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OPTIMALITY CONDITIONS AND DUALITY THEOREMS FOR NONLIPSCHITZ OPTIMIZATION PROBLEMS

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Abstract: In this paper, we are concerned with a multiobjective optimization problem (P). Using a notion of approximation derived from Jourani and Thibault , we give necessary and sufficient optimality conditions for (P). We establishe also duality theorems.

1 – Introduction

Many authors studied optimality conditions for vector optimization problems where the objectives are defined by single-valued mappings and obtained optimality conditions in terms of Lagrange–Kuhn–Tucker multipliers. Lin [15] has given optimality conditions for differentiable vector optimization problems by using the Motzkin's theorem. Censor [5] gives optimality conditions for differentiable convex vector optimization by using the theorem of Dubovitskii–Milyutin. When the objective functions are locally Lipschitzian, Minami [17] obtained Kuhn–Tucker type or Fritz–John type optimality conditions for weakly efficient solutions in terms of the generalized gradient. Also in the literature, some optimality conditions for set-valued optimization problems are studied (see Corley [6], Luc [16], Amahroq and Taa [3], ...).

Let us first recall that a feasible point x^* is called an efficient solution of (P) if, for any feasible x, $f_i(x) \leq f_i(x^*)$ for all $i \in I$ implies $f_i(x) = f_i(x^*)$ for all i; whereas a feasible point x^* is called a weakly efficient solution if no feasible x satisfies $f_i(x) < f_i(x^*)$ for all i. Characterization of efficient solutions (and weakly

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efficient solutions) for a constrained multi-objective programming problem are of practical interest, since multipliers associated with the dual optimality conditions have useful economic interpretations, for instance in Welfare Economics [4].

Consider the following multi-objective nonsmooth programming problem

$$(P): \begin{cases} \text{Min } \left(f_1(x), ..., f_p(x)\right) \\ \text{subject to}: \quad g_j(x) \leq 0, \quad j = 1, ..., m \,, \end{cases}$$

where the functions f_i and g_j, defined on a Banach space X, $i \in I = \{1, ..., p\}$, $j \in J = \{1, ..., m\}$, admit approximations.

In general, problem (P) is nonconvex and the Kuhn–Tucker optimality conditions (see Theorem 3.1) established by Amahroq and Gadhi [2] are only necessary. Under what assumptions, are the Kuhn–Tucker conditions also sufficient for the optimality of problem (P)? In [13], Kim and Lee considered the optimization problem (P) when the data are Locally Lipschitz. They give duality theorems by using the concepts of pseudoinvexity and quasiinvexity.

In this note, we extend Kim and Lee's findings by seeing if they are valid for larger class of problems with $\varphi = (f_1, ..., f_p)$ and $g = (g_1, ..., g_m)$ admitting approximations [1]. Based on necessary optimality conditions given by Amahroq and Gadhi [2] (see Theorem 3.1), our approach consists of formulating the Mond–Weir dual problem (D) and establishing duality theorems for (P) and (D) without any constraint qualification. We give also sufficient optimality conditions for (P).

Such a notion of approximation allows applications to continuous functions. Note that for a continuous function, symmetric subdifferentials [19], upper semicontinuous convexificators [9], and upper semicontinuous approximate Jacobians [10] are approximations. Naturally, for a locally Lipschitz function, most known subdifferentials such as the subdifferentials of Clarke, Michel–Penot, Ioffe–Mordukhovich and Treiman can be chosen as approximations.

The outline of the paper is as follows: preliminary results are described in Section 2; necessary and sufficient optimality conditions are given in Section 3; Sections 4 is reserved for duality results.

2 – Preliminaries

Let X and Y be two Banach spaces. We denote by $L(X,Y)$ the set of continuous linear mappings between X and Y, $\mathbb{B}_Y$ the closed unit ball of Y centered at the origin, $\mathbb{S}_Y$ the unit sphere of Y and X^* the continuous dual of X. We write $\langle .,. \rangle$ for the canonical bilinear form with respect to the duality $\langle X^*, X \rangle$.

In all the sequel we will need the following definition. It was introduced for the first time by Jourani and Thibault [12] and revised after by Allali and Amahroq [1]. Here we adopt the latest definition of approximation [1].

Definition 2.1 ([1])**.** Let f be a mapping from X into Y, $\overline{x} \in X$ and $A_f(\overline{x}) \subset L(X,Y)$. $A_f(\overline{x})$ is said to be an approximation of f at $\overline{x}$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) - f(\overline{x}) \in A_f(\overline{x})(x-\overline{x}) + \varepsilon \|x-\overline{x}\| \mathbb{B}_Y \tag{2.1}$$

for all $x \in \overline{x} + \delta\mathbb{B}_X$. □

It is easy to see that $f+g$ has $A_f(\overline{x}) + A_g(\overline{x})$ as an approximation at $\overline{x}$ whenever $A_f(\overline{x})$ and $A_g(\overline{x})$ are approximations of f and g at $\overline{x}$.

Note that $A_f(\overline{x})$ is a singleton if and only if f is Fréchet differentiable at $\overline{x}$. In [1], it is shown that when f is a locally Lipschitz function, it admits as an approximation the Clarke subdifferential of f at $\overline{x}$; i.e.

$$A_f(\overline{x}) = \partial f(\overline{x}) := cl\, co\Big\{ \mathrm{Lim}\, \nabla f(x_n)\,;\ x_n \in \mathrm{dom}\nabla f \text{ and } x_n \to \overline{x} \Big\}\,.$$

In order to give an example of non locally Lipschitz function, let us recall the following definition.

Definition 2.2 ([19])**.** Let $f\colon X \to \overline{R} := [-\infty,+\infty]$ be an extended-real--valued function and $\overline{x} \in \mathrm{dom}(f)$. The symmetric subdifferential of f at $\overline{x}$ is defined by

$$\partial^0 f(\overline{x}) := \partial f(\overline{x}) \sqcup [-\partial(-f)(\overline{x})]$$

where $\partial f(\overline{x}) := \limsup\limits_{x \xrightarrow{f} \overline{x},\, \varepsilon \searrow 0} \widehat{\partial}_\varepsilon f(x)$ and $\widehat{\partial}_\varepsilon f(x)$ is the ε-Fréchet subdifferential of f at x. For more details see [19]. □

Note that sufficient conditions for the upper semicontinuity of $\partial^0 f(.)$ can be found in [8] and [14]. It has been proved by Amahroq and Gadhi [2] that if f is continuous then $\partial^0 f(\overline{x})$ is an approximation of f at $\overline{x}$.

Remark 2.1. By a similar argument to that used in [1], Theorem 2.3 of [10] (Mean value theorem) implies that the upper semicontinuous hull of an approximate Jacobian [10, 11],

$$\overrightarrow{\partial^* f(\overline{x})} := \partial^* f(\overline{x}) \cup \Big\{ M \in \mathbb{R}^{n\times n} :\ x_k \to \overline{x},\ M_k \in \partial^* f(x_k)\,,\ M_k \to M \Big\}\,,$$

is an approximation.

Here, $\partial^* f(\overline{x})$ denotes the approximate Jacobian of f at $\overline{x}$. For more details on this notion, we refer the interested reader to [10, 11]. $\square$

Let $C := \{x \in X : \ g_j(x) \leq 0, \ j = 1, ..., \ m\}$. Assuming that g admits an approximation at $\overline{x}$, the following regularity condition is an adaptation of Amahroq and Gadhi's regularity [2] to our case.

Definition 2.3 ([2]). The problem (P) is said to be regular at $\overline{x} \in C$ if there exist a neighborhood U of $\overline{x}$ and $\delta, \gamma > 0$ such that :
$\forall y^* \in [0, +\infty[^m$, $\forall x \in U$, $\forall x^* \in A_g(x)$, $\exists \xi \in \delta \mathbb{B}_X$ such that

$$\langle y^*, g(x)\rangle + \langle y^* \circ x^*, \xi\rangle \ \geq \ \gamma \, \|y^*\| \ . \ \square$$

For the rest of the paper (Section 3 and Section 4), we suppose that X is separable and that the functions $f_i, \ i = 1, ..., p$, and $g_j, \ j = 1, ..., m$, admit approximations $A_{f_i}(\overline{x})$ and $A_{g_j}(\overline{x})$ at $\overline{x}$. Moreover, the functions f_i and g_j are assumed to have the following properties

- If $x_n^* \in \mu_n^* A_{g_j}(x_n)$, where $x_n^* \overset{w^*}{\to} x^*$ in X^*, $\mu_n^* \to \mu^*$ in $\mathbb{R}$ and $x_n \to \overline{x}$ in X, then $x^* \in \mu^* A_{g_j}(\overline{x})$.
- There exists $\delta > 0$ such that for every $x \in \overline{x} + \delta \mathbb{B}_X$, the function f_i admits an approximation $A_{f_i}(x)$ at x and $A_{f_i}(\overline{x})$ is bounded w^*-closed.
- There exists $\delta > 0$ such that for every $x \in \overline{x} + \delta \mathbb{B}_X$ the function g_j admits an approximation $A_{g_j}(x)$ at x.
- For each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \overline{x} + \delta \mathbb{B}_X$

$$A_{f_i}(x) \ \subset \ A_{f_i}(\overline{x}) + \varepsilon \, \mathbb{B}_{X^*} \ .$$

- For each $\varepsilon > 0$ and for each $\mu_n^* \to \mu^*$ in $[0, 1]$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\mu_n^* A_{g_j}(x) \ \subset \ \mu^* A_{g_j}(\overline{x}) + \varepsilon \, \mathbb{B}_{X^*} \quad \text{for all} \ \ n \geq n_0 \ \text{ and } \ x \in \overline{x} + \delta \mathbb{B}_X \ .$$

3 – Necessary and sufficient optimality conditions

The following theorem is a direct consequence of Theorem 2 of [2]. It gives necessary optimality conditions for the multi-objective optimization problem (P).

Theorem 3.1 (Necessary optimality conditions). *Suppose that $\overline{x}$ is an efficient solution of (P). Under the above regularity condition, there exist vectors $p^* = \left(\lambda_1^*, ..., \lambda_p^*\right) \in \mathbb{R}^p$, $\|p^*\| = 1$ and $y^* = (\mu_1^*, ..., \mu_m^*) \in \mathbb{R}^m$ such that*

$$0 \in \sum_{i=1}^{p} \lambda_i^* A_{f_i}(\overline{x}) + \sum_{j=1}^{m} \mu_j^* A_{g_j}(\overline{x}) \ , \tag{3.1}$$

$$\mu_j^* g_j(\overline{x}) = 0, \quad g_j(\overline{x}) \leq 0 \quad \text{for any} \ \ j = 1, ..., m \ , \tag{3.2}$$

$$\left(\lambda_1^*, ..., \lambda_p^*, \mu_1^*, ..., \mu_m^*\right) \geq 0 \ . \tag{3.3}$$

Remark 3.1. With appropriate data, Zowe and Kurcyusz's regularity [20] implies the regularity of [2]. For more details, we refer the reader to [2]. □

To give sufficient optimality conditions, we shall need additional assumption on the data. The definitions that we propose below are more comprehensive than Giorgi and Guerraggio's [7]; however, they are identical when the data are Lipschitz and the Clarke's subdifferential is taken as the approximation.

Definition 3.1. Suppose that a function $g : X \to \mathbb{R}$ admit an approximation $A_g(x) \subset L(X, \mathbb{R})$ for all $x \in X$. Then:

1. Let $\eta : X \times X \to X$ be a mapping; we say that g is (η, A_g)-pseudoinvex if

$$g(y) < g(x) \quad \text{implies} \quad \langle \xi, \eta(y,x) \rangle < 0 \ ,$$

$\forall x, y \in X, \ \forall \xi \in A_g(x)$.

2. Let $\eta : X \times X \to X$ be a mapping; we say that g is (η, A_g)-quasiinvex if

$$g(y) \leq g(x) \quad \text{implies} \quad \langle \xi, \eta(y,x) \rangle \leq 0 \ ,$$

$\forall x, y \in X, \ \forall \xi \in A_g(x)$.

3. Let $\eta : X \times X \to X$ be a mapping; we say that g is strictly (η, A_g)-pseudoinvex if

$$g(y) \leq g(x) \quad \text{implies} \quad \langle \xi, \eta(y,x) \rangle < 0 \ ,$$

$\forall x, y \in X$ with $x \neq y$, $\forall \xi \in A_g(x)$. □

Theorem 3.2 (Sufficient optimality conditions). *Let $\overline{x} \in C$ and suppose that*

1. *The functions f_i are (η, A_{f_i})-pseudoinvex for all $i = 1, ..., p$,*

2. *The functions g_j are $\left(\eta, A_{g_j}\right)$-quasiinvex for all $j = 1, ..., m$,*

3. *It is true all the thesis of Theorem 3.1.*

Then, $\overline{x}$ is a weakly efficient solution of (P).

Proof: By contradiction, suppose that $\overline{x}$ is not a weakly efficient solution of (P). Then, there exists $x \in X$ such that

$$\begin{cases} \Big(f_1(x) - f_1(\overline{x}), ..., f_p(x) - f_p(\overline{x})\Big) \in -\operatorname{Int}\mathbb{R}^p_+ \ , \\ \qquad g_j(x) \le 0 \quad \text{for all} \ \ j = 1, ..., m \ . \end{cases}$$

Consequently, for every $i \in \{1, ..., p\}$,

$$f_i(x) - f_i(\overline{x}) < 0 \ .$$

On the one hand, from the (η, A_{f_i})-pseudoinvexity of f_i, we have

$$\langle \xi, \eta(x, \overline{x}) \rangle < 0 \quad \text{for all} \ \ \xi \in A_{f_i}(\overline{x}) \ . \tag{3.4}$$

On the other hand, from hypothesis 3, there exist $a_i \in A_{f_i}(\overline{x})$ and $b_j \in A_{g_j}(\overline{x})$ such that

$$\begin{cases} \lambda_i^* \ge 0, \quad \left(\lambda_1^*, ..., \lambda_p^*\right) \ne 0 \ , \\ \displaystyle\sum_{i=1}^{p} \lambda_i^* a_i + \sum_{j=1}^{m} \mu_j^* b_j = 0 \ . \end{cases} \tag{3.5}$$

Combining (3.4) and (3.5),

$$\left\langle -\sum_{j=1}^{m} \mu_j^* b_j \ , \ \eta(x, \overline{x}) \right\rangle < 0 \ . \tag{3.6}$$

Combining (3.2) and (3.3), one has

$$\mu_j^* \Big(g_j(x) - g_j(\overline{x})\Big) \le 0 \ .$$

from the (η, A_{g_j})-quasiinvexity of g_j, we get

$$\left\langle \sum_{j=1}^{m} \mu_j^* b_j \ , \ \eta(x, \overline{x}) \right\rangle \le 0 \ ,$$

which is a contradiction. ∎

4 – Duality Theorems

In this section, we suppose also that the functions f_i, $i = 1, ..., p$, and g_j, $j = 1, ..., m$, admit approximations $A_{f_i}(x)$ and $A_{g_j}(x)$ at every point x. Using the necessary optimality conditions of Theorem 3.1, we formulate the Mond–Weir dual problem (D) [18] and establish duality theorems for (P) and (D).

Consider the Mond–Weir dual problem (D) of (P),

$$(D): \quad \text{Max}\ \varphi(v) = \Big(f_1(v), ..., f_p(v)\Big)$$

$$\text{s.t.} \quad 0 \in \sum_{i=1}^{p} \lambda_i^* A_{f_i}(v) + \sum_{j=1}^{m} \mu_j^* A_{g_j}(v)\ , \tag{4.1}$$

$$\mu_j^* g_j(v) \geq 0, \quad \text{for any}\ \ j = 1, ..., m\ , \tag{4.2}$$

$$\Big(\lambda_1^*, ..., \lambda_p^*, \mu_1^*, ..., \mu_m^*\Big) \geq 0\,, \quad \Big(\lambda_1^*, ..., \lambda_p^*\Big) \neq (0, ..., 0)\ . \tag{4.3}$$

Remark 4.1. In formulating (D), we do not use the equality in (4.2). ▫

Remark 4.2. In the hypotheses of Theorem 3.1, the set of feasible points of (D) is nonempty. ▫

In the following result, we establish weak duality relations between problems (P) and (D). The argument is similar to that used by Kim and Lee in [13], but we give the proof in a more general situation.

Theorem 4.1 (Weak Duality). *Suppose that for all $i \in \{1, ..., p\}$ and $j \in \{1, ..., m\}$, f_i is (η, A_{f_i})-pseudoinvex and g_j is strictly $\Big(\eta, A_{g_j}\Big)$-pseudoinvex. Then, for any feasible point x of (P) and any feasible point (v, λ, μ) of (D), there exists $s \in \{1, ..., p\}$ such that*

$$f_s(x) \ \geq\ f_s(v)\ .$$

Proof: By contrary, suppose that there exist a feasible point x of (P) and a feasible point (v, λ, μ) such that

$$f_i(x) < f_i(v) \quad \text{for all}\ \ i = 1, ..., p\ .$$

Remark that $x \neq v$. By the (η, A_{f_i})-pseudoinvexity of f_i, we get

$$\langle \xi_i, \eta(x, v)\rangle < 0 \quad \text{for all}\ \ \xi_i \in A_{f_i}(v)\,, \ \ i = 1, ..., p\ . \tag{4.4}$$

From (4.3) and (4.4), it follows that

$$\left\langle \sum_{i=1}^{p} \lambda_i \xi_i \, , \; \eta(x,v) \right\rangle < 0 \, . \tag{4.5}$$

Using relation (4.1), (4.5) becomes

$$\sum_{j=1}^{m} \mu_j \langle \zeta_j, \eta(x,v) \rangle > 0, \qquad \text{for suitable } \; \zeta_1 \in A_{g_1}(v), \; ..., \; \zeta_m \in A_{g_m}(v) \; . \tag{4.6}$$

Observe that $\mu = (\mu_1, ..., \mu_m) \neq 0$. (Otherwise, we get a contradiction with (4.6))

Now, let $M = \{j : \mu_j > 0\}$. As a consequence of (4.2), we have

$$g_j(v) \geq 0 \qquad \text{for all } \; j \in M \; .$$

Since $g_j(x) \leq 0$, one has

$$g_j(x) \leq g_j(v) \qquad \text{for all } \; j \in M \; .$$

From the strict (η, A_{g_j})-pseudoinvexity of g_j,

$$\langle \zeta_j, \eta(x,v) \rangle < 0 \qquad \text{for all } \; \zeta_j \in A_{g_j}(v), \;\; j \in M \; .$$

By definition of M, $\mu_j = 0$ for any $j \notin M$. Thus,

$$\sum_{j=1}^{m} \mu_j \langle \zeta_j, \eta(x,v) \rangle = \sum_{j=1, j\in M}^{m} \mu_j \langle \zeta_j, \eta(x,v) \rangle < 0, \qquad \text{for all } \; \zeta_j \in A_{g_j}(v) \; . \tag{4.7}$$

Combining (4.6) and (4.7), we get a contradiction. ∎

Theorem 4.1 and Theorem 4.2 are extensions of [13, Theorem 2.1] and [13, Theorem 2.2] obtained for Lipschitz functions.

Theorem 4.2 (Strong Duality). *Let $\overline{x}$ be a weakly efficient solution for (P) such that (P) is regular at $\overline{x}$. Then, there exist $\lambda^* \in \mathbb{R}^p$ and $\mu^* \in \mathbb{R}^m$ such that $(\overline{x}, \lambda^*, \mu^*)$ is a feasible point of (D) and their objective values are equal. Moreover, if f_i is (η, A_{f_i})-pseudoinvex and g_j is strictly (η, A_{g_j})-pseudoinvex, then $(\overline{x}, \lambda^*, \mu^*)$ is a weakly efficient solution of (D).*

Proof: Let x^* be a weakly efficient solution of (P). Let

$$\psi(x) = \max_{1\le i\le p} \Big[f_i(x) - f_i(\overline{x}) \Big] .$$

Then, following the approach of Minami [17], we can check easily that $\overline{x}$ is an optimal solution of the following scalar optimization problem

$$\operatorname{Min} \psi(x) , \quad \text{s.t. } \; g_j(x) \le 0, \quad j = 1, ..., m .$$

From Corollary 1 in [2],

$$A_\psi(\overline{x}) = \left\{ \sum_{i=1}^{p} \alpha_i^* \zeta_i : \; (\alpha_1^*, ..., \alpha_p^*) \in \mathbb{R}^p, \; \alpha_i^* \ge 0, \; \sum_{i=1}^{p} \alpha_i^* = 1, \; \zeta_i \in A_{f_i}(\overline{x}) \right\}$$

is an approximation of ψ at $\overline{x}$.

By Theorem 2 in [2], there exist $\tau^* > 0$ and $\mu_1^*, \; ..., \; \mu_m^* \ge 0$ such that

$$(4.8) \qquad 0 \in \tau^* A_\psi(\overline{x}) + \sum_{j=1}^{m} \mu_j^* A_{g_j}(\overline{x}) \quad \text{and} \quad \mu_j^* g_j(\overline{x}) = 0, \quad j = 1, ..., m .$$

Thus, there exists $\left(\alpha_1^*, \; ..., \; \alpha_p^*\right) \in \mathbb{R}^p$ such that $\alpha_i^* \ge 0$, $\sum_{i=1}^{p} \alpha_i^* = 1$, and

$$0 \in \sum_{i=1}^{p} \tau^* \alpha_i^* A_{f_i}(\overline{x}) + \sum_{j=1}^{m} \mu_j^* A_{g_j}(\overline{x}) \quad \text{and} \quad \mu_j^* g_j(\overline{x}) = 0, \quad j = 1, ..., m .$$

Setting $\lambda^* = \tau^*(\alpha_1^*, ..., \alpha_p^*)$, we get that $(\overline{x}, \lambda^*, \mu^*)$ is a feasible point of (D) and the objective values of (P) and (D) are equal.

Now, suppose that f_i is (η, A_{f_i})-pseudoinvex and g_j is strictly (η, A_{g_j})-pseudoinvex. Then, from Theorem 4.1, for every feasible point (v, λ, μ) of (D) there exists $s \in \{1, ..., \; p\}$ such that

$$f_s(\overline{x}) \ge f_s(v) \; .$$

Finally, since $(\overline{x}, \lambda^*, \mu^*)$ is a feasible point of (D), then it is a weakly efficient solution of (D). The proof is thus finished. ∎

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