is based on a classical method which consists to subdivide the interval $[0, T]$ in a sequence of subintervals I_n and to reformulate our problem with delay (SOSPD) to a family of problems without delay (SOSP) on each I_n and next we use an existence result given in [4] on each subinterval to get a solution on I_n. Finally, we prove the convergence of this family of solutions to a solution of (SOSPD). We point out that a different approach with more restrictive assumptions, is given in [10, 11] to prove the existence of solutions for the same problem (SOSPD). It consists to make use the existence of solutions for the first order problem (FOSPD) and the fixed point approach.

2 – Preliminaries and fundamental results

Throughout the paper, H will denote a *real separable Hilbert space*. Let S be a nonempty closed subset of H, we denote by $d_S(\cdot)$ or $d(S, \cdot)$ the usual distance function to the subset S. We recall that the proximal normal cone of S at $x \in S$

is defined by

$$N^P(S;x) = \Big\{\xi \in H\colon\ \exists\, \alpha > 0\colon\ x \in \operatorname{Proj}(x + \alpha\xi, S)\Big\}\ ,$$

where

$$\operatorname{Proj}(u, S) := \Big\{y \in S\colon\ d_S(u) = \|u - y\|\Big\}\ .$$

Recall now that for a given $r \in\ ]0, +\infty]$, a subset S is uniformly r-proximally regular if and only if for all $y \in S$ and all $\xi \in N^P(S; y)$, $\xi \neq 0$ one has

$$\left\langle \frac{\xi}{\|\xi\|},\, x - y \right\rangle \le \frac{1}{2\,r}\,\|x - y\|^2\ ,$$

for all $x \in S$ (see [21]). We make the convention $\frac{1}{r} = 0$ for $r = +\infty$ (in this case, the uniform r-prox-regularity is equivalent to the convexity of S).

In order to make clear the concept of r-prox-regular sets, we state the following concrete examples: The union of two disjoint intervals $[a, b]$ and $[c, d]$ is r-prox-regular with $r = \frac{c-b}{2}$. The finite union of disjoint closed convex sets in $\mathbb{R}^2$ is r-prox-regular with r depends on the distances between the sets (we refer the reader to [5] for a different application of this concept in Variational Inequalities. A general study of the class of r-prox-regular sets with more concrete examples is given in a forthcoming paper by the first author).

Let $K : H \rightrightarrows H$ be a set-valued mapping from H to H. We will say that K is Hausdorff-continuous (resp. Hausdorff–Lipschitz with ratio $\lambda > 0$) if for any $x \in H$ one has

$$\lim_{x' \to x} \mathcal{H}(K(x), K(x')) = 0$$

$\Big($resp.

$$\mathcal{H}(K(x), K(x')) \le \lambda\|x - x'\|, \qquad \text{for all}\ \ x, x' \in H\ \ \Big)\ .$$

Here $\mathcal{H}$ stands the Hausdorff distance relative to the norm associated with the Hilbert space H defined by

$$\mathcal{H}(A, B) := \max\Big\{\sup_{a\in A} d_B(a),\ \sup_{b\in B} d_A(b)\Big\}\ .$$

Let $\varphi\colon X \rightrightarrows Y$ be a set-valued mapping defined between two topological vector spaces X and Y, we say that φ is upper semi-continuous (in short u.s.c.) at $x \in \operatorname{dom}(\varphi) := \{x' \in X\colon\ \varphi(x') \neq \emptyset\}$ if for any open set O containing $\varphi(x)$ there exists a neighborhood V of x such that $\varphi(V) \subset O$.

We will deal with a finite delay $\tau > 0$. If $u : [-\tau, T] \to H$, then for every $t \in [0, T]$, we define the function $u_t(s) = u(t + s)$, $s \in [-\tau, 0]$ and the Banach space $\mathcal{C}_T := \mathcal{C}([-\tau, T], H)$ (resp. $\mathcal{C}_0 := \mathcal{C}([-\tau, 0], H)$) of all continuous mappings from $[-\tau, T]$ (resp. $[-\tau, 0]$) to H with the norm given by

$$\|\varphi\|_{\mathcal{C}_T} := \max\Big\{\|\varphi(s)\| \colon\ s \in [-\tau, T]\Big\}$$

$\Big($resp.

$$\|\varphi\|_{\mathcal{C}_0} := \max\Big\{\|\varphi(s)\| \colon\ s \in [-\tau, 0]\Big\} \quad \Big).$$

Clearly, if $u \in \mathcal{C}_T$, then $u_t \in \mathcal{C}_0$, and the mapping $u \to u_t$ is continuous in the sense of the uniform convergence.

The following propositions summarize some important consequences of the uniform prox-regularity needed in the sequel of the paper. For the proofs we refer to [9].

Proposition 2.1 ([9])**.** *Let S be a nonempty closed subset of H and $x \in S$. Then the following hold*

1) *$\partial^P d_S(x) = N_S^P(x) \cap \mathbb{B}$, where $\partial^P d_S(x)$ is the proximal subdifferential of the distance function (see [9] for the definition of the proximal subdifferential).*

2) *If S is uniformly r-prox-regular, then for any $x \in H$ with $d_S(x) < r$, then $\mathrm{Proj}_S(x) \neq \emptyset$ and unique, and the proximal subdifferential $\partial^P d_S(x)$ is a closed convex set in H.*

3) *If S is uniformly r-prox-regular, then for any $x \in S$ and any $\xi \in \partial^P d_S(x)$ one has*

$$\langle \xi, x' - x \rangle \le \frac{2}{r}\, \|x' - x\|^2 + d_S(x') \quad \text{for all } \ x' \in H \ \text{ with } \ d_S(x') < r\ . \ \blacksquare$$

As a direct consequence of part (2) in the previous proposition we have $\partial^P d_S(x) = \partial^C d_S(x)$ and hence $N^C(S;x) = N^P(S;x)$, whenever S is uniformly r-prox-regular set. So, we will denote $N(S;x) := N^C(S;x) = N^P(S;x)$ and $\partial d_S(x) := \partial^P d_S(x) = \partial^C d_S(x)$ for such class of sets.

Proposition 2.2 ([4, 7])**.** *Let $r \in\,]0, \infty]$ and Ω be an open subset in H and let $C \colon \Omega \rightrightarrows H$ be a Hausdorff-continuous set-valued mapping. Assume that C has uniformly r-prox-regular values. Then, the set-valued mapping given by $(z, x) \rightrightarrows \partial d_{C(z)}(x)$ from $\Omega \times H$ (endowed with the strong topology) to H (endowed with the weak topology) is upper semicontinuous, which is equivalent to the u.s.c. of the function $(z, x) \mapsto \sigma(\partial d_{C(z)}(x), p)$ for any $p \in H$. Here $\sigma(S, p)$ denotes the support function associated with S, i.e., $\sigma(S, p) = \sup_{s \in S} \langle s, p \rangle$.* $\blacksquare$

3 – First order perturbed nonconvex sweeping process with delay

In all this section, let $T > 0$, $I := [0,T]$, $r \in]0,+\infty]$, and $C\colon I \rightrightarrows H$ be an absolutely continuous set-valued mapping, that is, for any $y \in H$ and any $t,t' \in I$

$$|d_{C(t)}(y) - d_{C(t')}(y)| \le |v(t) - v(t')| \ , \tag{3.1}$$

with $v\colon I \to \mathbb{R}$ is an absolutely continuous function, i.e., there exists $g \in L^1(I,\mathbb{R})$ such that $v(t) = v(0) + \int_0^t g(s)\,ds$, for all $t \in I$. Note that the function v does not depend on y. The following proposition provides an approximate solution for the (FOSPD) under consideration.

Theorem 3.1. *Assume that $C(t)$ is uniformly r-prox-regular for every $t \in I$. Let $F\colon I\times\mathcal{C}_0 \rightrightarrows H$ be a set-valued mapping with convex compact values in H such that $F(t,\cdot)$ is u.s.c. on $\mathcal{C}_0$ for any fixed $t \in I$ and $F(\cdot,\varphi)$ admits a measurable selection on I for any fixed $\varphi \in \mathcal{C}_0$. Assume that $F(t,\varphi) \subset l\mathbb{B}$ for all $(t,\varphi) \in I\times\mathcal{C}_0$, for some $l > 0$. Then, for any $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$ and for any n large enough there exists a continuous mapping $u_n\colon [-\tau,T] \to H$ which enjoys the following properties:*

1) $\dot u_n(t) \in -N(u_n(\theta_n(t)); C(\theta_n(t))) + F(\rho_n(t), \mathcal{T}(\rho_n(t))u_n)$, *a.e.* $t \in I$, *where* $\theta_n,\rho_n\colon I \to I$ *with* $\theta_n(t) \to t$ *and* $\rho_n(t) \to t$ *for all* $t \in I$.

2) $\|\dot u_n(t)\| \le (l+1)(\dot v(t)+1)$, *a.e.* $t \in I$.

Proof: We prove the conclusion of our theorem when F is globally u.s.c. on $I \times \mathcal{C}_0$ and then as in [13], we can proceed by approximation to prove it when $F(t,\cdot)$ is u.s.c. on $\mathcal{C}_0$ for any fixed $t \in I$ and $F(\cdot,\varphi)$ admits a measurable selection on I for any fixed $\varphi \in \mathcal{C}_0$.

First, observing that (3.1) ensures for $t \le t'$

$$|d_{C(t')}(y) - d_{C(t)}(y)| \le \int_t^{t'} |\dot v(s)|\,ds \ , \tag{3.2}$$

we may suppose (replacing $\dot v$ by $|\dot v|$ if necessary) that $\dot v(t) \ge 0$ for all $t \in I$. We construct via discretization the sequence desired of continuous mappings $\{u_n\}_n$ in $\mathcal{C}_T$.

For every $n \in \mathbb{N}$, we consider the following partition of I:

$$t_i^n := \frac{iT}{2^n} \quad (0 \le i \le 2^n) \qquad \text{and} \qquad I_i^n :=]t_i^n, t_{i+1}^n] \ \text{ if } \ 0 \le i \le 2^n - 1 \ . \tag{3.3}$$

Put

$$(3.4)\qquad \mu_n := \frac{T}{2^n}\,, \qquad \epsilon_i^n := \int_{t_i^n}^{t_{i+1}^n} \dot v(s)\,ds \quad \text{and} \quad \epsilon_n := \max_{0\le i<2^n} \{\mu_n + \epsilon_i^n\}\ .$$

As $\epsilon_n \to 0$, we can fix $n_0 \ge 1$ satisfying for every $n \ge n_0$

$$(3.5)\qquad 2\,\mu_n < \frac{r}{(2\,l+1)} \quad \text{and} \quad 2\,\epsilon_n < \min\left\{1, \frac{r}{(4\,l+3)}\right\}\,,$$

(3.6) First, we put $u_n(s) := \varphi(s)$, for all $s \in [-\tau, 0]$ and for all $n \ge n_0$.

For every $n \ge n_0$, we define by induction:

$$(3.7)\qquad u_n(t_{i+1}^n) := u_{i+1}^n = \operatorname{Proj}_{C(t_{i+1}^n)}\Big(u_i^n - \mu_n\, f_0(t_i^n, \mathcal{T}(t_i^n)u_n)\Big)\,,$$

where $f_0(t_i^n, \mathcal{T}(t_i^n)u_n)$ is the minimal norm element of $F(t_i^n, \mathcal{T}(t_i^n)u_n)$, i.e.,

$$(3.8)\qquad \|f_0(t_i^n, \mathcal{T}(t_i^n)u_n)\| = \min\Big\{\|y\|\colon\ y \in F(t_i^n, \mathcal{T}(t_i^n)u_n)\Big\} \le l$$

and

$$\mathcal{T}(t_i^n)\,u_n := (u_n)_{t_i^n}\ .$$

The above construction is possible despite the nonconvexity of the images of C. Indeed, we can show that for every $n \ge n_0$ we have

$$d_{C(t_{i+1}^n)}\Big(u_i^n - \mu_n\, f_0(t_i^n, \mathcal{T}(t_i^n)u_n)\Big) \le l\mu_n + v(t_{i+1}^n) - v(t_i^n) \le (l+1)\,\epsilon_n \le \frac{r}{2}$$

and hence as C has uniformly r-prox-regular values, by Proposition 2.1 one can choose for all $n \ge n_0$ a point $u_{i+1}^n = \operatorname{Proj}_{C(t_{i+1}^n)}(u_i^n - \mu_n f_0(t_i^n, \mathcal{T}(t_i^n)u_n))$. Note that from (3.7) and (3.2) one deduces for every $0 \le i < 2^n$

$$(3.9)\qquad \Big\|u_{i+1}^n - \Big(u_i^n - \mu_n\, f_0(t_i^n, \mathcal{T}(t_i^n)u_n)\Big)\Big\| \le l\mu_n + \epsilon_i^n \le (l+1)\,(\mu_n + \epsilon_i^n)\ .$$

(3.10) By construction we have $u_i^n \in C(t_i^n)$, for all $0 \le i < 2^n$.

For every $n \ge n_0$, these $(u_i^n)_{0\le i\le 2^n}$ and $(f_0(t_i^n, \mathcal{T}(t_i^n)u_n)_{0\le i\le 2^n}$ are used to construct two mappings u_n and f_n from I to H by defining their restrictions to each interval I_i^n as follows:

for $t = 0$, set $f_n(t) := f_0^n$ and $u_n(t) := u_0^n = \varphi(0)$,

for all $t \in I_i^n$ $(0 \le i \le 2^n)$, set $f_n(t) := f_i^n$ and

$$(3.11)\qquad u_n(t) := u_i^n + \frac{a(t) - a(t_i^n)}{\epsilon_i^n + \mu_n}\,(u_{i+1}^n - u_i^n + \mu_n f_i^n) + (t - t_i^n) f_i^n\ ,$$

where $f_i^n := f_0(t_i^n, \mathcal{T}(t_i^n)u_n)$ and $a(t) := v(t) + t$ for all $t \in I$. Hence for every t and t' in I_i^n $(0 \le i \le 2^n)$ one has

$$u_n(t') - u_n(t) \;=\; \frac{a(t') - a(t)}{\epsilon_i^n + \mu_n}\,(u_{i+1}^n - u_i^n + \mu_n f_i^n) + (t' - t) f_i^n \;.$$

Thus, in view of (3.9), if $t, t' \in I_i^n$ $(0 \le i < 2^n)$ with $t \le t'$, one obtains

$$\|u_n(t') - u_n(t)\| \;\le\; (l{+}1)\Big(a(t'){-}a(t)\Big) + l(t'{-}t) \;\le\; (2\,l{+}1)\Big(a(t'){-}a(t)\Big)\;, \tag{3.12}$$

and, by addition this also holds for all $t, t' \in I$ with $t \le t'$. This inequality entails that u_n is absolutely continuous.

Coming back to the definition of u_n in (3.11), one observes that for $0 \le i < 2^n$

$$\dot u_n(t) = \frac{\dot a(t)}{\epsilon_i^n + \mu_n}\,(u_{i+1}^n - u_i^n + \mu_n f_i^n) + f_i^n \qquad \text{for a.e. } \; t \in I_i^n \;.$$

Then one obtains, in view of (3.9), for a.e. $t \in I$

$$\|\dot u_n(t) - f_n(t)\| \;\le\; (l+1)\Big(\dot v(t) + 1\Big)\;, \tag{3.13}$$

which proves the part 2) of the theorem.

Now, let θ_n, ρ_n be defined from I to I by $\theta_n(0) = 0$, $\rho_n(0) = 0$, and

$$\theta_n(t) = t_{i+1}^n, \quad \rho_n(t) = t_i^n \qquad \text{if } \; t \in I_i^n \;\; (0 \le i < 2^n)\;. \tag{3.14}$$

Then, by (3.7), the construction of u_n and f_n, and the properties of proximal normal cones to subsets, we have for a.e. $t \in I$

$$f_n(t) \;\in\; F\Big(\rho_n(t), \mathcal{T}(\rho_n(t))u_n\Big)$$

and

$$\dot u_n(t) - f_n(t) \;\in\; -N\Big(C(\theta_n(t)); u_n(\theta_n(t))\Big)\;. \tag{3.15}$$

These last inclusions ensure part 1) of the theorem and then the proof is complete. ∎

Now, we are able to state our first existence result for (FOSPD).

Theorem 3.2. *Assume that the assumptions of Theorem 3.1 are satisfied. Assume that $C(t)$ is strongly compact for every $t \in I$. Then for every $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$, there exists a continuous mapping $u\colon [-\tau, T] \to H$ such that u is absolutely continuous on I and satisfies:*

$$\text{(FOSPD)} \qquad \begin{cases} \dot u(t) \in -N_{C(t)}(u(t)) + F(t, u_t) & \text{a.e. on } \; I\,; \\ u(t) \in C(t), & \forall\, t \in I\,; \\ u(s) = \mathcal{T}(0)\,u(s) = \varphi(s), & \forall\, s \in [-\tau, 0]\,; \end{cases}$$

and

$$\|\dot u(t)\| \le (l+1)\Big(\dot v(t) + 1\Big) \qquad \text{a.e. on } \; I\;.$$

Proof: Let $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$. By Theorem 3.1 there exists a sequence of continuous mappings $\{u_n\}$ enjoying the properties 1) and 2) in Theorem 3.1. Let $n_0 \in \mathbb{N}$ satisfying (3.5). Then by (3.1), (3.4), and (3.12) we get for any $n \geq n_0$ and any $t \in I$

$$\begin{aligned} d(u_n(t), C(t)) &\leq \|u_n(t) - u_n(t_i^n)\| + d(u_n(t_i^n), C(t)) \\ &\leq (2\,l+1)\Big(a(t) - a(t_i^n)\Big) + \Big(v(t) - v(t_i^n)\Big) \\ &\leq (2\,l+1)\,(\epsilon_i^n + \mu_n) + \epsilon_i^n \;\leq\; 2(l+1)\,\epsilon_n \;. \end{aligned} \tag{3.16}$$

Since $C(t)$ is strongly compact and $\epsilon_n \to 0$, (3.16) implies that the set $\{u_n(t) : n \geq n_0\}$ is relatively strongly compact in H for all $t \in I$. Thus, by Arzela–Ascoli's theorem we can extract a subsequence of the sequence $\{u_n\}_n$ still denoted $\{u_n\}_n$, which converges uniformly on $[-\tau, T]$ to a continuous function u which clearly satisfies $u_0 = \varphi$. Now by letting $n \to +\infty$ we get for all $t \in I$

$$u(t) \in C(t) \;. \tag{3.17}$$

On one hand, it follows from our construction in the proof of Theorem 3.1 that for all $t \in I$

$$\mathcal{H}\Big(C(\theta_n(t)), C(t)\Big) \leq |v(\theta_n(t)) - v(t)| \leq \epsilon_n \;\to\; 0 \;, \tag{3.18}$$

and by (3.12), (3.5), and the uniform convergence of $\{u_n\}_n$ to u over I we get

$$\|u_n(\theta_n(t)) - u(t)\| \leq \|u_n(\theta_n(t)) - u(\theta_n(t))\| + \|u(\theta_n(t)) - u(t)\| \;\to\; 0 \;. \tag{3.19}$$

Now, using the same technique in [13] and the relations (3.5) and (3.12) we obtain

$$\lim_{n\to\infty} \|\mathcal{T}(\rho_n(t))u_n - \mathcal{T}(t)u_n\| = 0 \quad \text{ in } \mathcal{C}_0 \;.$$

Therefore, as the uniform convergence of u_n to u in $[-\tau, T]$ implies that $\mathcal{T}(t)u_n$ converges to $\mathcal{T}(t)u$ uniformly on $[-\tau, 0]$, we conclude that

$$\mathcal{T}(\rho_n(t))u_n \longrightarrow \mathcal{T}(t)u = u_t \quad \text{ in } \mathcal{C}_0 \;. \tag{3.20}$$

On the other hand, from $f_n(t) \in F(\rho_n(t), \mathcal{T}(\rho_n(t))u_n)$ and (3.12), the sequences (f_n) and $(\dot{u}_n)$ are bounded sequences in $L^\infty(I, H)$. Then by extracting subsequences (because $L^\infty(I, H)$ is the dual space of the separable space $L^1(I, H)$), we may suppose without loss of generality that f_n and $\dot{u}_n$ weakly-$\star$ converge in $L^\infty(I, H)$ to some mappings f and ω respectively. Then, for all $t \in I$ one has

$$u(t) \;=\; \lim_{n\to\infty} u_n(t) \;=\; \varphi(0) + \lim_{n\to\infty} \int_0^t \dot{u}_n(s)\,ds \;=\; x_0 + \int_0^t \omega(s)\,ds \;,$$

which proves that u is absolutely continuous and $\dot{u}(t) = \omega(t)$ for a.e. $t \in I$.

Using now Mazur's lemma, we obtain:

$$\dot u(t) - f(t) \in \bigcap_n \overline{\mathrm{co}}\Big\{\dot u_k(t) - f_k(t)\colon\, k \geq n\Big\} \qquad \text{a.e. } \ t \in I \ .$$

Fix such t in I and any ξ in H, the last relation above yields

$$\langle \dot u(t) - f(t),\, \xi\rangle \,\leq\, \inf_n \sup_{k\geq n} \langle \dot u_k(t) - f_k(t),\, \xi\rangle \ .$$

By (3.13), (3.15), and Proposition 2.1 part (1) we have for a.e. $t \in I$

$$\dot u_n(t) - f_n(t) \,\in\, -N\Big(C(\theta_n(t));\, u_n(\theta_n(t))\Big) \cap \delta(t)\mathbb{B} \,=\, -\delta(t)\,\partial d_{C(\theta_n(t))}(u_n(\theta_n(t)))\ ,$$

where $\delta(t) := (l+1)\,(\dot v(t)+1)$. Hence, according to this last inclusion and proposition 2.2 we get

$$\begin{aligned}\langle \dot u(t) - f(t),\, \xi\rangle \,&\leq\, \delta(t)\, \limsup_n \sigma\Big(-\partial d_{C(\theta_n(t)}(u_n(\theta_n(t)));\, \xi\Big)\\ &\leq\, \delta(t)\,\sigma\Big(-\partial d_{C(t)}(u(t));\, \xi\Big)\end{aligned}$$

Since $\partial d_{C(t)}(u(t))$ is closed convex, we obtain

$$\dot u(t) - f(t) \,\in\, -\delta(t)\,\partial d_{C(t)}(u(t)) \subset -N_{C(t)}(u(t))$$

and then

$$\dot u(t) \,\in\, -N_{C(t)}(u(t)) + f(t)$$

because $u(t) \in C(t)$. Finally, from (3.20) and the global upper semicontinuity of F and the convexity of its values and with the same techniques used above we can prove that

$$f(t) \,\in\, F(t, \mathcal{T}(t)u) = F(t, u_t) \qquad \text{a.e. } \ t \in I \ .$$

Thus, the existence is proved. ∎

Under different assumptions another existence result for (FOSPD) is also proved in the following theorem.

Theorem 3.3. *Assume that the assumptions of Theorem 3.1 are satisfied. Assume also that $F(t,\varphi) \subset \mathcal{K} \subset l\mathbb{B}$ for every $(t,\varphi) \in I \times \mathcal{C}_0$, where $\mathcal{K}$ is a strongly compact set in H. Then for every $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$, there exists a continuous mapping $u\colon [-\tau, T] \to H$ such that u is absolutely continuous on I and satisfies:*

$$\begin{cases} \dot u(t) \in -N_{C(t)}(u(t)) + F(t, u_t) & \text{a.e. on } \ I\,;\\ u(t) \in C(t), & \forall\, t \in I\,;\\ u(s) = \mathcal{T}(0)\,u(s) = \varphi(s), & \forall\, s \in [-\tau, 0]\,;\end{cases}$$

and

$$\|\dot u(t)\| \leq (l+1)\Big(\dot v(t) + 1\Big) \qquad \text{a.e. on } \ I \ .$$

Proof: Let $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$. By Theorem 3.1 there exists a sequence of continuous mappings $\{u_n\}$ enjoying the properties 1) and 2) in Theorem 3.1. Let $n_0 \in \mathbb{N}$ satisfying (3.5). Let us show that the sequence $(u_n)_n$ satisfies the Cauchy property in the space of continuous mappings $\mathcal{C}(I,H)$ endowed with the norm of uniform convergence. Fix $m,n \in \mathbb{N}$ such that $m \geq n \geq n_0$ and fix also $t \in I$ with $t \neq t_{m,i}$ for $i = 0,...,2^m$ and $t \neq t_{n,j}$ for $j = 0,...,2^n$. Observe by (3.2), (3.4), and (3.12) that

$$\begin{aligned} d_{C(\theta_n(t))}(u_m(t)) &= d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(\theta_m(t))) \\ &\leq v(\theta_n(t)) - v(\theta_m(t)) + \|u_m(\theta_m(t)) - u_m(t)\| \\ &\leq \int_{\theta_m(t)}^{\theta_n(t)} \dot v(s)\,ds \,+\, (2\,l{+}1)\left[\int_t^{\theta_m(t)} \dot v(s)\,ds \,+\, (\theta_m(t){-}t)\right] \\ &\leq \epsilon_n + (2\,l+1)\,\epsilon_m \end{aligned} \tag{3.21}$$

and hence, by (3.5) we get $d_{C(\theta_n(t))}(u_m(t)) < r$. Set $\delta(t) := (l{+}1)\,\dot a(t)$. Then, (3.15), (3.21), and Proposition 2.1 part (3) entail

$$\begin{aligned} &\Big\langle \dot u_n(t) - f_n(t),\ u_n(\theta_n(t)) - u_m(t)\Big\rangle \;\leq \\ &\quad\leq\; \frac{2\,\delta(t)}{r}\,\|u_n(\theta_n(t)) - u_m(t)\|^2 +\, \delta(t)\,d_{C(\theta_n(t))}(u_m(t)) \\ &\quad\leq\; \frac{2\,\delta(t)}{r}\Big[\|u_n(t) - u_m(t)\| + \|u_n(\theta_n(t)) - u_n(t)\|\Big]^2 +\, \delta(t)\Big(\epsilon_n + (2\,l{+}1)\,\epsilon_m\Big)\;, \end{aligned}$$

and this yields by (3.4) and (3.12)

$$\begin{aligned} &\Big\langle \dot u_n(t) - f_n(t),\ u_n(\theta_n(t)) - u_m(t)\Big\rangle \;\leq \\ &\qquad\leq\; \frac{2\,\delta(t)}{r}\Big[\|u_n(t) - u_m(t)\| + (2\,l{+}1)\,\epsilon_n\Big]^2 +\, \delta(t)\,(2\,l{+}1)\,(\epsilon_n + \epsilon_m)\;. \end{aligned} \tag{3.22}$$

Now, let us define $g_n(t) := \int_0^t f_n(s)\,ds$ for all $t \in I$. Observe that for all $t \in I$ the set $\{g_n(t) \colon n \geq n_0\}$ is contained in the strongly compact set $T\mathcal{K}$ and so it is relatively strongly compact in H. Then, as $\|f_n(t)\| \leq l$ a.e. on I, Arzela–Ascoli's theorem yields the relative strong compactness of the set $\{g_n \colon n \geq n_0\}$ with respect to the uniform convergence in $C(I,H)$ and so we may assume without loss of generality that (g_n) converges uniformly to some mapping g. Also, we may suppose that (f_n) weakly converges in $L^1(I,H)$ to some mapping f. Then, for all $t \in I$,

$$g(t) \;=\; \lim_n g_n(t) \;=\; \lim_n \int_0^t f_n(s)\,ds \;=\; \int_0^t f(s)\,ds\;,$$

which gives that g is absolutely continuous and $\dot g = f$ a.e. on I.

Put now $w_n(t) := u_n(t) - g_n(t)$ for all $n \geq n_0$ and all $t \in I$ and put $\eta_n := \max\{\epsilon_n, \|g_n - g\|_\infty\}$. Then by (3.13) and (3.22) one gets

$$\begin{aligned}
&\Big\langle \dot{w}_n(t),\, w_n(\theta_n(t)) - w_m(t) \Big\rangle \;= \\
&\qquad = \Big\langle \dot{w}_n(t),\, u_n(\theta_n(t)) - u_m(t) \Big\rangle + \Big\langle \dot{w}_n(t),\, g_n(\theta_n(t)) - g_m(t) \Big\rangle \\
&\qquad \leq \frac{2\,\delta(t)}{r} \Big[\|w_n(t) - w_m(t)\| + \|g_n(t) - g_m(t)\| + (2\,l{+}1)\,\epsilon_n \Big]^2 \\
&\qquad\quad + \delta(t)\,(2\,l{+}1)\,(\epsilon_n + \epsilon_m) + \delta(t)\,\|g_n(\theta_n(t)) - g_m(t)\| \\
&\qquad \leq \frac{2\,\delta(t)}{r} \Big[\|w_n(t) - w_m(t)\| + (\eta_n + \eta_m) + (2\,l{+}1)\,\eta_n \Big]^2 \\
&\qquad\quad + 2\,\delta(t)\,(2\,l{+}1)\,(\eta_n + \eta_m) \;.
\end{aligned}$$

This last inequality ensures by (3.13)

$$\begin{aligned}
\Big\langle \dot{w}_n(t),\, w_n(t){-}w_m(t) \Big\rangle \;&\leq\; \Big\langle \dot{w}_n(t),\, w_n(t){-}w_n(\theta_n(t)) \Big\rangle \;+\; 2\,\delta(t)\,(2\,l{+}1)\,(\eta_n{+}\,\eta_m) \\
&\quad +\; \frac{2\,\delta(t)}{r} \Big[\|w_n(t) - w_m(t)\| + (\eta_n{+}\,\eta_m) + (2\,l{+}1)\,\eta_n \Big]^2 \\
&\leq\; 4\,\delta(t)\,(2\,l{+}1)\,(\eta_n{+}\,\eta_m) \\
&\quad +\; \frac{2\,\delta(t)}{r} \Big[\|w_n(t) - w_m(t)\| + (\eta_n{+}\,\eta_m) + (2\,l{+}1)\,\eta_n \Big]^2.
\end{aligned}$$

In the same way, we also have

$$\begin{aligned}
\Big\langle \dot{w}_m(t),\, w_m(t){-}w_n(t) \Big\rangle \;&\leq\; 4\,\delta(t)\,(2\,l{+}1)\,(\eta_n{+}\,\eta_m) \\
&\quad +\; \frac{2\,\delta(t)}{r} \Big[\|w_n(t){-}w_m(t)\| + (\eta_n{+}\,\eta_m) + (2\,l{+}1)\,\eta_m \Big]^2.
\end{aligned}$$

It then follows from both last inequalities that we have for some positive constant α independent of m, n and t (note that $\|w_n(t)\| \leq lT + \|\varphi(0)\| + \int_0^T \dot{v}(s)\,ds$)

$$2\,\Big\langle \dot{w}_m(t) - \dot{w}_n(t),\, w_m(t) - w_n(t) \Big\rangle \;\leq\; \alpha\,\delta(t)\,(\eta_n{+}\,\eta_m) + 8\,\frac{\delta(t)}{r}\,\|w_m(t) - w_n(t)\|^2 \;,$$

and so, for some positive constants β and γ independent of m, n and t

$$\frac{d}{dt}\Big(\|w_m(t) - w_n(t)\|^2 \Big) \;\leq\; \beta\,\dot{a}(t)\,\|w_m(t) - w_n(t)\|^2 + \gamma\,\dot{a}(t)\,(\eta_n{+}\,\eta_m) \;.$$

As $\|w_m(0) - w_n(0)\|^2 = 0$, the Gronwall inequality yields for all $t \in I$

$$\|w_m(t) - w_n(t)\|^2 \;\leq\; \gamma(\eta_n{+}\,\eta_m)\, \int_0^t \Big[\dot{a}(s)\,\exp\Big(\beta \int_s^t \dot{a}(u)\,du \Big) \Big] ds$$

and hence for some positive constant K independent of m, n and t we have

$$\|w_m(t) - w_n(t)\|^2 \leq K(\eta_n + \eta_m) .$$

The Cauchy property in $\mathcal{C}(I,H)$ of the sequence $(w_n)_n = (u_n - g_n)_n$ is thus established and hence this sequence converges uniformly to some mapping w. Therefore the sequence $(u_n)_n$ constructed in Theorem 3.1 converges uniformly to $u := w + g$. Following the same arguments in the proof of Theorem 3.2 we prove the conclusion of the theorem, i.e., the limit mapping u is continuous on $[-\tau, T]$ and absolutely continuous on I and satisfies

$$\begin{cases} \dot{u}(t) \in -N_{C(t)}(u(t)) + F(t, u_t) & \text{a.e. on } I\,; \\ u(t) \in C(t), & \forall\, t \in I\,; \\ u(s) = \mathcal{T}(0)\, u(s) = \varphi(s), & \forall\, s \in [-\tau, 0]\,; \end{cases}$$

and

$$\|\dot{u}(t)\| \leq (l+1)\,(\dot{v}(t) + 1) \qquad \text{a.e. on } I \;. \blacksquare$$

Remark 3.1. Our results in this section generalizes many results given in [13, 15, 24]. Theorem 3.2 extends the one given in [13] to the case of absolutely continuous set-valued mappings with nonconvex values, and Theorem 3.3 extends Theorem 3.1 in [24] and Theorem 2.1 in [15] given only in the finite dimensional setting. Note that the proof here is completely different of those given in [15, 24] and it allows us to obtain the result in the infinite dimensional setting. It is interesting to point out that our assumptions on F are different to those supposed in Theorem 2.1 in [15]. They supposed that F has compact values and satisfies the linear growth condition and in our Theorem 3.3, F is supposed to be contained in a compact set. In a forthcoming paper, we extend our results to the case when F satisfies some linear growth condition. □

We end this section with a uniqueness result. We need first the following lemma. Its proof follows directly from the third part of Proposition 2.1.

Lemma 3.1. *Let $r \in]0, +\infty]$ and let S be a uniformly r-prox-regular subset of H. Then for any $x_1, x_2 \in S$ and any $\xi_i \in \partial d_S(x_i)$ $(i = 1, 2)$, one has*

$$\langle \xi_1 - \xi_2,\, x_1 - x_2 \rangle \;\geq\; -\frac{4}{r}\, \|x_1 - x_2\|^2 \;. \blacksquare$$

Theorem 3.4. *Under the hypothesis of either Theorem 3.2 or Theorem 3.3 and if in addition, there exist a positive function $g \in L^1(I,\mathbb{R})$ satisfying:*

1) $g(t) \leq \frac{4}{r}\,(l+1)\,(\dot v(t)+2)$, *for a.e.* $t \in I$,

2) $\forall\, t \in I,\ \ \forall\, x_1, x_2 \in \mathcal{C}_T,\ \ \forall y_i \in F(t, \mathcal{T}(t)x_i),\ \ i=1,2$ *one has*

$$\Big\langle y_1(t)-y_2(t),\, x_1(t)-x_2(t)\Big\rangle \;\geq\; g(t)\,\|x_1(t)-x_2(t)\|^2\ ,$$

then (FOSPD) *has a unique solution.*

Proof: Let u_0, u_1 be two solutions of (FOSPD) with the initial values φ_1, φ_2 in $\mathcal{C}_0$ with $\varphi_i(0) \in C(0)$ $(i=1,2)$, i.e., for $i=1,2$, one has

$$\begin{cases} \dot u_i(t) \in -N_{C(t)}(u_i(t)) + f_i(t), & \text{a.e. } \ t \in I\,,\\ f_i(t) \in F(t, \mathcal{T}(t)u_i), & \text{a.e. } \ t \in I\,,\\ u_i(t) \in C(t), & \text{for all } \ t \in I\,,\\ u_i = \varphi_i, & \text{on } \ [-\tau, 0]\ , \end{cases}$$

with

$$\|\dot u_i(t)\| \leq (l+1)\,(\dot v(t)+1), \qquad \text{a.e. } \ t \in I\ .$$

As for $i=1,2$ and for a.e. $t \in I$, one has $\|f_i(t)\| \leq l$, then one gets

$$-\dot u_i(t) + f_i(t) \;\in\; N_{C(t)}(u_i(t)) \cap (l{+}1)\,(\dot v(t)+2)\,\mathbb{B} \qquad \text{a.e. } \ t \in I\ .$$

This ensures by Proposition 2.1–(1) that for $i=1,2$ and for a.e. $t \in I$

$$-\dot u_i(t) + f_i(t) \;\in\; (l{+}1)\,(\dot v(t)+2)\,\partial d_{C(t)}(u_i(t))\ .$$

Using the previous lemma one obtains for a.e. $t \in I$

$$\Big\langle -\Big(\dot u_1(t){-}\dot u_2(t)\Big) + \Big(f_1(t){-}f_2(t)\Big),\; u_1(t){-}u_2(t)\Big\rangle \;\geq\; \frac{4}{r}\,(l{+}1)\,(\dot v(t){+}2)\,\|u_1(t){-}u_2(t)\|^2\,.$$

Now by the second hypothesis of the theorem, one has

$$\Big\langle f_1(t)-f_2(t),\, u_1(t)-u_2(t)\Big\rangle \;\geq\; g(t)\,\|u_1(t)-u_2(t)\|^2\ .$$

Additionning the two last inequalities one gets

$$-\Big\langle \dot u_1(t)-\dot u_2(t),\, u_1(t)-u_2(t)\Big\rangle \;\geq\; \Big[g(t) - \frac{4}{r}\,(l{+}1)\,(\dot v(t)+2)\Big]\|u_1(t)-u_2(t)\|^2\ ,$$

which can be rewritten as

$$\frac{d}{dt}\left(\frac{1}{2}\|u_1(t)-u_2(t)\|^2\right) \le \alpha(t)\,\|u_1(t)-u_2(t)\|^2 ,$$

with $\alpha(t) = \left[\frac{4}{r}\,(l+1)\,(\dot v(t)+2) - g(t)\right] \ge 0$ a.e. on I. This ensures by Gronwall inequality that for a.e. $t \in I$

$$\|u_1(t)-u_2(t)\|^2 \le \|u_1(0)-u_2(0)\|^2 \exp\left[2\int_0^t \alpha(s)\,ds\right] .$$

Hence, if u_1 and u_2 are two solutions of (FOSPD) with the same initial value $\varphi_1 = \varphi_2 \in \mathcal{C}(0)$, we get $u_1 = u_2$ on $[-\tau, 0]$ and $u_1(0) = u_2(0) \in C(0)$. Therefore, by the last inequality we obtain $u_1 = u_2$ on $[0,T]$ and so $u_1 = u_2$ on $[-\tau, T]$. Thus the uniqueness is established. ∎

4 – Second Order Perturbed Sweeping Process with delay

In all this section we let $r \in \,]0,\infty]$, $x_0 \in H$, $u_0 \in K(x_0)$, $\mathcal{V}_0$ be an open neighborhood of x_0 in H, and $\zeta > 0$ such that $x_0 + \zeta\mathcal{B} \subset \mathcal{V}_0$, and let $K\colon \mathrm{cl}(\mathcal{V}_0) \to H$ be a Lipschitz set-valued mapping with ratio $\lambda > 0$ taking nonempty closed uniformly r-prox-regular values in H.

First we state the following result used in our main proofs. It is a direct consequence of Theorem 3.1 in [4] by taking $G(t,x,u) = \{0\}$.

Theorem 4.1. *Assume that:*

- **(i)** $\forall\, x \in \mathrm{cl}(\mathcal{V}_0)$, $K(x) \subset \mathcal{K}_1 \subset l\mathbb{B}$, *$\mathcal{K}_1$ is a convex compact set in H, and* $l > 0$;
- **(ii)** $F\colon [0,\infty[\times H \times H \to H$ *is scalarily upper semi-continuous on* $[0,\frac{\zeta}{l}]\times \mathrm{gph}(K)$ *with nonempty convex weakly compact values;*
- **(iii)** $F(t,x,u) \subset \varrho(1+\|x\|+\|u\|)\mathbb{B}$, $\forall\,(t,x,u) \in [0,\frac{\zeta}{l}]\times \mathrm{gph}(K)$.

Then, for any $T \in \,]0,\frac{\zeta}{l}]$, there exists a Lipschitz mapping $x\colon I=[0,T] \to \mathrm{cl}(\mathcal{V}_0)$ such that:

$$\begin{cases} \ddot x(t) \in -N_{K(x(t))}(\dot x(t)) + F(t, x(t), \dot x(t)), & \text{a.e. on } I\,; \\ \dot x(t) \in K(x(t)), & \forall\, t \in I\,; \\ x(0) = x_0\,, \quad \dot x(0) = u_0\,, \end{cases}$$

with $\|\dot x(t)\| \le l$, $\|\ddot x(t)\| \le l\lambda + 2\,(1+\alpha+l)\,\varrho$ *a.e. on I.* ∎

Remark 4.1. We point out that the solution mapping x obtained in Theorem 4.1 is differentiable everywhere on I. $\square$

Now let us state the existence result for the second order perturbed sweeping process with delay (SOSPD).

Theorem 4.2. *Assume that (i) and the following conditions hold:*

(ii)$'$ F: $[0,+\infty[\times\mathcal{C}_0\times\mathcal{C}_0 \rightrightarrows H$ *is scalarily upper semi-continuous on* $[0,\frac{\zeta}{l}]\times\mathcal{C}_0\times\mathcal{C}_0$, *taking convex weakly compact values in* H, *and*

(iii)$'$ $F(t,\varphi,\phi) \subset \varrho(1+\|\varphi(0)\|+\|\phi(0)\|)\,\mathbb{B}$, $\forall\,(t,\varphi,\phi)\in[0,\frac{\zeta}{l}]\times\mathcal{C}_0\times\mathcal{C}_0$.

Then for every $T\in\,]0,\frac{\zeta}{l}]$ *and for every* $\phi\in\mathcal{C}_0$ *verifying* $\phi(0)=u_0$, *there exists a Lipschitz mapping* $x\colon[0,T]\to\mathrm{cl}(\mathcal{V}_0)$ *such that:*

$$\begin{cases} \ddot{x}(t)\in -N_{K(x(t))}(\dot{x}(t))+F(t,\mathcal{T}(t)x,\mathcal{T}(t)\dot{x}), & \text{a.e. on } [0,T]\,;\\ \dot{x}(t)\in K(x(t)), & \forall\,t\in[0,T]\,;\\ \mathcal{T}(0)x=\varphi \ \text{ and } \ \mathcal{T}(0)\dot{x}=\phi & \text{on } [-\tau,0]\,, \end{cases}$$

with $\varphi(t)=x_0+\displaystyle\int_0^t\phi(s)\,ds$, *for all* $t\in[-\tau,0]$, *and* $\|\dot{x}(t)\|\le l$ *and* $\|\ddot{x}(t)\|\le l\lambda+2(1+\alpha+l)\,\varrho$ *a.e. on* $[0,T]$.

Proof: Without loss of generality, we may take $T=1$. Let $\phi\in\mathcal{C}_0$ satisfying $\phi(0)=u_0$, and put $\varphi(t):=x_0+\int_0^t\phi(s)\,ds$ for all $t\in[-\tau,0]$. Let $(\mathcal{P}_n)$ be a subdivision of $[0,1]$ defined by the points: $t_i^n:=\frac{i}{n}$ $(i=0,1,...,n)$. For every $(t,x,u)\in[-\tau,t_1^n]\times\mathrm{gph}(K)$, we define f_0^n: $[-\tau,t_1^n]\times\mathrm{cl}(\mathcal{V}_0)\to H$ and g_0^n: $[-\tau,t_1^n]\times K(\mathrm{cl}(\mathcal{V}_0))\to H$ by

$$f_0^n(t,x)=\begin{cases}\varphi(t), & t\in[-\tau,0]\,,\\ \varphi(0)+n\,t(x-\varphi(0)), & t\in[0,t_1^n]\ ,\end{cases}$$

and

$$g_0^n(t,u)=\begin{cases}\phi(t), & t\in[-\tau,0]\,,\\ \phi(0)+n\,t(u-\phi(0)), & t\in[0,t_1^n]\ .\end{cases}$$

We have $f_0^n(\frac{1}{n},x)=x$ and $g_0^n(\frac{1}{n},u)=u$ for all $(x,u)\in\mathrm{gph}(K)$. Observe that the mapping $(x,u)\mapsto(\mathcal{T}(t_1^n)f_0^n(\cdot,x),\mathcal{T}(t_1^n)g_0^n(\cdot,u))$ from $\mathrm{gph}(K)$ to $\mathcal{C}_0\times\mathcal{C}_0$ is nonexpansive. Indeed, we have for all $(x,y)\in H\times H$

$$\begin{aligned}
\left\|\mathcal{T}\Big(\tfrac{1}{n}\Big)\, f_0^n(\cdot,x)-\mathcal{T}\Big(\tfrac{1}{n}\Big)\, f_0^n(\cdot,y)\right\|_{\mathcal{C}_0} &= \sup_{s\in[-\tau,0]}\left\|f_0^n\Big(s+\tfrac{1}{n},x\Big)-f_0^n\Big(s+\tfrac{1}{n},y\Big)\right\| \\
&= \sup_{s\in\left[-\tau+\frac{1}{n},\frac{1}{n}\right]}\left\|f_0^n(s,x)-f_0^n(s,y)\right\| \\
&= \sup_{0\le s\le \frac{1}{n}}\left\|n\,s(x-\varphi(0))-n\,s(y-\varphi(0))\right\| \\
&= \sup_{0\le s\le \frac{1}{n}}\|n\,s(x-y)\| \\
&= \|x-y\|\;.
\end{aligned}$$

In the same way, we get for all $(u,v)\in H\times H$

$$\left\|\mathcal{T}\Big(\tfrac{1}{n}\Big)\, g_0^n(\cdot,u)-\mathcal{T}\Big(\tfrac{1}{n}\Big)\, g_0^n(\cdot,v)\right\|_{\mathcal{C}_0} = \|u-v\|\;.$$

Hence the mapping $(x,u)\mapsto(\mathcal{T}(t_1^n)f_0^n(\cdot,x),\mathcal{T}(t_1^n)g_0^n(\cdot,u))$ from $\mathrm{gph}(K)$ to $\mathcal{C}_0\times\mathcal{C}_0$ is nonexpansive and so the set-valued mapping $F_0^n\colon[0,\frac{1}{n}]\times\mathrm{gph}(K)\rightrightarrows H$ defined by: $F_0^n(t,x,u)=F(t,\mathcal{T}(\frac{1}{n})f_0^n(\cdot,x),\mathcal{T}(\frac{1}{n})\,g_0^n(\cdot,u))$ is scalarily upper semi-continuous on $[0,\frac{1}{n}]\times\mathrm{gph}(K)$ because F is also scalarily upper semi-continuous on $[0,\frac{1}{n}]\times\mathcal{C}_0\times\mathcal{C}_0$, with nonempty convex weakly compact values in H and satisfies

$$F_0^n(t,x,u) \;=\; F\Big(t,\,\mathcal{T}\Big(\tfrac{1}{n}\Big)\,f_0^n(\cdot,x),\,\mathcal{T}\Big(\tfrac{1}{n}\Big)\,g_0^n(\cdot,u)\Big)\subset\varrho\Big(1+\|x\|+\|u\|\Big)\;,$$

for all $(t,x,u)\in[0,\frac{1}{n}]\times\mathrm{gph}(K)$ because $\mathcal{T}(\frac{1}{n})f_0^n(0,x)=x$ and $\mathcal{T}(\frac{1}{n})\,g_0^n(0,u)=u$. Hence F_0^n verifies conditions of Theorem 4.1 [4], provides a Lipschitz differentiable solution $y_0^n\colon[0,\frac{1}{n}]\to\mathrm{cl}(\mathcal{V}_0)$ to the problem

$$\begin{cases}
\ddot{y}_0^n(t)\in-N_{K(y_0^n(t))}(\dot{y}_0^n(t))+F\Big(t,\,\mathcal{T}\Big(\tfrac{1}{n}\Big)f_0^n(\cdot,y_0^n(t)),\,\mathcal{T}\Big(\tfrac{1}{n}\Big)f_1^n(\cdot,\dot{y}_0^n(t))\Big), & \text{a.e. on } \Big[0,\tfrac{1}{n}\Big]\,; \\
\dot{y}_0^n(t)\in K(y_0^n(t))\,, \qquad \forall\, t\in\Big[0,\tfrac{1}{n}\Big]\,; & \\
y_0^n(0)=x_0=\varphi(0)\,,\qquad \dot{y}_0^n(0)=u_0=\phi(0)\;. &
\end{cases}$$

Further we have $\|\dot{y}_0^n(t)\|\le l$ and $\|\ddot{y}_0^n(t)\|\le l\lambda+2(1+\alpha+l)\,\varrho$.

Set

$$y_n(t) \;=\; \begin{cases}\varphi(t), & \forall\, t\in[-\tau,0]\,,\\ y_0^n(t), & \forall\, t\in\Big[0,\tfrac{1}{n}\Big]\;.\end{cases}$$

Then, y_n is well defined on $[-\tau, \frac{1}{n}]$, with $y_n = \varphi$ on $[-\tau, 0]$ and

$$\dot{y}_n(t) = \begin{cases} \phi(t) & \forall\, t \in [-\tau, 0]\,, \\ \dot{y}_0^n(t) & \forall\, t \in \left]0, \frac{1}{n}\right[\,, \end{cases}$$

and

$$\begin{cases} \ddot{y}_n(t) \in -N_{K(y_n(t))}(\dot{y}_n(t)) + F\Big(t,\, \mathcal{T}\Big(\frac{1}{n}\Big) f_0^n(\cdot, y_n(t)),\, \mathcal{T}\Big(\frac{1}{n}\Big) g_0^n(\cdot, \dot{y}_n(t))\Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. on } \left[0, \frac{1}{n}\right]; \\ \dot{y}_n(t) \in K(y_n(t))\,, \qquad\qquad\qquad\qquad \forall\, t \in \left[0, \frac{1}{n}\right]; \\ y_n(0) = x_0 = \varphi(0)\,, \qquad \dot{y}_n(0) = u_0\,, \end{cases}$$

with $\|\dot{y}_n(t)\| \le l$ and $\|\ddot{y}_n(t)\| \le l\lambda + 2(1 + \alpha + l)\,\varrho$ a.e. $t \in [0, \frac{1}{n}]$.

Suppose that y_n is defined on $[-\tau, \frac{k}{n}]$ $(k \ge 1)$ with $y_n = \varphi$ on $[-\tau, 0]$ and satisfies:

$$y_n(t) = \begin{cases} y_0^n(t) = x_0 + \displaystyle\int_0^t \dot{y}_n(s)\, ds & \forall\, t \in \left[0, \frac{1}{n}\right], \\ y_1^n(t) := y_n\Big(\frac{1}{n}\Big) + \displaystyle\int_{\frac{1}{n}}^t \dot{y}_n(s)\, ds & \forall\, t \in \left[\frac{1}{n}, \frac{2}{n}\right], \\ \cdots & \\ y_{k-1}^n(t) := y_n\Big(\frac{k-1}{n}\Big) + \displaystyle\int_{\frac{k-1}{n}}^t \dot{y}_n(s)\, ds & \forall\, t \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \end{cases}$$

and y_n is a Lipschitz solution of

$$\begin{cases} y_n(t) = y_{k-1}^n(t) := y_n\Big(\frac{k-1}{n}\Big) + \displaystyle\int_{\frac{k-1}{n}}^t \dot{y}_n(s)\, ds & \forall\, t \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \\ \ddot{y}_n(t) \in -N_{K(y_n(t))}(\dot{y}_n(t)) + F\Big(t,\, \mathcal{T}\Big(\frac{k}{n}\Big) f_{k-1}^n(\cdot, y_n(t)),\, \mathcal{T}\Big(\frac{k}{n}\Big) g_{k-1}^n(\cdot, \dot{y}_n(t))\Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } \left[\frac{k-1}{n}, \frac{k}{n}\right], \\ \dot{y}_n(t) \in K(y_n(t)), \qquad\qquad \forall\, t \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \end{cases}$$

where f_{k-1}^n and g_{k-1}^n are defined for any $(x, u) \in \mathrm{gph}(K)$ as follows

$$f_{k-1}^n(t, x) = \begin{cases} y_n(t), & t \in \left[-\tau, \frac{k-1}{n}\right], \\ y_n\Big(\frac{k-1}{n}\Big) + n\Big(t - \frac{k-1}{n}\Big)\Big(x - y_n\Big(\frac{k-1}{n}\Big)\Big), & t \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \end{cases}$$

and

$$g_{k-1}^n(t,u) = \begin{cases} \dot{y}_n(t), & t \in \left[-\tau, \frac{k-1}{n}\right], \\ \dot{y}_n\left(\frac{k-1}{n}\right) + n\left(t - \frac{k-1}{n}\right)\left(u - \dot{y}_n\left(\frac{k-1}{n}\right)\right), & t \in \left[\frac{k-1}{n}, \frac{k}{n}\right]. \end{cases}$$

Similarly we can define $f_k^n, g_k^n : [-\tau, \frac{k+1}{n}] \times H \to H$ as

$$f_k^n(t,x) = \begin{cases} y_n(t), & t \in \left[-\tau, \frac{k}{n}\right], \\ y_n\left(\frac{k}{n}\right) + n\left(t - \frac{k}{n}\right)\left(x - y_n\left(\frac{k}{n}\right)\right), & t \in \left[\frac{k}{n}, \frac{k+1}{n}\right], \end{cases}$$

and

$$g_k^n(t,u) = \begin{cases} \dot{y}_n(t), & t \in \left[-\tau, \frac{k}{n}\right], \\ \dot{y}_n\left(\frac{k}{n}\right) + n\left(t - \frac{k}{n}\right)\left(u - \dot{y}_n\left(\frac{k}{n}\right)\right), & t \in \left[\frac{k}{n}, \frac{k+1}{n}\right], \end{cases}$$

for any $(x,u) \in \mathrm{gph}(K)$. Note that $\mathcal{T}(\frac{k+1}{n}) f_k^n(0,x) = f_k^n(\frac{k+1}{n}, x) = x$ and $\mathcal{T}(\frac{k+1}{n})\, g_k^n(0,u) = g_k^n(\frac{k+1}{n}, u) = u$, for all $(x,u) \in \mathrm{gph}(K)$.

Note also that, for all $(x,u),(y,v) \in \mathrm{gph}(K)$, we have

$$\begin{aligned} \left\|\mathcal{T}\left(\tfrac{k+1}{n}\right) f_k^n(\cdot,x) - \mathcal{T}\left(\tfrac{k+1}{n}\right) f_k^n(\cdot,y)\right\|_{\mathcal{C}_0} &= \sup_{s\in[-\tau,0]} \left\|f_k^n\left(s+\tfrac{k+1}{n}, x\right) - f_k^n\left(s+\tfrac{k+1}{n}, y\right)\right\| \\ &= \sup_{s\in\left[-\tau+\frac{k+1}{n}, \frac{k+1}{n}\right]} \|f_k^n(s,x) - f_k^n(s,y)\|, \end{aligned}$$

and

$$\begin{aligned} \left\|\mathcal{T}\left(\tfrac{k+1}{n}\right) g_k^n(\cdot,u) - \mathcal{T}\left(\tfrac{k+1}{n}\right) g_k^n(\cdot,v)\right\|_{\mathcal{C}_0} &= \sup_{s\in[-\tau,0]} \left\|g_k^n\left(s+\tfrac{k+1}{n}, u\right) - g_k^n\left(s+\tfrac{k+1}{n}, v\right)\right\| \\ &= \sup_{s\in\left[-\tau+\frac{k+1}{n}, \frac{k+1}{n}\right]} \|g_k^n(s,u) - g_k^n(s,v)\|. \end{aligned}$$

We distinguish two cases

(1) if $-\tau + \frac{k+1}{n} < \frac{k}{n}$, we have

$$\begin{aligned} \sup_{s\in\left[-\tau+\frac{k+1}{n}, \frac{k+1}{n}\right]} \|f_k^n(s,x) - f_k^n(s,y)\| &= \sup_{s\in\left[\frac{k}{n}, \frac{k+1}{n}\right]} \|f_k^n(s,x) - f_k^n(s,y)\| \\ &= \sup_{\frac{k}{n}\le s\le \frac{k+1}{n}} \left\|n\left(s - \tfrac{k}{n}\right)(x-y)\right\| = \|x-y\| \end{aligned}$$

and

$$\begin{aligned}
\sup_{s\in\left[-\tau+\frac{k+1}{n},\frac{k+1}{n}\right]} \|g_k^n(s,v)-g_k^n(s,v)\| &= \sup_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]} \|g_k^n(s,u)-g_k^n(s,v)\| \\
&= \sup_{\frac{k}{n}\le s\le\frac{k+1}{n}} \left\|n\Big(s-\frac{k}{n}\Big)\,(u-v)\right\| \;=\; \|u-v\| \;.
\end{aligned}$$

(**2**) if $\frac{k}{n} \le -\tau + \frac{k+1}{n} \le \frac{k+1}{n}$, we have

$$\begin{aligned}
\sup_{s\in\left[-\tau+\frac{k+1}{n},\frac{k+1}{n}\right]} \|f_k^n(s,x)-f_k^n(s,y)\| &\le \sup_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]} \|f_k^n(s,x)-f_k^n(s,y)\| \\
&= \sup_{\frac{k}{n}\le s\le\frac{k+1}{n}} \left\|n\Big(s-\frac{k}{n}\Big)\,(x-y)\right\| \;=\; \|x-y\|
\end{aligned}$$

and

$$\begin{aligned}
\sup_{s\in\left[-\tau+\frac{k+1}{n},\frac{k+1}{n}\right]} \|g_k^n(s,v)-g_k^n(s,v)\| &\le \sup_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]} \|g_k^n(s,u)-g_k^n(s,v)\| \\
&= \sup_{\frac{k}{n}\le s\le\frac{k+1}{n}} \left\|n\Big(s-\frac{k}{n}\Big)\,(u-v)\right\| \;=\; \|u-v\| \;.
\end{aligned}$$

So the mapping $(x,u) \mapsto (\mathcal{T}(\frac{k+1}{n})f_k^n(\cdot,x), \mathcal{T}(\frac{k+1}{n})g_k^n(\cdot,u))$ from $\mathrm{gph}(K)$ to $\mathcal{C}_0\times\mathcal{C}_0$ is nonexpansive. Hence, the set-valued mapping $F_k^n\colon [0,1]\times\mathrm{gph}(K) \rightrightarrows H$ defined by

$$F_k^n(t,x,u) \;:=\; F\Big(t,\,\mathcal{T}\Big(\tfrac{k+1}{n}\Big)f_k^n(\cdot,x),\,\mathcal{T}\Big(\tfrac{k+1}{n}\Big)\,g_k^n(.,u)\Big)$$

is scalarly upper semi-continuous on $[0,1]\times\mathrm{gph}(K)$ with nonempty convex weakly compact values. As above we can easily check that F_k^n satisfies the growth condition:

$$F_k^n(t,x,u) \subset \varrho\Big(1+\|x\|+\|u\|\Big)\,, \qquad \forall\,(t,x,u)\in[0,1]\times\mathrm{gph}(K)\;.$$

Applying Theorem 4.1 gives a Lipschitz solution $y_k^n : \left[\frac{k}{n},\frac{k+1}{n}\right] \to \mathrm{cl}(\mathcal{V}_0)$ to the problem

$$\begin{cases}
\ddot{y}_k^n(t) \in -N_{K(y_k^n(t))}(\dot{y}_k^n(t)) + F_k^n(t,y_k^n(t),\dot{y}_k^n(t)) & \text{a.e. on } \left[\frac{k}{n},\frac{k+1}{n}\right], \\
y_k^n\Big(\frac{k}{n}\Big) = y_n\Big(\frac{k}{n}\Big)\,, & \\
\dot{y}_k^n(t) \in K(y_k^n(t) & \forall\,t\in\left[\frac{k}{n},\frac{k+1}{n}\right]\,,
\end{cases}$$

with $\|\dot{y}_k^n(t)\| \le l,\ \ \|\ddot{y}_k^n(t)\| \le l\lambda + 2(1+\alpha+l)\,\varrho\,.$

Consequently, there exists $h_k^n \in L^1([\frac{k}{n}, \frac{k+1}{n}], H)$ such that

$$\begin{cases} h_k^n(t) \in F\Big(t,\, \mathcal{T}\Big(\frac{k+1}{n}\Big) f_k^n(\cdot, y_k^n(t)),\, \mathcal{T}\Big(\frac{k+1}{n}\Big) g_k^n(\cdot, \dot{y}_k^n(t))\Big) & \text{a.e. on } \Big[\frac{k}{n}, \frac{k+1}{n}\Big]\,, \\ \ddot{y}_k^n(t) \in -N_{K(y_k^n(t))}(\dot{y}_k^n(t)) + h_k^n(t) & \text{a.e. on } \Big[\frac{k}{n}, \frac{k+1}{n}\Big]\,, \\ y_k^n\Big(\frac{k}{n}\Big) = y_n\Big(\frac{k}{n}\Big)\,, & \\ \dot{y}_k^n(t) \in K(y_k^n(t)) & \forall\, t \in \Big[\frac{k}{n}, \frac{k+1}{n}\Big]\,. \end{cases}$$

Thus, by induction, we can construct a continuous function $y_n \colon [-\tau, 1] \to \mathrm{cl}(\mathcal{V}_0)$ with $y_n = \varphi$ on $[-\tau, 0]$ such that its restriction on each interval $[\frac{k}{n}, \frac{k+1}{n}]$ is a solution to

$$\begin{cases} \ddot{x}(t) \in -N_{K(x(t))}(\dot{x}(t)) + F\Big(t, \mathcal{T}\Big(\frac{k+1}{n}\Big) f_k^n(\cdot, x(t)), \mathcal{T}\Big(\frac{k+1}{n}\Big) g_k^n(\cdot, \dot{x}(t))\Big) & \text{a.e. } \Big[\frac{k}{n}, \frac{k+1}{n}\Big] \\ x\Big(\frac{k}{n}\Big) = y_n\Big(\frac{k}{n}\Big)\,, & \\ \dot{x}(t) \in K(x(t)) & \forall\, t \in \Big[\frac{k}{n}, \frac{k+1}{n}\Big]\,. \end{cases}$$

Indeed, set

$$y_n(t) := \begin{cases} \varphi(t) & \forall\, t \in [-\tau, 0]\,, \\ y_0^n(t) & \forall\, t \in \Big[0, \frac{1}{n}\Big]\,, \\ \cdots & \\ y_k^n(t) & \forall\, t \in \Big[\frac{k}{n}, \frac{k+1}{n}\Big]\,, \end{cases}$$

$$\dot{y}_n(t) := \begin{cases} \phi(t) & \forall\, t \in [-\tau, 0]\,, \\ \dot{y}_0^n(t) & \forall\, t \in \Big]0, \frac{1}{n}\Big[\,, \\ \cdots & \\ \dot{y}_k^n(t) & \forall\, t \in \Big]\frac{k}{n}, \frac{k+1}{n}\Big[\,, \end{cases}$$

and

$$h_n(t) := h_k^n(t) \qquad \text{on} \quad \Big]\frac{k}{n}, \frac{k+1}{n}\Big]\,.$$

Also, for notational convenience, we set $\theta_n(t) := \frac{k+1}{n}$ and $\delta_n(t) := \frac{k}{n}$, for $t \in]\frac{k}{n}, \frac{k+1}{n}]$. Then, we get

(4.1)

$$\begin{cases} h_n(t) \in F\Big(t,\, \mathcal{T}(\theta_n(t))\, f^n_{n\theta_n(t)-1}(\cdot, y_n(t)),\, \mathcal{T}(\theta_n(t))\, g^n_{n\theta_n(t)-1}(\cdot, \dot{y}_n(t))\Big) & \text{a.e. }]0,1]\,, \\ \ddot{y}_n(t) \in -N_{K(y_n(t))}(\dot{y}_n(t)) + h_n(t) & \text{a.e. }]0,1]\,, \\ y_n(0) = x_0 = \varphi(0)\,, \quad \dot{y}_n(0) = u_0 \in K(x_0)\,, & \\ \dot{y}_n(t) \in K(y_n(t) & \forall\, t \in [0,1]\,, \end{cases}$$

with

$$\|\dot{y}_n(t)\| \le l\,, \quad \|\ddot{y}_n(t)\| \le l\lambda + 2\,(1{+}\alpha{+}l)\,\varrho \qquad \text{a.e.} \ \ ]0,1]$$

and a.e $t \in [0,1]$,

$$F\Big(t,\, \mathcal{T}(\theta_n(t))\, f^n_{n\theta_n(t)-1}(\cdot, y_n(t)),\, \mathcal{T}(\theta_n(t))\, g^n_{n\theta_n(t)-1}(\cdot, \dot{y}_n(t))\Big) \ \subset$$
$$\subset \ \varrho\Big(1 + \|y_n(t)\| + \|\dot{y}_n(t)\|\Big) \ .$$

Step 2. *Uniform convergence of* (y_n):

Let (y_n) and (h_n) be as in (4.1), we have

$$\|h_n(t)\| \ \le \ \varrho\Big(1 + \|y_n(t)\| + \|\dot{y}_n(t)\|\Big) \ \le \ \varrho\Big(1 + \|x_0\| + Tl + l\Big)$$

a.e. $t \in [0,1]$, $\forall\, n$ so (h_n) is a bounded sequence in $\mathcal{L}^1([0,1],H)$. By extracting a subsequence, we may assume that (h_n) converge weakly to some $h \in L^1([0,1],H)$. Further, (y_n) is relatively compact in $\mathcal{C}([0,1],H)$, so we may suppose that (y_n) converges in $\mathcal{C}([0,1],H)$ to some $z \in \mathcal{C}([0,1],H)$ with

$$z(t) \ = \ \varphi(0) + \int_0^t \dot{z}(s)\, ds \qquad \forall\, t \in [0,1] \ .$$

For each $t \in [-\tau, 0]$, we set

$$y(t) \ = \ \begin{cases} \varphi(t) & \forall\, t \in [-\tau, 0]\,, \\ z(t) & \forall\, t \in [0,1] \ . \end{cases}$$

Then, $y \in \mathcal{C}_1$ and y_n converges to y in $\mathcal{C}_1$.

Step 3. We claim that $\mathcal{T}(\theta_n(t)) f^n_{n\theta_n(t)-1}(\cdot, y_n(t))$ and $\mathcal{T}(\theta_n(t)) g^n_{n\theta_n(t)-1}(\cdot, \dot{y}_n(t))$ pointwise converge on]0,1] to $\mathcal{T}(t)y$ and $\mathcal{T}(t)\dot{y}$ respectively in the Banach space $\mathcal{C}_0$. The proof is similar to the one given in Theorem 2.1 in [15].

Step 4. *Existence of solutions*:

We have for a.e. $t \in [0,1]$

$$h_n(t) \in F\Big(t,\, \mathcal{T}(\theta_n(t))\, f^n_{n\theta_n(t)-1}(\cdot, y_n(t)),\, \mathcal{T}(\theta_n(t))\, g^n_{n\theta_n(t)-1}(\cdot, \dot{y}_n(t))\Big) \ .$$

As h_n converges weakly to h in $\mathcal{L}^1([0,1],H)$ and for all $t \in\,]0,1]$

$$\Big\|\mathcal{T}(\theta_n(t))\, f^n_{n\theta_n(t)-1}(\cdot, y_n(t)) - \mathcal{T}(t)\, y\Big\|_{\mathcal{C}_0} \ \to \ 0$$

and

$$\Big\|\mathcal{T}(\theta_n(t))\, g^n_{n\theta_n(t)-1}(\cdot, \dot{y}_n(t)) - \mathcal{T}(t)\, \dot{y}\Big\|_{\mathcal{C}_0} \ \to \ 0 \ ,$$

and as the multifunction F is scalarly upper semi-continuous with convex weakly compact values, by a classical closure result (see, for instance [16], we get $h(t) \in F(t, \mathcal{T}(t)y, \mathcal{T}(t)\dot{y})$.

Further, as $\|\dot{y}_n(t)\| \leq l$, we may assume that $\dot{y}_n$ converges weakly to $\dot{y}$ and similarly $\ddot{y}_n$ converges weakly to $\ddot{y}$. By what precedes one has $(\ddot{y}_n - h_n)$ weakly converges to $\ddot{y} - h$ in $L^1([0,1], H)$ and so Mazur's lemma ensures that for almost every $t \in [0,1]$

$$\ddot{y}(t) - h(t) \in \bigcap_n \overline{\text{co}}\Big\{\ddot{y}_k(t) - h_k(t)\colon\ k \geq n\Big\} .$$

Fix such t in I and any μ in H, then the last relation gives

$$\Big\langle \ddot{y}(t) - h(t),\, \mu \Big\rangle \leq \inf_n \sup_{k \geq n} \Big\langle \ddot{y}_n(t) - h_n(t),\, \mu \Big\rangle$$

and hence according to (4.1) one has

$$\ddot{y}_n(t) - h_n(t) \in -N_{K(y_n(t))}(\dot{y}_n(t)) \qquad \text{a.e. on } \ ]0,1]$$

with $\|\ddot{y}_n(t) - h_n(t)\| \leq l\lambda + 2(1{+}\alpha{+}l) + \varrho(1{+}\|x_0\|{+}Tl{+}l) = \delta$. Then, for a.e. $t \in]0,1]$

$$\ddot{y}_n(t) - h_n(t) \in -\delta\, \partial d_{K(y_n(t))}(\dot{y}_n(t))$$

hence, one obtains

$$\begin{aligned} \Big\langle \ddot{y}(t) - h(t),\, \mu \Big\rangle &\leq \limsup_n \sigma\Big(-\delta\, \partial d_{K(y_n(t))n}(\dot{y}_n(t)),\, \mu\Big) \\ &\leq \sigma\Big(-\delta\, \partial d_{K(y(t))}(\dot{y}(t)),\, \mu\Big) . \end{aligned}$$

As the set $\partial d_{K(y(t))}(\dot{y}(t))$ is closed convex, we obtain

$$\ddot{y}(t) - h(t) \in -\delta\, \partial d_{K(y(t))}(\dot{y}(t))$$

and then

$$\ddot{y}(t) - h(t) \in -N_{K(y(t))}(\dot{y}(t))$$

because $\dot{y}(t) \in K(x(t))$.

Thus,

$$\ddot{y}(t) \in -N_{K(y(t))}(\dot{y}(t)) + F\Big(t, \mathcal{T}(t)y, \mathcal{T}(t)\dot{y}\Big) .$$

The proof then is complete. ∎

Remark 4.2.

1. Note that some existence results have been given when the perturbation F depends only on the variables t and $\dot{x}_t$, see for example [11, 19]. So, our result in Theorem 4.2 is more general than those proved in [11, 19] because our perturbation F depends on all the variables t, x_t, and $\dot{x}_t$.

2. In [19] the set-valued mapping K is assumed with convex values and the perturbation F is assumed to be uniformly continuous and depending only on t and $\dot{x}_t$, and with nonconvex values, which is another variant of existence results for (SOSPD), because our perturbation F in Theorem 4.2 has convex values and upper semi-continuous. Our techniques used here can be also used to adapt the proof of the result in [19] to obtain a generalization when K has nonconvex values and under the same assumptions on F. $\square$

ACKNOWLEDGEMENT – The authors would like to thank the referee for his careful and thorough reading of the paper.

REFERENCES

[1] Aubin, J.P. and Cellina, A. – *Differential Inclusions – Set-Valued Maps and Viability Theory,* Springer-Verlag, Berlin, 1984.

[2] Benabdellah, H. – Existence of solutions to the nonconvex sweeping process, *J. Diff. Equations,* 164(2) (2000), 286–295.

[3] Benabdellah, H.; Castaing, C.; Salvadori, A. and Syam, A. – Nonconvex Sweeping Process, *J. Appl. Anal.,* 2(2) (1996), 217–240.

[4] Bounkhel, M. – General existence results for second order nonconvex sweeping process with unbounded perturbations, *Portugaliae Mathematica,* 60(3) (2003), 269–304.

[5] Bounkhel, M.; Tadj, L. and Hamdi, A. – Iterative Schemes to Solve Non convex Variational Problems, *J. Ineq. Pure & Appl. Math.,* 4(1) (2003), Article 14.

[6] Bounkhel, M. and Laouir-Azzam, D. – Théorèmes d'existence pour des inclusions differentielles du second ordre, *C. R. Acad. Sci., Serie I,* 336 (2003), 657–659.

[7] Bounkhel, M. and Laouir-Azzam, D. – Existence results on the second order nonconvex sweeping processes with perturbations (to appear in *Set-Valued Analysis*).

[8] Bounkhel, M. and Thibault, L. – On various notions of regularity of sets in nonsmooth analysis, *Nonlinear Anal.: Theory, Methods and Applications,* 48(2) (2002), 223–246.

[9] Bounkhel, M. and Thibault, L. – *Further characterizations of regular sets in Hilbert spaces and their applications to nonconvex sweeping process* (submitted).

[10] Bounkhel, M. and Yarou, M. – Existence results for nonconvex sweeping process with perturbations and with delay: Lipschitz case, *Arab Journal of Mathematics,* 8(2) (2002), 1–12.

[11] Bounkhel, M. and Yarou, M. – *Existence results for first and second order nonconvex sweeping processes with perturbations and with delay: Fixed point approach* (submitted).

[12] Castaing, C. – Quelques problèmes d'évolution du second ordre, *Sém. d'Anal. Convexe,* Montpellier, exposé No. 5, (1988).

[13] Castaing, C. and Monteiro-Marques, M.D.P. – Topological properties of solutions sets for sweeping processes with delay, *Portugaliae Mathematica,* 54(4) (1997), 485–507.

[14] Castaing, C. and Monteiro Marques, M.D.P. – Sweeping process by nonconvex closed moving set with perturbation, *C. R. Acad. Sci., Serie I,* 319 (1994), 127–132.

[15] Castaing, C.; Salvadori, A. and Thibault, L. – Functional evolution equations governed by nonconvex sweeping process, *J. Nonl. Conv. Anal.,* 2(1) (2001).

[16] Castaing, C. and Valadier, M. – Convex Analysis and Measurable Multifunctions, *Lecture Notes in Mathematics,* 580, Springer-Verlag (1977).

[17] Colombo, G. and Goncharov, V.V. – The sweeping process without convexity, *Set-Valued Analysis,* 7(4) (1999), 357–374.

[18] Gavioli, A. and Malaguti, L. – Viable solutions of differential inclusions with memory in Banach spaces, *Portugaliae Mathematica,* 57(2) (2000), 203–217.

[19] Duc Ha, T.X. and Monteiro-Marques, M.D.P. – Nonconvex second-order differential inclusions with memory, *Set-Valued Analysis,* 3 (1995), 71–86.

[20] Monteiro-Marques, M.D.P. – *Differential Inclusions in Nonsmooth Mechanical Problem, Shoks and Dry Friction*, Birkhäuser, 1995.

[21] Poliquin, R.A.; Rockafellar, R.T. and Thibault, L. – Local differentiability of distance functions, *Trans. Amer. Math. Soc.,* 352(11), (2000), 5231–5249.

[22] Ruess, W.M. and Summers, W.H. – Operators Semigroups for functional differential equations with delay, *Trans. Amer. Math. Soc.,* 341(2) (1994), 695–719.

[23] Syam, A. – *Contributions aux Inclusions Différentielles*, Thèse, Université de Montpellier 2, Montpellier, 1993.

[24] Thibault, L. – *Sweeping process with regular and nonregular sets*, Preprint, Montpellier, 1999.

M. Bounkhel,
College of Science, Department of Mathematics,
P.O. Box 2455, Riyadh 11451 – SAUDI ARABIA
E-mail: `bounkhel@ksu.edu.sa`

and

M. Yarou,
Department of Mathematics, University of Jijel,
B.P.: 98, 18000 – ALGERIA