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EULER SCHEME FOR SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-LIPSCHITZ COEFFICIENTS

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Abstract: Firstly, we investigate existence and uniqueness of solutions of stochastic differential equations when the coefficients are random Lipschitz or of class C^1. Secondly, we prove the strong convergence of the associated Euler scheme. The usual rates of convergence are obtained.

1 – Introduction and notations

The theory of stochastic differential equations (SDE's) provide a useful tool to introduce stochasticity into models and to characterize the evolution of many processes in finance, biology and others. In many cases, the solutions are not given explicitly, therefore numerical approximations are used to study the properties of these models. Unfortunately current results concerning the convergence of such schemes impose conditions on the drift and diffusion coefficients of these equations, namely the linear growth and global Lipschitz conditions (see Skorohod 1965, Kloeden and Platen 1992 and Mao 1997). We note that Yamada (1978) relaxed the global Lipschitz condition, whilst Kaneko and Nakao (1988) have shown that the Euler scheme converges in the strong sense, to the solution of the stochastic differential equation whenever path-wise uniqueness of the solution

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holds. However, both results require the linear growth condition whilst the latter provides no information on the order of the approximation. Recently, Mao et al (2002) have shown the convergence in Probability, of the Euler scheme under specific conditions on the coefficients.

In this paper, we study existence and uniqueness of solutions of stochastic differential equations even where the coefficients are not necessarily Lipschitz. We study also their approximation in L^p by the well known Euler scheme under Novikov's conditions (see conditions in sections 3 and 4). The usual rates of convergence are obtained.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbf{P})$ be a filtered space and $\{W_t\,,\, 0 \le t \le T\}$ be an $\mathbb{R}^l$-valued Brownian motion. We consider the multidimensional stochastic differential equation:

$$X_t \,=\, X_0 + \int_0^t B(s, X_s)\, ds \,+ \int_0^t A(s, X_s)\, dW_s \,, \tag{1}$$

where $X_0 \in \mathbb{R}^d$, $B : \Omega\times\mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d$ and $A : \Omega\times\mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^l$ are two functions satisfying some hypotheses that we will precise later in sections 3 and 4.

Through this paper, we adopt the following notations. Let $\Pi_n = \{0 = t_{n,0} < t_{n,1} < ... < t_{n,n} = T\}$ be a subdivision on the interval $[0, T]$ for each integer n with the discretization step $\delta_n = \sup_k (t_{n,k+1} - t_{n,k})$. Let $\mathbf{1}_C$ be the characteristic function of a subset C defined by $\mathbf{1}_C(x) = 1$ if $x \in C$ and $\mathbf{1}_C(x) = 0$ if $x \notin C$, $\mathcal{C}([0, T]; \mathbb{R}^k)$ be the space of continuous functions defined on $[0, T]$, with values in $\mathbb{R}^k$. Let define $\rho^n(t) = \sup\{t_{n,k} \le t\,,\, k = 0...n\}$ and $Y^{\rho^n}(t) = Y(\rho^n_t)$. We will design the inner product and the norm in $\mathbb{R}^k$ respectively by $\langle\, .\,, .\,\rangle$ and $\|\,.\,\|$.

This paper is organized as follows. In section 2, we state in Lemma 2.1, a stochastic version of the well known Gronwall Lemma. We apply this result to investigate some estimates on the Euler scheme associated to the stochastic differential equation (1), it is the subject of Proposition 2.1. In section 3, we consider a SDE where the random coefficients are supposed to be Lipschitz and the Lipschitz constants are random functions satisfying some hypotheses of integrability and continuity. Theorem 3.1 gives us existence and uniqueness of the solution. Furthermore, it is proved that the Euler scheme converges strongly and the rates are given. At the end, in section 4, the coefficients of our SDE, are considered to be of class C^1. Under some conditions on the solution, we prove in Theorem 4.1 the convergence of the Euler scheme and give the rates of convergence.

2 – Some preliminaries results

In this section, we prove some intermediate results which will be useful in the next.

Lemma 2.1. *Let $\{G_t\,, 0 \le t \le T\}$ be an increasing and continuous progressively measurable process with $G_0 = 0$, $\{F_q(t)\,;\ 0 \le t \le T,\ q \ge 0\}$ be a family of nonnegative increasing random functions. We suppose that there exists reals $\nu > 0$ and $p \ge 0$, such that the nonnegative random function $\{U_t\,, 0 \le t \le T\}$ satisfies for every stopping time $0 \le \tau \le T$ and every real $q \in [p, p+\nu[$, the following inequality:*

$$\mathbf{E}(U_\tau)^q \;\le\; K_q\, \mathbf{E} \int_0^\tau (U_s)^q\, dG_s + \mathbf{E} F_q(\tau)\;, \tag{2}$$

where K_q is a positive constant. Then for every $t \in [0,T]$, $q \in [p, p+\nu[$, $r > 1$, r' its conjugate and $a > 0$ such that $ar' > K_{qr'}$ and $qr' < p+\nu$, we have for every $R > 0$, $\eta \in [0,1]$, $e > 1$ and e' its conjugate:

$$\mathbf{E}(U_t)^q \;\le\; \mathbf{E} F_q(t) + z_q(T)\;, \tag{3}$$

where

$$\begin{aligned} z_q(T) \;=\; & \eta\, \frac{r'}{ar' - K_{qr'}} \Big\{\mathbf{E} F_{qr'}(T)\Big\}^{\frac{1}{r'}} \Big\{\mathbf{E} \exp(ar\, G_T)\Big\}^{\frac{1}{r}} \\ & + (1-\eta) \left\{ C(R, K_q)\, \mathbf{E} F_q(T) + \frac{1}{R} \left[\mathbf{E}(G_T)^e\right]^{\frac{1}{e}} \left[\mathbf{E}\left(\int_0^T (U_s)^q\, dG_s\right)^{e'}\right]^{\frac{1}{e'}} \right\}, \end{aligned}$$

with $C(R, K_q)$ a positive constant depending essentially on R. If furthermore, for $q \in [p, 2p+\nu[$, we have that:

$$\begin{aligned} \mathbf{E} \sup_{0\le t\le T} (U(t))^q \;\le\;& K_q\, \mathbf{E} \int_0^T (U_s)^q\, dG_s \;+\; K_q \left(\mathbf{E} \int_0^T (U_s)^{2q}\, dG_s\right)^{\frac{1}{2}} \\ & + \mathbf{E} F_q(T) + (\mathbf{E} F_{2q}(T))^{\frac{1}{2}}\;, \end{aligned} \tag{4}$$

then for every $q \in [p, p+\nu[$:

$$\mathbf{E} \sup_{0\le t\le T} (U(t))^q \;\le\; \mathbf{E} F_q(T) + (\mathbf{E} F_{2q}(T))^{\frac{1}{2}} + z_q(T) + (z_{2q}(T))^{\frac{1}{2}}\;. \tag{5}$$

In particular, when the relation (2) is satisfied with $F_q \equiv 0$ and for some $e > 1$ and e' its conjugate, $\mathbf{E}(G_T)^e < +\infty$ and $\mathbf{E}\left\{\int_0^T (U_s)^q\, dG_s\right\}^{e'} < +\infty$, then $U(t) = 0$ a.s for every $t \in [0,T]$.

Remark 2.1. When the stochastic process G is not progressively measurable, the formula (2) has to be fulfilled for every random time $\tau \in [0,T]$. ▫

Proof: Let define:

$$\tau_t \;=\; \inf\Big\{s \in [0,T],\ G_s \geq t\Big\}\ ,$$

and $\inf \emptyset = +\infty$. From (2), we obtain that:

$$\mathbf{E}(U(T\wedge\tau_t))^q \;\leq\; K_q\,\mathbf{E}\int_0^{T\wedge\tau_t} (U(s))^q\,dG_s \;+\; \mathbf{E}F_q(T\wedge\tau_t)\ .$$

The function F_q is increasing, then $F_q(T\wedge\tau_t) \leq F_q(T)$. By change of variables, we have:

$$\mathbf{E}\int_0^{T\wedge\tau_t} (U(s))^q\,dG_s \;\leq\; \mathbf{E}\int_0^t (U(T\wedge\tau_s))^q\,ds\ .$$

Then the function $b(t) := \mathbf{E}(U(T\wedge\tau_t))^q$ satisfies the following inequality:

$$b(t) \;\leq\; K_q\int_0^t b(s)\,ds \;+\; \mathbf{E}F_q(T)\ .$$

From Gronwall's Lemma, we deduce that:

$$b(t) \;\leq\; \exp(K_q\,t)\;\mathbf{E}F_q(T)\ . \tag{6}$$

Now by change of variables and application of Hölder's inequality, we obtain that:

$$\begin{aligned}
\mathbf{E}\int_0^t (U_s)^q\,dG_s \;&=\; \mathbf{E}\int_0^{G_t} (U(T\wedge\tau_s))^q\;ds\\
&=\; \int_0^{+\infty} \mathbf{E}\Big\{\mathbf{1}_{(G_t\geq s)}\,(U(T\wedge\tau_s))^q\Big\}\;ds\\
&\leq\; \int_0^{+\infty} \Big\{\mathbf{E}(\mathbf{1}_{(G_t\geq s)})^r\Big\}^{\frac{1}{r}}\;\Big\{\mathbf{E}(U(T\wedge\tau_s))^{qr'}\Big\}^{\frac{1}{r'}}\;ds\\
&\leq\; \int_0^{+\infty} \{\mathbf{P}(G_t\geq s)\}^{\frac{1}{r}}\;\Big\{\mathbf{E}(U(T\wedge\tau_s))^{qr'}\Big\}^{\frac{1}{r'}}\;ds\ ,
\end{aligned}$$

with $r>1$ and r' its conjugate satisfying $qr' < p+\nu$. By Markov's inequality, we have that for each positive real a:

$$\mathbf{P}(G_t\geq s) \;\leq\; \exp(-a\,r\,s)\;\mathbf{E}\exp(a\,r\,G_t)\ .$$

We apply (6) and the last inequality to obtain that:

$$\mathbf{E}\int_0^t (U_s)^q\,dG_s \;\le\; \Big[\mathbf{E}F_{qr'}(T)\Big]^{\frac{1}{r'}}\,\Big[\mathbf{E}\exp\left(ar\,G_t\right)\Big]^{\frac{1}{r}}\int_0^{\infty}\exp\Big[-\Big(a-\frac{K_{qr'}}{r'}\Big)s\Big]\,ds\;.$$

We take then a such that $ar' > K_{qr'}$, by consequence:

$$\int_0^{\infty}\exp\Big[-\Big(a-\frac{K_{qr'}}{r'}\Big)s\Big]\,ds \;<\; +\infty\;.$$

From (2) and the previous estimates, we have:

$$\mathbf{E}(U_t)^q \;\le\; \mathbf{E}F_q(t)+\frac{r'}{ar'-K_{qr'}}\Big[\mathbf{E}F_{qr'}(T)\Big]^{\frac{1}{r'}}\Big[\mathbf{E}\exp(ar\,G_t)\Big]^{\frac{1}{r}}\;. \tag{7}$$

We also remark by change of variables and application of Hölder's and Markov's inequalities that for every $R>0$:

$$\begin{aligned}\mathbf{E}\int_0^t (U(s))^q\,dG_s \;&=\; \mathbf{E}\bigg[\mathbf{1}_{(G_t\le R)}\int_0^{G_t}(U(T\wedge\tau_s))^q\,ds\bigg]+\mathbf{E}\bigg[\mathbf{1}_{(G_t\ge R)}\int_0^t (U(s))^q\,dG_s\bigg]\\ &\le\; \mathbf{E}\int_0^R (U(T\wedge\tau_s))^q\,ds+\frac{1}{R}\,[\mathbf{E}(G_T)^e]^{\frac{1}{e}}\left[\mathbf{E}\left(\int_0^t (U(s))^q\,dG_s\right)^{e'}\right]^{\frac{1}{e'}},\end{aligned}$$

with $e>1$ and e' its conjugate. From (6), we have that:

$$\mathbf{E}\int_0^R (U(T\wedge\tau_s))^q\,ds \;\le\; R\,\exp(K_q\,R)\;\mathbf{E}F_q(T)\;.$$

Then

$$\begin{aligned}\mathbf{E}(U_t)^q \;&\le\; \mathbf{E}F_q(t)+R\,\exp(K_q\,R)\,\mathbf{E}F_q(T)\\ &\quad+\frac{1}{R}\,[\mathbf{E}(G_T)^e]^{\frac{1}{e}}\left[\mathbf{E}\left(\int_0^T (U_s)^q\,dG_s\right)^{e'}\right]^{\frac{1}{e'}}\;.\end{aligned} \tag{8}$$

Henceforth by combining (7) and (8), it suffices to see that:

$$\mathbf{E}(U(t))^q \;=\; \eta\,\mathbf{E}(U(t))^q+(1-\eta)\,\mathbf{E}(U(t))^q\;.$$

If furthermore, (4) is satisfied, from previous results, we obtain (5). At the end, when $F_q \equiv 0$, we obtain from (8) that:

$$\mathbf{E}(U(t))^q \leq \frac{1}{R}\,(\mathbf{E}(G_T)^e)^{\frac{1}{e}} \left[\mathbf{E}\left(\int_0^t (U(s))^q\, dG_s\right)^{e'}\right]^{\frac{1}{e'}} .$$

For every real $R > 0$. We let R tend to infinity and the result is obtained. ∎

In what follows, we will state the second result of this section. Let $B\colon \Omega\times\mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d$ and $A\colon \Omega\times\mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^l$ be two measurable random functions such that for every $x \in \mathbb{R}^d$, the processes $(B(t,x),\, t \geq 0)$ and $(A(t,x),\, t \geq 0)$ are progressively measurable. For an integer n, Let X^n be the sequence of stochastic processes given by:

$$X_t^n \;=\; X_0 + \int_0^t B(\rho_s^n, X_s^{n,\rho^n})\, ds + \int_0^t A(\rho_s^n, X_s^{n,\rho^n})\, dW_s \; . \tag{9}$$

We define for integers n, m, the following quantities:

1. The process $\xi^{n,m} = (\xi_s^{n,m},\, 0 \leq s \leq T)$:

$$\xi_s^{n,m} \;=\; \mathbf{1}_{(X_s^{n,\rho^n} \neq X_s^{m,\rho^m})} \frac{\left\|B(\rho_s^n, X_s^{n,\rho^n}) - B(\rho_s^n, X_s^{m,\rho^m})\right\|}{\|X_s^{n,\rho^n} - X_s^{m,\rho^m}\|} + \mathbf{1}_{(X_s^{n,\rho^n} \neq X_s^{m,\rho^m})} \frac{\left\|A(\rho_s^n, X_s^{n,\rho^n}) - A(\rho_s^n, X_s^{m,\rho^m})\right\|^2}{\|X_s^{n,\rho^n} - X_s^{m,\rho^m}\|^2} + 1 \; .$$

2. The process $\phi^{n,m,q} = (\phi_s^{n,m,q},\, 0 \leq s \leq T)$:

$$\phi_s^{n,m,q} \;=\; \xi_s^{n,m}\Big(\|X_s^n - X_s^{n,\rho^n}\|^{2q} + \|X_s^m - X_s^{m,\rho^m}\|^{2q}\Big) \; .$$

3. The process $\varphi^{n,m,q} = (\varphi_s^{n,m,q},\, 0 \leq s \leq T)$:

$$\varphi_s^{n,m,q} \;=\; \left\|B(\rho_s^n, X_s^{n,\rho^n}) - B(\rho_s^m, X_s^{n,\rho^n})\right\|^{2q} + \left\|A(\rho_s^n, X_s^{n,\rho^n}) - A(\rho_s^m, X_s^{n,\rho^n})\right\|^{2q} .$$

4. The process $F^{n,m,q} = (F_s^{n,m,q},\, 0 \leq s \leq T)$:

$$F_s^{n,m,q} \;=\; \int_0^s (\phi_u^{n,m,q} + \varphi_u^{n,m,q})\, du \; .$$

5. The process $\gamma^{n,m} = (\gamma^{n,m}_s,\, 0 \le s \le T)$:

$$\gamma^{n,m}_s \;=\; \int_0^s \xi^{n,m}_u \, du \;.$$

6. The process $S^{n,m}_a = (S^{n,m}_{a,s},\, 0 \le s \le T)$:

$$S^{n,m}_{a,s} \;=\; \mathbf{E}\exp(a\,\gamma^{n,m}_s) \;.$$

Proposition 2.1. *Let p be an integer, $r > 1$ and r' its conjugate. Then there exists a real $a > 4r'p^2$ and a constant $C(p,r) > 0$ such that for every n, m, $\eta \in [0,1]$, $e > 1$, e' its conjugate and $R > 0$, we have:*

$$\mathbf{E} \sup_{0\le t\le T} \|X^n_t - X^m_t\|^{2p} \;\le\; C(p,r)\left\{ z^{n,m,p}_T + (z^{n,m,2p}_T)^{\frac{1}{2}} \right\}\,, \tag{10}$$

where

$$\begin{aligned} z^{n,m,p}(T) = \eta\left\{\mathbf{E}F^{n,m,pr'}(T)\right\}^{\frac{1}{r'}}\left\{S^{n,m}_{ar,T}\right\}^{\frac{1}{r}} + \left\{1 + (1-\eta)\,C_R\right\}\mathbf{E}F^{n,m,p}(T) \\ + (1-\eta)\frac{1}{R}\left\{\mathbf{E}\,(\gamma^{n,m}_T)^e\right\}^{\frac{1}{e}}\left\{\mathbf{E}\Big(\int_0^T \xi^{n,m}_s\Big(\|X^{n,\rho^n}_s\|^{2p} + \|X^{m,\rho^m}_s\|^{2p}\Big)ds\Big)^{e'}\right\}^{\frac{1}{e'}}, \end{aligned}$$

with C_R is a positive constant.

Proof: We suppose at first that the term on the right hand in (10) is finite. Let define the processes $U^{n,m}_s = X^n_s - X^m_s$, $B^n(s) = B(\rho^n_s, X^{n,\rho^n}_s)$, $A^n(s) = A(\rho^n_s, X^{n,\rho^n}_s)$ and the stopping time:

$$T^{n,m}_N \;=\; \inf\Big\{ t \in [0,T]\,;\; \|B^n_t\| + \|B^m_t\| + \|A^n_t\| + \|A^m_t\| \ge N \Big\}\;.$$

For an integer p and $t \in [0,T]$, we apply Itô's formula with the function $x \to \|x\|^{2p}$ and obtain that:

$$\begin{aligned} \|U^{n,m}(t\wedge T^{n,m}_N)\|^{2p} \;=\; & 2\,p\int_0^{t\wedge T^{n,m}_N} \|U^{n,m}_s\|^{2(p-1)}\left\langle U^{n,m}_s,\, B^n_s - B^m_s\right\rangle ds \\ & +\, 2\,p\int_0^{t\wedge T^{n,m}_N} \|U^{n,m}_s\|^{2(p-1)}\left\langle U^{n,m}_s,\, (A^n_s - A^m_s)\,dW_s\right\rangle \\ & +\, p\int_0^{t\wedge T^{n,m}_N} \|U^{n,m}_s\|^{2(p-1)}\,\|A^n_s - A^m_s\|^2\, ds \\ & +\, 2\,p(p-1)\int_0^{t\wedge T^{n,m}_N} \|U^{n,m}_s\|^{2(p-2)}\left\|\left\langle U^{n,m}_s,\, A^n_s - A^m_s\right\rangle\right\|^2 ds\;. \end{aligned}$$

From Schwartz's inequality, we have:

$$
\begin{aligned}
\|U^{n,m}(t\wedge T_N^{n,m})\|^{2p} \;\le\; & 2\,p\int_0^{t\wedge T_N^{n,m}} \|U_s^{n,m}\|^{2p-1}\,\|B_s^n - B_s^m\|\;ds \\
& +\,2\,p\int_0^{t\wedge T_N^{n,m}} \|U_s^{n,m}\|^{2(p-1)}\left\langle U_s^{n,m},\,(A_s^n - A_s^m)\,dW_s\right\rangle \\
& +\,(p+2\,p(p-1))\int_0^{t\wedge T_N^{n,m}} \|U_s^{n,m}\|^{2p-2}\,\|A_s^n - A_s^m\|^2\,ds\ .
\end{aligned}
$$

By taking the expectation, we obtain:

$$
\begin{aligned}
\mathbf{E}\|U^{n,m}(t\wedge T_N^{n,m})\|^{2p} \;\le\; & 2\,p\,\mathbf{E}\int_0^{t\wedge T_N^{n,m}} \|U_s^{n,m}\|^{2p-1}\,\|B_s^n - B_s^m\|\,ds \\
& +\,(p+2\,p(p-1))\,\mathbf{E}\int_0^{t\wedge T_N^{n,m}} \|U_s^{n,m}\|^{2p-2}\,\|A_s^n - A_s^m\|^2\,ds\ . \qquad (11)
\end{aligned}
$$

In order to further bound the first term in the right hand of (11) we note that:

$$
\begin{aligned}
\|B_s^n - B_s^m\| \;\le\; & \left\|B(\rho_s^n, X_s^{n,\rho^n}) - B(\rho_s^m, X_s^{n,\rho^n})\right\| + \left\|B(\rho_s^m, X_s^{n,\rho^n}) - B(\rho_s^m, X_s^{m,\rho^m})\right\| \\
\le\; & \left\|B(\rho_s^n, X_s^{n,\rho^n}) - B(\rho_s^m, X_s^{n,\rho^n})\right\| + K_s^{n,m}(B)\,\|X_s^{n,\rho^n} - X_s^{m,\rho^m}\| \\
\le\; & \left\|B(\rho_s^n, X_s^{n,\rho^n}) - B(\rho_s^m, X_s^{n,\rho^n})\right\| \\
& +\,K_s^{n,m}(B)\left(\|X_s^n - X_s^{n,\rho^n}\| + \|U_s^{n,m}\| + \|X_s^m - X_s^{m,\rho^m}\|\right)\,, \qquad (12)
\end{aligned}
$$

where

$$
K_s^{n,m}(B) \;=\; \mathbf{1}_{(X_s^{n,\rho^n} \neq X_s^{m,\rho^m})}\,\frac{\left\|B(\rho_s^m, X_s^{n,\rho^n}) - B(\rho_s^m, X_s^{m,\rho^m})\right\|}{\|X_s^{n,\rho^n} - X_s^{m,\rho^m}\|}\ . \qquad (13)
$$

Substituting relation (12) in the integrand of the first integral of (11) and next applying Young's inequality, there exists a constant C_p such that:

$$
\begin{aligned}
\|U_s^{n,m}\|^{2p-1}\,\|B_s^n - B_s^m\| \;\le\; & \|U_s^{n,m}\|^{2p}\left(1 + K_s^{n,m}(B)\right) \\
& +\,C_p\Big[\left\|B(\rho_s^n, X_s^{n,\rho^n}) - B(\rho_s^m, X_s^{n,\rho^n})\right\|^{2p} \\
& +\,K_s^{n,m}(B)\left(\|X_s^n - X_s^{n,\rho^n}\|^{2p} + \|X_s^m - X_s^{m,\rho^m}\|^{2p}\right)\Big]\,.
\end{aligned}
$$

In the same way the integrand in the last term of the right hand of (11) is bounded above:

$$\begin{aligned}\|U_s^{n,m}\|^{2p-2}\,\|A_s^n - A_s^m\|^2 \;\le\; &\|U_s^{n,m}\|^{2p}\Big(1+(K_s^{n,m}(A))^2\Big)\\ &+ C_p\Big[\Big\|A(\rho_s^n, X_s^{n,\rho^n}) - A(\rho_s^m, X_s^{n,\rho^n})\Big\|^{2p}\\ &+ (K_s^{n,m}(A))^2\Big(\|X_s^n - X_s^{n,\rho^n}\|^{2p} + \|X_s^m - X_s^{m,\rho^m}\|^{2p}\Big)\Big]\,,\end{aligned}$$

where $K_s^{n,m}(A)$ is defined by (13) when replacing B by A. Therefore for every stopping time $0 \le \tau \le T$ and by using the notations introduced before:

$$\mathbf{E}\Big\|U^{n,m}(\tau\wedge T_N^{n,m})\Big\|^{2p} \;\le\; 4\,p^2\,\mathbf{E}\int_0^\tau \|U_s^{n,m}\|^{2p}\,\xi_s^{n,m}\,ds \;+\; C_p\,\mathbf{E}F^{n,m,p}(T)\ .$$

We apply Fatou's Lemma and let N tend to infinity to obtain:

$$\mathbf{E}\|U^{n,m}(\tau)\|^{2p} \;\le\; 4\,p^2\,\mathbf{E}\int_0^\tau \|U_s^{n,m}\|^{2p}\,\xi_s^{n,m}\,ds \;+\; C_p\,\mathbf{E}F^{n,m,p}(T)\ . \tag{14}$$

By applying Itô's formula again, Burkholder's and Schwartz's inequalities yield:

$$\begin{aligned}\mathbf{E}\sup_{0\le t\le T}\|U^{n,m}(t)\|^{2p} \;\le\; &2\,p\,\mathbf{E}\int_0^T \|U_s^{n,m}\|^{(2p-1)}\,\|B_s^n - B_s^m\|\,ds\\ &+ 2\,p\Big(\mathbf{E}\int_0^T \|U_s^{n,m}\|^{(4p-2)}\,\|A_s^n - A_s^m\|^2\,ds\Big)^{\frac12}\\ &+ (p + 2\,p(p-1))\,\mathbf{E}\int_0^T \|U_s^{n,m}\|^{(2p-2)}\,\|A_s^n - A_s^m\|^2\,ds\ .\end{aligned}$$

We use the same arguments as before to obtain that:

$$\begin{aligned}\mathbf{E}\sup_{0\le t\le T}\|U^{n,m}(t)\|^{2p} \;\le\; &C_p(T)\Big\{K^{n,m,p}(T) + (K^{n,m,2p}(T))^{\frac12}\Big\}\\ &+ C_p(T)\Big\{\mathbf{E}F^{n,m,p}(T) + (\mathbf{E}F^{n,m,2p}(T))^{\frac12}\Big\}\ ,\end{aligned} \tag{15}$$

where

$$K^{n,m,p}(T) \;=\; \mathbf{E}\int_0^T \|U_s^{n,m}\|^{2p}\,\xi_s^{n,m}\,ds\ .$$

By combining (14), (15) and Lemma 2.1, we obtain our estimate (10). ∎

3 – The SDE's coefficients are random Lipschitz

In this section, we consider the following stochastic differential equation:

$$X_t \;=\; X_0 + \int_0^t B(s,X_s)\,ds \;+\; \int_0^t A(s,X_s)\,dW_s \;, \tag{16}$$

where $X_0 \in \mathbb{R}^d$ and $(W_t\,,\, t\in[0,T])$ is an $\mathbb{R}^l$-valued Brownian motion. Let $q>0$ and set the following hypotheses $(\mathbf{H}_q)$:

1. The random functions $B\colon \Omega\times\mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d$ and $A\colon \Omega\times\mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^l$ are measurable.

2. There exists a constant $M>0$ such that for every $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, we have:

$$\|B(t,x)\| + \|A(t,x)\| \;\le\; M \;.$$

3. For each $x\in\mathbb{R}^d$, the processes $B(t,x)$ and $A(t,x)$ are progressively measurable.

4. There exists two nonnegative random functions $\Gamma(t)$ and $\Gamma'(t)$ such that for every real vectors (x,x'), we have:

$$\|B(t,x)-B(t,x')\| \;\le\; \Gamma(t)\,\|x-x'\| \;,$$

and

$$\|A(t,x)-A(t,x')\|^2 \;\le\; \Gamma'(t)\,\|x-x'\|^2 \;.$$

5. The random functions $\Gamma(t)$ and $\Gamma'(t)$ satisfy:

$$\varlimsup_{n\to\infty} \int_0^T \mathbf{E}\Big[\Gamma(\rho^n(s)) + \Gamma'(\rho^n(s))\Big]^2\,ds \;<\; +\infty \;.$$

6. For every $R>0$, the sequences $(\omega_R^{n,q}(z,T))_{n\ge0}$ $(z=A,B)$ converge towards zero, where

$$\omega_R^{n,q}(z,T) \;=\; \sup_{x,\|x\|\le R} \mathbf{E}\int_0^T \Big\|z(\rho_s^n,x) - z(s,x)\Big\|^q\,ds \;.$$

7. For some $a>4q^2$:

$$\varlimsup_{n\to\infty} \mathbf{E}\,\exp\Big\{a\int_0^T \Big[\Gamma(\rho^n(s)) + \Gamma'(\rho^n(s))\Big]\,ds\Big\} \;<\; +\infty \;.$$

Remark 3.1. We remark that:

1. The hypothesis $(\mathbf{H}_q; 2)$ may be replaced by the following:

$$\mathbf{E}\int_0^T \Big(\|B(t,0)\|^q + \|A(t,0)\|^q\Big)\, ds \;<\; +\infty\ .$$

2. One of the following conditions implies $(\mathbf{H}_q; 5)$: The random functions Γ and Γ' are increasing and

$$\mathbf{E}\Big[\Gamma(T) + \Gamma'(T)\Big]^2 \;<\; +\infty\ ,$$

or the random functions Γ and Γ' are uniformly continuous in t w.r.t $\omega\in\Omega$ and

$$\int_0^T \mathbf{E}\Big[\Gamma(s) + \Gamma'(s)\Big]^2 ds \;<\; +\infty\ .$$

3. The following condition implies $(\mathbf{H}_q; 6)$: The random functions A and B are uniformly continuous in t w.r.t ω and x.

4. One of the following conditions implies $(\mathbf{H}_q; 7)$: The random functions Γ and Γ' are increasing and

$$\mathbf{E}\ \exp\Big\{a\,T\,\Big(\Gamma(T) + \Gamma'(T)\Big)\Big\} < +\infty\ ,$$

or the random functions Γ and Γ' are uniformly continuous in t w.r.t $\omega \in \Omega$ and

$$\mathbf{E}\ \exp\Big\{a\int_0^T \Big[\Gamma(s) + \Gamma'(s)\Big]\, ds\Big\} \;<\; +\infty\ .\ \square$$

Let X^n be the Euler scheme associated to the equation (16), defined by:

$$X_t^n \;=\; X_0 + \int_0^t B(\rho_s^n, X_s^{n,\rho^n})\, ds \;+\; \int_0^t A(\rho_s^n, X_s^{n,\rho^n})\, dW_s\ . \tag{17}$$

Then the main result of this section is:

Theorem 3.1. *Let $p \geq 1$ be an integer. Under the assumptions $(\mathbf{H}_{q_0}\,;\,1\!-\!6)$ for $q_0 > 4p$, there exists a unique solution for the equation (16) and the Euler scheme X^n converges towards this solution in $L^{2p}(\Omega, \mathcal{C}([0,T];\mathbb{R}^d))$. If furthermore the assumption $(\mathbf{H}_{q_0}\,;\,7)$ is satisfied, then there exists a constant $K_p(T) > 0$ such that for every $R > 0$ and $n \in \mathbf{N}$, we have:*

$$\mathbf{E}\sup_{0\leq t\leq T} \|X_t^n - X_t\|^{2p} \;\leq\; K_p(T)\,\Big\{z_T^{R,n,p} + (z_T^{R,n,2p})^{\frac{1}{2}}\Big\} \tag{18}$$

where

$$z_T^{R,n,p} \;=\; |\delta_n|^p + \frac{1}{R} + \omega_R^{n,p}(B,T) + \omega_R^{n,p}(A,T)\ .$$

Remark 3.2. If the random functions A and B are Hölder continuous in t with respectively $\alpha_A(x,\omega)$ and $\alpha_B(x,\omega)$ their Hölder constants such that:

$$\sup_{x\in\mathbb{R}^d} \mathbf{E}\Big\{\|\alpha_A(x)\|^{2q} + \|\alpha_B(x)\|^{2q}\Big\} \;<\; +\infty\ ,$$

then

$$\mathbf{E}\sup_{0\le t\le T} \|X_t^n - X_t\|^{2p} \;\le\; K_p(T)\,|\delta_n|^{p\nu}\ ,$$

where $\nu = \inf(1,\beta_A,\beta_B)$ with β_A and β_B are respectively the Hölder orders of A and B. ▫

Proof: To prove this Theorem, we will use the result of Proposition 2.1.

Existence: In Proposition 2.1, we take $\eta = 0$ and $e = e' = 2$. Then from hypothesis $(\mathbf{H}_{q_0}\,;\,4)$, we have:

$$\xi_s^{n,m} \;\le\; \Gamma(\rho_s^m) + \Gamma'(\rho_s^m) + 1\ ,$$

and

$$\phi_s^{n,m,q} \;\le\; \Big(\Gamma(\rho_s^m) + \Gamma'(\rho_s^m) + 1\Big)\Big(\|X_s^n - X_s^{n,\rho^n}\|^{2q} + \|X_s^m - X_s^{m,\rho^m}\|^{2q}\Big)\ .$$

From the assumption $(\mathbf{H}_{q_0}\,;\,2)$ and (17), we have:

$$\sup_{0\le t\le T} \mathbf{E}\|X_t^n\|^{2q} \;\le\; C_q(T)\ , \tag{19}$$

and

$$\sup_{0\le s\le T} \mathbf{E}\|X_s^n - X_s^{n,\rho^n}\|^{2q} \;\le\; C_q(T)\,|\delta_n|^q\ . \tag{20}$$

We remark also that:

$$\begin{aligned}
\varphi_s^{n,m,q} \;&=\; \Big\|B(\rho_s^n, X_s^{n,\rho^n}) - B(\rho_s^m, X_s^{n,\rho^n})\Big\|^{2q} + \Big\|A(\rho_s^n, X_s^{n,\rho^n}) - A(\rho_s^m, X_s^{n,\rho^n})\Big\|^{2q} \\
&\le\; 2^{2q}\Big\|B(\rho_s^n, X_s^{n,\rho^n}) - B(s, X_s^{n,\rho^n})\Big\|^{2q} + 2^{2q}\Big\|B(s, X_s^{n,\rho^n}) - B(\rho_s^m, X_s^{n,\rho^n})\Big\|^{2q} \\
&\quad + 2^{2q}\Big\|A(\rho_s^n, X_s^{n,\rho^n}) - A(s, X_s^{n,\rho^n})\Big\|^{2q} + 2^{2q}\Big\|A(s, X_s^{n,\rho^n}) - A(\rho_s^m, X_s^{n,\rho^n})\Big\|^{2q} \\
&=:\; 2^{2q}\Big(J_1(s) + J_2(s) + J_3(s) + J_4(s)\Big)\ .
\end{aligned}$$

Then for every $R' > 0$ and by applying Markov's inequality, we obtain from (19), $(\mathbf{H}_{q_0}\,;6)$ and the boundedness of B that:

$$\mathbf{E}\int_0^T J_1(s)\,ds \;=$$

$$= \mathbf{E}\int_0^T \left\| B(\rho_s^n, X_s^{n,\rho^n}) - B(s, X_s^{n,\rho^n}) \right\|^{2q} \left(\mathbf{1}_{(\|X_s^{n,\rho^n}\| \le R')} + \mathbf{1}_{(\|X_s^{n,\rho^n}\| \ge R')} \right) ds \qquad (21)$$

$$\le \omega_{R'}^{n,q}(B,T) + \frac{1}{R'}\, C_q(T)\ .$$

The same calculus may be done for J_i, $i = 2, 3, 4$. By consequence from Proposition 2.1, we obtain for $\eta = 0$:

$$\mathbf{E}\sup_{0\le t\le T} \|X_t^n - X_t^m\|^{2p} \;\le\; C(p,T)\left\{ z^{n,m,p}(T) + (z^{n,m,2p}(T))^{\frac{1}{2}} \right\}\ ,$$

where

$$z^{n,m,p}(T) \;\le\; (1 + C_R)\left(|\delta_n|^p + |\delta_m|^p + \Psi_{R'}^{n,m,p}(T) + \frac{1}{R'} \right) + \frac{1}{R}\ ,$$

with

$$\Psi_{R'}^{n,m,q}(T) \;=\; \omega_{R'}^{n,q}(B,T) + \omega_{R'}^{n,q}(A,T) + \omega_{R'}^{m,q}(B,T) + \omega_{R'}^{m,q}(A,T)\ .$$

We let n, m, R' and R, in this order, tend to infinity then it is easily seen that the sequence X^n is a Cauchy sequence in $L^{2p}(\Omega, \mathcal{C}([0,T];\mathbb{R}^d))$. So there exists a process X in the same space to which the sequence X^n converges. Further there exists also a subsequence X^{n_k} which converges uniformly a.s to X. So X is a solution of the equation (16).

Uniqueness: Let X, X' be two solutions of the equation (16). By applying Itô formula and using the assumption $(\mathbf{H}_{q_0}\,;4)$, we have for every stopping time $0 \le \tau \le T$:

$$\mathbf{E}\|X_\tau - X'_\tau\|^{2q} \;\le\; K(q,T)\,\mathbf{E}\int_0^\tau \|X_s - X'_s\|^{2q}\,(\Gamma_s + \Gamma'_s)\,ds\ .$$

We take $U_t := X_t - X'_t$ and $G_t := \int_0^t (\Gamma_s + \Gamma'_s)\,ds$ to obtain from Lemma 2.1 that $\|X_t - X'_t\|^{2q} = 0$ a.s for every $t \in [0,T]$. By Itô's formula again, Hölder's and Burkholder's inequalities yield:

$$\mathbf{E}\sup_{0\le t\le T} \|X_t - X'_t\|^{2q} \;\le\; K(q,T)\,\mathbf{E}\int_0^T \|X_s - X'_s\|^{2q}\,(\Gamma_s + \Gamma'_s)\;ds$$
$$+\; K'(q,T)\left(\mathbf{E}\int_0^T \|X_s - X'_s\|^{4q}\;\Gamma'_s\;ds \right)^{\frac{1}{2}}\ .$$

Then

$$\mathbf{E} \sup_{0\le t\le T} \|X_t - X'_t\|^{2q} \;=\; 0 \;.$$

Approximation: The convergence of the Euler scheme to the solution is already shown. If in addition, $(\mathbf{H}_{q_0}\,;7)$ is satisfied, we take $\eta=1$ in Proposition 2.1. Then from previous result and Fatou's Lemma, the result is obtained. ∎

4 – The SDE's coefficients are of class C^1

In this section, we consider the following stochastic differential equation:

$$X_t \;=\; X_0 + \int_0^t b(X_s)\,ds \;+ \int_0^t g(X_s)\,dW_s \;, \tag{22}$$

where $X_0 \in \mathbb{R}^d$, $b\colon \mathbb{R}^d \to \mathbb{R}^d$ and $g\colon \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^l$ are two regular functions. To state the main result of this section, we introduce the following hypotheses $(\mathbf{H}^*{}_q)$:

1. The equation (22) has a unique solution.

2. The functions b and g are of class C^1.

3. The functions b and g satisfy:

$$\mathbf{E}\int_0^T \Big\{ \Big(\|b'(X_s)\| + \|g'(X_s)\|^2\Big) \Big(\|b(X_s)\|^q + \|g(X_s)\|^q\Big) \Big\}^2 ds \;<\; +\infty \;,$$

and

$$\mathbf{E}\int_0^T \Big\{ \|b'(X_s)\|^2 + \|g'(X_s)\|^4 \Big\}\, ds \;<\; +\infty \;,$$

where b' and g' denote respectively the derivatives of b and g.

4. For some $a > 4q^2$:

$$\mathbf{E}\,\exp\Big\{ a\int_0^T \Big(\|b'(X_s)\| + \|g'(X_s)\|^2\Big)\, ds \Big\} \;<\; +\infty \;.$$

We consider also the Euler scheme X^n associated to the solution X, defined by:

$$X^n_t \;=\; X_0 + \int_0^t b(X^{n,\rho^n}_s)\,ds \;+ \int_0^t g(X^{n,\rho^n}_s)\,dW_s \;. \tag{23}$$

Theorem 4.1. *Let $p \geq 1$ be an integer. Under hypotheses $(\mathbf{H}^*{}_{q_0}\,;1-3)$ with $q_0 > 2p$, the Euler scheme X^n converges to X in $L^{2p}(\Omega, \mathcal{C}([0,T];\mathbb{R}^d))$ norm. If furthermore, the hypothesis $(\mathbf{H}^*{}_{q_0}\,;4)$ is satisfied, then there exists a constant $K_p(T) > 0$ such that for every n:*

$$\mathbf{E} \sup_{0 \leq t \leq T} \|X_t^n - X_t\|^p \leq K_p(T)\, |\delta_n|^{p/2} \,. \tag{24}$$

Proof: We prove firstly that the Euler scheme converges to the unique solution X. In order to use the result of Proposition 2.1, we take $B(t,\omega,x) = b(x)$ and $A(t,\omega,x) = g(x)$. Then for $\eta = 0$ and $e = 2$ in Proposition 2.1, we obtain that:

$$\mathbf{E} \sup_{0 \leq t \leq T} \|X_t^n - X_t^m\|^{2p} \leq C_p(T) \left\{ z^{n,m,p}(T) + (z^{n,m,2p}(T))^{\frac{1}{2}} \right\} ,$$

where

$$\begin{aligned} z^{n,m,p}(T) \;=\; & (1 + C_R)\, \mathbf{E} F^{n,m,p}(T) \\ & + \frac{1}{R} \left[\mathbf{E} (\gamma^{n,m}(T))^2 \right]^{\frac{1}{2}} \left\{ \mathbf{E} \left[\int_0^T \xi_s^{n,m} \Big(\|X_s^{n,\rho^n}\|^{2p} + \|X_s^{m,\rho^m}\|^{2p} \Big) ds \right]^2 \right\}^{\frac{1}{2}} , \end{aligned}$$

and

$$F^{n,m,p}(T) = \int_0^T \xi_s^{n,m} \Big(\|X_s^n - X_s^{n,\rho^n}\|^{2p} + \|X_s^m - X_s^{m,\rho^m}\|^{2p} \Big)\, ds \,.$$

From (22) and (23), we obtain that:

$$\begin{aligned} \mathbf{E} F^{n,m,p}(T) \leq\; & \left[\mathbf{E} \int_0^T \Big(\xi_s^{n,m} \Big(\|b(X_s^{n,\rho^n})\|^{2p} + \|g(X_s^{n,\rho^n})\|^{2p} \Big) \Big)^2 ds \right]^{\frac{1}{2}} |\delta_n|^p \\ & + \left[\mathbf{E} \int_0^T \Big(\xi_s^{n,m} \Big(\|b(X_s^{m,\rho^m})\|^{2p} + \|g(X_s^{m,\rho^m})\|^{2p} \Big) \Big)^2 ds \right]^{\frac{1}{2}} |\delta_m|^p \,. \end{aligned}$$

Since b and g are differentiable, the intermediate value Theorem yields:

$$\begin{aligned} \xi_t^{n,m} \;\leq\; & \int_0^1 \left[\Big\| b'\Big(\theta X_t^{n,\rho^n} + (1-\theta) X_t^{m,\rho^m} \Big) \Big\| + \Big\| g'\Big(\theta X_t^{n,\rho^n} + (1-\theta) X_t^{m,\rho^m} \Big) \Big\|^2 \right] d\theta \\ =: & \; \Psi_t^{n,m} \,. \end{aligned}$$

Then for every $R > 0$:

$$(z^{n,m,p}(T))^2 \leq (1 + C_p(R)) \left[\Pi_1^{n,m,p}\, |\delta_n|^{2p} + \Pi_2^{n,m,p}\, |\delta_m|^{2p} \right] + \frac{1}{R}\, \Pi_3^{n,m,p}\, \Pi_4^{n,m,p} \,,$$

where

$$\Pi_1^{n,m,p} = \mathbf{E}\int_0^T \Big[\Psi_s^{n,m}\Big(\|b(X_s^{n,\rho^n})\|^{2p} + \|g(X_s^{n,\rho^n})\|^{2p}\Big)\Big]^2 ds \ ,$$

$$\Pi_2^{n,m,p} = \mathbf{E}\int_0^T \Big[\Psi_s^{n,m}\Big(\|b(X_s^{m,\rho^m})\|^{2p} + \|g(X_s^{m,\rho^m})\|^{2p}\Big)\Big]^2 ds \ ,$$

$$\Pi_3^{n,m,p} = \mathbf{E}\int_0^T [\Psi_s^{n,m}]^2\, ds$$

and

$$\Pi_4^{n,m,p} = \mathbf{E}\int_0^T \Big[\Psi_s^{n,m}\Big(\|X_s^{n,\rho^n}\|^{2p} + \|X_s^{n,\rho^n}\|^{2p}\Big)\Big]^2 ds \ .$$

We remark that the terms $\Pi_1^{n,m,p}$, $\Pi_2^{n,m,p}$, $\Pi_3^{n,m,p}$ and $\Pi_4^{n,m,p}$ are of the form

$$\Phi^{n,m} = \mathbf{E}\int_0^T K(X_s^{n,\rho^n}, X_s^{m,\rho^m})\, ds \ ,$$

with K is a positive and continuous real function. To prove that the sequence X^n is of Cauchy, it suffices to show that

$$\sup_{n,m} \Phi^{n,m} < +\infty \ . \tag{25}$$

To do this, let define for a real $L > 0$ large enough, the functions b_L and g_L by $b_L = b$ and $g_L = g$ on the set $\{x \in \mathbb{R}^d,\ \|x\| \le L\}$ such that b_L and g_L are of class C^1. Let X^L be the solution of the following stochastic differential equation:

$$X_t^L = X_0 + \int_0^t b_L(X_s^L)\, ds + \int_0^t g_L(X_s^L)\, dW_s \ , \tag{26}$$

and let $X^{L,n}$ be the Euler scheme associated to the solution X^L. The functions b_L and g_L are globally Lipschitz, then the solution X^L exists and it is unique. We conclude also that for fixed L, the sequence $X^{L,n}$ converges uniformly in L^{2p} and a.s to X^L. To prove (25), we apply Fatou's Lemma:

$$\begin{aligned}
\overline{\lim_{n,m\to\infty}}\, \Phi_T^{n,m} &:= \overline{\lim_{n,m\to\infty}}\, \mathbf{E}\int_0^T K(X_s^{n,\rho^n}, X_s^{m,\rho^m})\, ds \ \le \\
&\le \underline{\lim}_{M,L\to\infty}\, \overline{\lim_{n,m\to\infty}}\, \mathbf{E}\int_0^T K(X_s^{L,n,\rho^n}, X_s^{L,m,\rho^m})\, \mathbf{1}_{\left\{\|X_s^{L,n,\rho^n}\| + \|X_s^{L,m,\rho^m}\| < M\right\}}\, ds \\
&\le \underline{\lim}_{M,L\to\infty}\, \mathbf{E}\int_0^T K(X_s^L, X_s^L)\, \mathbf{1}_{\{2\|X_s^L\| < M\}}\, ds \\
&\le \mathbf{E}\int_0^T K(X_s, X_s)\, ds \ < \ +\infty \ .
\end{aligned}$$

Now we state the rate of convergence of the Euler Scheme. In this case, we take $\eta = 1$ and $r = r' = 2$ in Proposition 2.1. Then

$$\mathbf{E} \sup_{0\le t\le T} \|X_t^n - X_t^m\|^{2p} \;\le\; C_p(T) \left\{ z^{n,m,p}(T) + (z^{n,m,2p}(T))^{\frac{1}{2}} \right\} ,$$

where

$$z^{n,m,p}(T) \;=\; \mathbf{E}F^{n,m,p}(T) + \left\{ S^{n,m}(a,T)\, \mathbf{E}F^{n,m,2p}(T) \right\}^{\frac{1}{2}} \;.$$

We proved previously that:

$$\mathbf{E}F^{n,m,p}(T) \;\le\; \Pi_1^{n,m,p}\, |\delta_n|^p + \Pi_2^{n,m,p}\, |\delta_m|^p \;.$$

Then

$$z^{n,m,p}(T) \;\le\; C_p(T) \Big(|\delta_n|^p + |\delta_m|^p \Big) \Big[1 + (S^{n,m}(a,T))^{\frac{1}{2}} \Big] \;.$$

To obtain the result, it suffices to prove that $\sup_{n,m} S^{n,m}(a,T) < +\infty$. By the same way as before, we have:

$$\overline{\lim_{n,m\to\infty}}\, S^{n,m}(a,T) \;\le\; \mathbf{E} \exp\!\left(a \int_0^T \Big(\|b'(X_s)\| + \|g'(X_s)\|^2 \Big)\, ds \right),$$

which is finite from $(\mathbf{H^*}_{q_0}\,;\,4)$. The Theorem is proved. ∎

5 – Example

It is generally acknowledged that the volatility of many financial return series is not constant over time and that these series exhibit prolonged periods of high and low volatility, often referred to as volatility clustering. Over the past two decades, the Stochastic Volatility (SV) model is one of the prominent classes of models that has been developed which capture this time-varying autocorrelated volatility process. The variance in this model is modelled as an unobserved component that follows some stochastic process. The most popular version of the SV model defines volatility as a logarithmic Ornstein–Uhlenbeck diffusion process which is used in the option pricing literature. In Hull and White (1987), they consider the price process x of a derivative asset as the solution of the following stochastic differential equation:

$$x_t \;=\; x_0 + \int_0^t \mu_1\, x_s\, ds \;+ \int_0^t \sigma_s\, x_s\, dW_s \;, \tag{27}$$

with the process $y := \sigma^2$ satisfying:

$$y_t = y_0 + \int_0^t \mu_2 \, y_s \, ds + \int_0^t \eta \, y_s \, dZ_s \;, \tag{28}$$

$x_0, y_0 \in \mathbb{R}_+$, $\mu_1, \mu_2, \eta \in \mathbb{R}_+$, $(W_t,\ t \in [0,T])$ and $(Z_t,\ t \in [0,T])$ are two correlated real Brownian motions with correlation ρ. Let x^n be the Euler scheme associated to the equation (27), defined by:

$$x_t^n = x_0 + \int_0^t \mu_1 \, x_s^{n,\rho^n} ds + \int_0^t \sigma_s^n \, x_s^{n,\rho^n} dW_s \;, \tag{29}$$

with $\sigma_s^n = \sigma(\rho_s^n)$. Then the Euler scheme x^n converges strongly towards the solution x with the usual rates of convergence. It suffices to apply theorem 3.1 and remark (3.1,4.) with $B(t,x) = \mu_1 \, x$ and $A(t,x) = \sigma_t \, x$ and to verify that the condition:

$$\mathbf{E} \exp\left\{ \int_0^T (\sigma_s)^2 \, ds \right\} < +\infty \;,$$

is satisfied which is the case.

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REFERENCES

[1] Hull, J. and White, A. – The pricing of options on assets with Stochastic Volatilities, *The Journal of Finance,* 42 (1987), 281–300.

[2] Kaneko, T. and Nakao, S. – A note of approximation for Stochastic Differential Equations, Séminaire de Probabilités, 22, *Lect. Notes Math.,* 1321 (1988).

[3] Kloeden, P.E. and Platen, E. – *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, 1992.

[4] Mao, X. – *Stochastic Differential Equations and Applications*, Horwood, 1997.

[5] Mao, X.; Marion, G. and Renshaw, E. – Convergence of the Euler scheme for a class of Stochastic Differential Equation, *Int. Math. Journal,* 1(1) (2002) 9–22.

[6] Skorohod, A.V. – *Studies in the Theory of Random Processes*, Addison–Wisely, 1965.

[7] Yamada, T. – Sur une construction des solutions d'équations différentielles stochastiques dans le cas non-Lipschitzien, Séminaire de Probabilités, 12, *Lect. Notes Math.,* 649 (1978).

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