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# ON A CLASS OF MONGE–AMPÉRE PROBLEMS WITH NON-HOMOGENEOUS DIRICHLET BOUNDARY CONDITION

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**Abstract:** We assume in the plane that $\Omega$ is a strictly convex domain, with its boundary $\partial\Omega$ sufficiently regular. We consider the Monge–Ampére equations in its general form $\det u_{ij} = g(|\nabla u|^2)h(u)$, where $u_{ij}$ denotes the Hessian of $u$, and $g$, $h$ are some given functions. This equation is subject to the non-homogeneous Dirichlet boundary condition $u = f$. A sharp necessary condition of solvability for this equation is given using the maximum principle in $\mathbb{R}^2$. We note that this maximum principle is extended to the N-dimensional case and two different applications have been given to illustrate this principle.

## 1 – Introduction

Let $u$ be a classical solution of the following Monge–Ampére equations

$$\det(u_{,ij}) = F(x, u, |\nabla u|^2) \quad \text{in } \Omega , \tag{1}$$

where $\Omega$ is assumed to be a bounded domain, strictly convex. In this note, we derive a new maximum principle for the general Monge–Ampére equations (1) with $F(x, u, |\nabla u|^2) = g(|\nabla u|^2)h(u)$ in $\mathbb{R}^N$, $N \geq 2$, which generalizes a recent result of Ma [11] (the particular case when $g.h = \text{const.}$ in $\Omega$).

In order to prove this maximum principle, we assume in the sequel that the functions $g$ and $h$ are subject to some appropriate conditions. These conditions lead to some differential inequality, which will be investigated in Section 2.

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Then employing the second maximum principle of E. Hopf [10], we conclude that the corresponding maximum value is attained on the boundary $\partial\Omega$ of $\Omega$.

For the first application, we shall treat the following non-homogeneous Dirichlet boundary condition

$$u = f \quad \text{on} \quad \partial\Omega \,, \tag{2}$$

where $\partial\Omega$ denotes the boundary of $\Omega$ sufficiently regular and $f$ is a positive function of class $C^1$. Monge–Ampére equations in conjunction with Dirichlet and Neumann conditions were investigated in [1,2,4,5,6,7,8]. For the second application, we consider the particular Dirichlet case $f = 0$. Ma [11] showed that the combination $P = |\nabla u|^2 - 2\sqrt{c}u$ is a constant, $u$ is radial and $\Omega$ is a ball. We extend this result for more general combination and prove for some particular values of $g$, that $\Omega$ is an N-ball and $u$ is radial.

Some applications are given involving different situations, where various bounds for $u$ and its gradient $|\nabla u|$ are obtained. The maximum principle for Monge–Ampére equations was already used by Ma [11, 12] and Safoui [13].

In the case of the Neumann boundary condition

$$\frac{\partial u}{\partial n} = \cos(\theta(x,u))\,(1 + |\nabla u|^2)^{\frac{1}{2}} \quad \text{on} \quad \partial\Omega \,, \tag{3}$$

where $\mathbf{n}$ is the outward normal vector and the wetting angle $\theta$ is an element of $(0, \frac{\pi}{2})$, Ma in [11], proved the following result, by assuming that the bounded domain $\Omega$ is strictly convex, the constant $c$ is positive and, the angle $\theta$ is an element of $(0, \frac{\pi}{2})$

**Theorem 1.** *Under the above hypotheses on $\Omega$, $c$, $\theta_0$, if u is a strictly convex solution of* (1), (3) *then the following relation is satisfied*

$$k_0 \le \max\Big\{c^{\frac{1}{2}} \cos(\theta_0),\, c^{\frac{1}{2}} tan(\theta_0)\Big\} \,, \tag{4}$$

*where $k_0 := \min_{x\in\partial\Omega} k(x)$ and $k(x)$ is the curvature of the boundary $\partial\Omega$ of $\Omega$ at x.*

In the case when $F :=$ const. and $f(x) = 0$, he showed the following theorem (see [11])

**Theorem 2.** *Under the above hypotheses on the domain $\Omega$ and constant c, if u is a strictly convex solution for the boundary value problems (1)–(2) then we have the following estimates*

$$\max_{x\in\bar\Omega} |\nabla u|^2 \le \frac{c}{{k_0}^2} \,, \tag{5}$$

$$-\frac{\sqrt{c}}{2k_0} \leq u \leq 0 \quad in \ \bar{\Omega} , \tag{6}$$

*where* $k_0 := \min_{x\in\partial\Omega} k(x)$, $k(x)$ *is the curvature of* $\partial\Omega$ *at* $x$.

For the proof of Theorem 2, he used the maximum principle [9,10] in $\mathbb{R}^2$ for the following combination

$$\Phi := |\nabla u|^2 - 2\, c^{\frac{1}{2}} u \ , \tag{7}$$

and the expression of the Monge–Ampére equations (1) in normal coordinates (see Section 3, (40)).

The purpose of this paper is, firstly, to generalize this maximum principle in $\mathbb{R}^N$ for a general combination of the form

$$\Phi := g(|\nabla u|^2) + h(u) \ , \tag{8}$$

where $g$ and $h$ are supposed to be positive. Secondly, to consider a more general equation

$$\det(u_{,ij}) = g(|\nabla u|^2)\, h(u) \quad \text{in} \ \bar{\Omega} , \tag{9}$$

with non-homogeneous boundary condition (2). This generalization gives us an upper bound for $u$ and its gradient $|\nabla u|$ in function of the geometry of $\Omega$ and the first and second derivatives of $f$.

Throughout the paper, we shall be concerned with a bounded domain $\Omega$ of $\mathbb{R}^N$, strictly convex. A comma will be used to denote differentiation. We make use the summation convention with repeated Latin indices running from 1 to $N$.

$$\begin{aligned} u_{,i} &:= \frac{\partial u}{\partial x_i} \ , \\ u_{,ij}u_{,ij} &:= \sum_{i=1}^{i=N} \sum_{j=1}^{j=N} \left[\frac{\partial^2 u}{\partial x_i\, \partial x_j}\right]^2, \\ u_n &:= \frac{\partial u}{\partial n} \ , \\ u_s &:= \frac{\partial u}{\partial s} \ , \\ (u_s)_n &:= \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial s}\right) \ , \\ (u_s)_n &:= (u_n)_s - K u_s \ , \\ u_{nn} &:= \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n}\right) \ , \\ u_{ss} &:= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) \ . \end{aligned} \tag{10}$$

## 2 – On a maximum principle

Hereafter, we shall assume that the solution $u$ of the Monge–Ampère equations defined by (8) is at least of class $C^2(\bar{\Omega})\cap C^3(\Omega)$ in a bounded domain $\Omega$ described in Section 1. In this Section, we will show, that the maximum principle of the combination $\Phi$ defined by the following equation

$$\Phi := g(u_{,i}u_{,i}) + h(u) \quad \text{in } \bar{\Omega} , \tag{11}$$

attains its maximum value on the boundary $\partial\Omega$, where the functions $h$ and $g$ are subject to some conditions. For the differential equation of the form

$$\Delta u + f(u) = 0 \quad \text{in } \bar{\Omega} ,$$

the corresponding function constructed for this type of equation depends essentially on the dimension $N$ and the imposed boundary conditions, for which in general the treatment in $\mathbb{R}^2$ differs from that of $\mathbb{R}^N$, where $N \geq 3$, since some differential equalities are valid in $\mathbb{R}^2$ and unfortunately not valid in $\mathbb{R}^N$, as

$$|\nabla u|^2 u_{,ij}\, u_{,ij} = |\nabla u|^2(\Delta u)^2 + u_{,i}\, u_{,ik}\, u_{,j}\, u_{,jk} - 2\,(\Delta u)\, u_{,i}\, u_{,j}\, u_{,ij} .$$

It is already known that, the combination $\Phi$ attains its maximum principle at three different places for an arbitrary $g$ and $f$ (see R. Sperb [14]). Assuming that $\Phi$ is nonconstant, the corresponding maximum is attained on the boundary $\partial\Omega$ as first possibility, at a critical point as second possibility and finally at an interior point of the domain $\Omega$. In our context, we choose $g$ and $h$ such that, the elliptic differential inequality formed is strictly positive.

**Theorem 3.** *Let $u$ be a strictly convex solution of* (9) *and $\Phi$ the combination defined by* (8), *then*

$$\frac{1}{2}\, u^{ij}\, \Phi_{,ij} + \cdots = g'(|\nabla u|^2)\Big(-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'}\Big) + 2\, g' \Delta u + N h' \tag{12}$$

*where the dots stand for terms of the form $V_{,k}\Phi_{,k}$ with specific vector fields $V_{,k}$ which are bounded except at critical points of $u$.*

To start the proof of Theorem 3, we construct an appropriate differential inequality for $\Phi$, except at a critical value of the solution $u$. Let $\Phi$ defined by (8) then

$$\Phi_{,i} = 2\, u_{,ik}\, u_{,k}\, g' + u_{,i}\, h' , \tag{13}$$

$$\Phi_{,ij} = 2\, g'(u_{,ijk}\, u_{,k} + u_{,jk}\, u_{,ik}) + 4(u_{,jl}\, u_{,l}\, u_{,ik}\, u_{,k})\, g'' + u_{,ij}\, h' + u_{,i}\, u_{,j}\, h'' . \tag{14}$$

Let $u_{ij}$ be the inverse of the Hessian matrix $H := u^{ij}$. As $u$ is strictly convex solution of (9), the matrix $u^{ij}$ is positive definite and consequently by computing

$$u^{ij}\Phi_{,ij} = 2\,g'(u^{ij}u_{,ijk}\,u_{,k} + u^{ij}u_{,jk}\,u_{,ik}) + 4(u^{ij}u_{,jl}\,u_{,l}\,u_{,ik}\,u_{,k})\,g'' + u^{ij}u_{,ij}\,h' + u^{ij}u_{,i}\,u_{,j}h'' \,, \tag{15}$$

we claim that $u^{ij}\Phi_{,ij}$ is strictly positive in $\bar{\Omega}$. Knowing that the following identities $u^{ij}u_{,ij}\,u_{,jl} = \Delta u$, $u^{ij}u_{,ij} = N$, $u^{ij}u_{,il}\,u_{,l}\,u_{,jk}\,u_{,k} = u_{,kl}\,u_{,k}\,u_{,l}$ and $(gh)[u^{ij}u_{,ijk}\,u_{,k}] = (gh)_{,j}u_{,j}$ are valid in $\mathbb{R}^N$, then we are able to prove that $\Phi$ satisfies an appropriate differential inequality. For this, we compute

$$u^{ij}\Phi_{,i}\,u_{,j} = u^{ij}\{2\,u_{,j}u_{,ik}\,u_{,k}\,g' + u_{,i}u_{,j}h'\} \,, \tag{16}$$

$$u_i\Phi_{,i} = 2\,g'u_{,i}\,u_{,ik}\,u_{,k} + u_{,i}\,u_{,i}\,h' \,. \tag{17}$$

From (16) and (17), we obtain

$$-\,u^{ij}u_{,i}\,u_{,j}h' + u^{ij}\Phi_{,i}\,u_{,j} = 2\,g'u^{ij}u_{,j}\,u_{,ik}\,u_{,k} = 2\,u_{,i}\,u_{,i}\,g' \,, \tag{18}$$

$$2\,u_{,ij}\,u_{,j}\,u_{,i}\,g' - u_{,i}\,\Phi_{,i} = -u_{,i}\,u_{,i}\,h' \,. \tag{19}$$

Hence by (18) and (19), we conclude that

$$u^{ij}\Phi_{,ij} + \cdots = 2\,g'\Big(|\nabla u|^2\Big)\Big(-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'}\Big) + 2\,g'\Delta\,u + Nh' \,. \tag{20}$$

Using the following arithmetic-geometric inequality

$$\Delta\,u \geq N(gh)^{\frac{1}{N}}$$

(or simply $\Delta\,u > 0$ since $g'$ is positive), we obtain

$$u^{ij}\Phi_{,ij} + \cdots \geq 2\,g'\Big(|\nabla u|^2\Big)\Big(-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'}\Big) + 2\,g'N(gh)^{\frac{1}{N}} + Nh' \,, \tag{21}$$

where $g$, $h$, $g'$ and $h'$ satisfy the following conditions

$$g' > 0\,, \qquad h' > 0 \,, \tag{22}$$

and

$$-\frac{h'}{g} + \frac{h'}{h} - \frac{h'g''}{(g')^2} - \frac{h''}{h'} > 0 \,. \tag{23}$$

Then the maximum of $\Phi$ is attained on the boundary $\partial\Omega$ of $\Omega$ at some point $P$. If inequalities (22) and (23) are reversed, then we conclude that the minimum value of $\Phi$ occurs on the boundary $\partial\Omega$, or at a critical point of $u$.

We have then established the following theorem which extend the result of Ma [11,12] to the $N$ dimensional case.

**Theorem 4.** *Let $\Phi$ be defined by* (8) *where $g$, $h$, $g'$ and $h'$ satisfy* (22), (23) *and $u$ supposed to be strictly convex. Then the maximum principle of the combination $\Phi$ is attained on the boundary $\partial\Omega$ of $\Omega$ at some point $P$.*

## 3 – Estimates of the solution $u$ and its gradient $|\nabla u|$ for the Dirichlet boundary condition

In this Section, we investigate in dimension 2 the following result which illustrates Theorem 4. The bounds obtained for $u$ and its gradient $|\nabla u|$ seems appear for the first time in the non-homogeneous Dirichlet case.

**Theorem 5.** *We assume that $u$ is a classical solution of the non-homogeneous Dirichlet problem* (2), (9), *strictly convex, at least of class $C^2(\bar{\Omega}) \cap C^3(\Omega)$. Let $\Omega$ be a bounded domain, convex in $\mathbb{R}^2$. Then we have*

$$\max_{\bar{\Omega}} |\nabla u|^2 \le \frac{1}{K}\left\{\frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2}{f_s}^2 + |f_{ss}|\right\}, \tag{24}$$

$$-h(u_{\min}) + h(f) \le g\left(\frac{1}{K}\left\{\frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2}{f_s}^2 + |f_{ss}|\right\}\right) \quad \text{in } \bar{\Omega}\,. \tag{25}$$

For the proof of Theorem 5, we need to use in dimension 2 some differential equality valid in $\mathbb{R}^2$. This fact consists on the computation of the normal derivative of the combination $\Phi$ in function of the mean curvature $K$, the first $u_s$ and second $u_{ss}$ tangential derivatives of $u$ in the plane. This result will be established as follows.

We begin by computing the normal derivative of $\Phi$ in $\mathbb{R}^2$

$$\begin{aligned}\frac{\partial\Phi}{\partial n} &= 2\,g'\Big\{u_{,1}(u_{,11}n_1 + u_{,12}n_2) + u_{,2}(u_{,21}n_1 + u_{,22}n_2)\Big\} + h'u_n\\ &= 2\,g'u_n\Big\{\Delta\,u + u_{,2}\frac{\partial}{\partial s}u_{,1} - u_{,1}\frac{\partial}{\partial s}u_{,2}\Big\} + h'u_n\\ &= 2\,g'u_n\Big\{\Delta\,u + u_s u_{ns} - u_{ss} - K|\nabla u|^2\Big\} + h'u_n\,,\end{aligned} \tag{26}$$

where $s$ denotes differentiation in the tangential direction on the boundary $\partial\Omega$ and $K$ stands for curvature of $\partial\Omega$ at some point $\hat{P}$.

In the terms $\frac{\partial}{\partial s}u_{,1}$ and $\frac{\partial}{\partial s}u_{,2}$ we have broken $u_{,1}$ and $u_{,2}$ into normal and tangential derivative components and used the identities

$$\frac{\partial u_{,1}}{\partial s} = -Kn_2 \quad \text{and} \quad \frac{\partial u_{,2}}{\partial s} = Kn_1\,. \tag{27}$$

Since the maximum of the combination $\Phi$ defined by (11) is attained on the boundary $\partial\Omega$ at $\hat{P}$, we must have

$$\begin{aligned} \frac{\partial \Phi}{\partial s}(\hat{P}) &= g' \frac{\partial}{\partial s}(|\nabla u|^2) + h' u_s \\ &= 2\, g'(u_n u_{ns} + u_s u_{ss}) + h' u_s = 0 \ . \end{aligned} \tag{28}$$

Now we need to use the differential equality (28) in order to eliminate the product $u_n u_{ns}$ in (26). In fact, involving (28) we deduce

$$u_n u_{ns} = \frac{h' u_s}{2g'} - u_s u_{ss} \ . \tag{29}$$

The Monge–Ampére equations (9) can be rewritten in $\mathbb{R}^2$ as

$$u_{nn}(K u_n + u_{ss}) = gh + [u_{sn}]^2 \ . \tag{30}$$

In this case, by using (26), (28), (29), and making use of the following inequality

$$u_{ns} = u_{sn} - K u_s \ , \tag{31}$$

we obtain

$$\max_{\bar{\Omega}} |\nabla u|^2 \leq \frac{1}{K}\left\{ \frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2}{f_s}^2 + |f_{ss}| \right\} , \tag{32}$$

where $M$ and $\tilde{M}$ are two positive bounds of Laplace $u$ and the mixed derivative $u_{sn}$ since $u$ is assumed to be of class $C^2$.

For this last differential inequality (32), we have only considered the case when the normal derivative of the solution $u$ is non-equal to zero. Since for the nullity case, we are conducted to the triviality of the solution $u$. We are concerned now with the estimation of the solution $u$, which will be illustrated by applying the statement of Theorem 4. We know that

$$- h(u) \leq g(A) - h(u) + g(|\nabla u|^2) \ , \tag{33}$$

where $A$ is defined by

$$A := \frac{1}{K}\left\{ \frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2}{f_s}^2 + |f_{ss}| \right\} , \tag{34}$$

At a critical point of $u$, we obtain

$$0 < -h(u_{\min}) \leq g(A) - h(f) \ , \tag{35}$$

from which we deduce

$$- h(u_{\min}) + h(f) \le g(A) \ . \tag{36}$$

Finally, we have explicitly

$$- h(u_{\min}) + h(f) \le g\left(\frac{1}{K}\left\{\frac{h'}{2g'} + M + \frac{1}{2}\tilde{M}^2 + \frac{1}{2}{f_s}^2 + |f_{ss}|\right\}\right) \ . \tag{37}$$

## 4 – On an over-determined Monge–Ampére problem

Ma in [11] proved in $\mathbb{R}^2$ the following result

**Theorem 6.** *Under the same hypothesis of* $c$, $\Omega$ *and* $u(x)$ *as in Theorem* 2, *if* $P(x) := |\nabla u|^2 - 2\sqrt{c}u$ *attains its maximum in* $\Omega$, *then*

$$\Omega = B_R(0) \ , \tag{38}$$

$$u = \frac{\sqrt{c}\,(x_1^2 + x_2^2)}{2} - \frac{\sqrt{c}\,R^2}{2} \ , \tag{39}$$

$$P = c\,R^2 \ , \tag{40}$$

*where* $R$ *is a positive constant.*

Our goal is to extend this result to the $N$-dimensional space for more general Monge–Ampére equations (9). In the next theorem, we establish our result.

**Theorem 7.** *We assume that* $u$ *is a classical solution of* (2), (9) *with* $f = 0$.

*If* $\Phi = g(|\nabla u|^2) + h(u)$, *where* $gg' = 1$ *and* $h(0) > 0$, *attains its maximum on the boundary* $\partial\Omega$, *then*

$$\Omega = B_r(0) \ , \tag{41}$$

$$u = \left(\frac{h'(0)}{2h(0)}\right)^{\frac{N-1}{2}} \Big(|x|^2 - h(0)\Big)^{\frac{N-1}{2}} - \left(\frac{h'(0)}{2h(0)}\right)^{\frac{N-1}{2}} \Big(r^2 - h(0)\Big)^{\frac{N-1}{2}} \ , \tag{42}$$

$$\Phi = \left(\frac{2h(0)}{h'(0)K}\right)^{\frac{2}{N-1}} + h(0) = const. \quad on \ \ \partial\Omega \ , \tag{43}$$

*where* $r$ *is a positive constant.*

In order to establish this statement, we compute the normal and tangential derivatives of $\Phi$ and we use the fact that its maximum is attained on the boundary $\partial\Omega$, we then obtain

$$\frac{\partial \Phi}{\partial n} \;=\; 2\, g'\Big(u_n u_{nn} + (\nabla_s u)(\nabla_s u)_n\Big) + h' u_n \;=\; 0 \;, \tag{44}$$

where $\nabla_s u$ denotes the tangential gradient of $u$ on the boundary $\partial\Omega$.

From (44), we deduce that the second normal derivative of $u$ can be evaluated explicitly on the boundary $\partial\Omega$ as

$$u_{nn} = -\frac{h'}{2g'} \;, \tag{45}$$

which is non-positive by the hypothesis on $h'$ and $g'$ (see (22)).

The general Monge Ampère equations (9) with $f = 0$ takes the form

$$K(P)\,|\nabla\, u|^{N-1}\, u_{nn}(P) \;=\; g({u_n}^2)\, h(0) \;. \tag{46}$$

In fact, this is due to the following Lemma investigated by Safoui in [13]

**Lemma 1** (Lemma 1.5 p:16 [13])**.** *Let $u$ be a function of class $C^2$ strictly convex in $\bar{\Omega}$ and constant on $\partial\Omega$, and let $P_0$ be an element of $\partial\Omega$ where $|\nabla u|^2$ realizes its maximum.*

*We have then at this point the relation*

$$\det(u_{,ij}) \;= \Gamma(P_0)\, u_n^{N-1}\, u_{nn} \;,$$

*where $\Gamma(P_0)$ denotes the curvature of Gauss of $\partial\Omega$ at the point $P_0$.*

This last differential equality (46) becomes in view of Lemma 2

$$u_{nn} = \frac{g({u_n}^2)\, h(0)}{K|\nabla\, u|^{N-1}} \;. \tag{47}$$

Combining (45) and (47), we get

$$\frac{\partial u}{\partial n} = \left(\frac{2\, h(0)}{h'(0)K}\right)^{\frac{1}{N-1}} \;. \tag{48}$$

Then we have

$$\Phi \;= \left(\frac{2\, h(0)}{h'(0)K}\right)^{\frac{2}{N-1}} + h(0) \;=\; \text{const.} \quad \text{on} \;\; \partial\Omega \;. \tag{49}$$

From (49), we obtain the value of the mean curvature $K$ of the boundary $\partial\Omega$ as follows

$$K = \left(\frac{h'(0)}{2h(0)}\right)\left(r^2 - h(0)\right)^{\frac{N-1}{2}}\,, \tag{50}$$

where $r$ is a positive constant.

To this end, the solution $u$ takes the form

$$u = \left(\frac{h'(0)}{2h(0)}\right)^{\frac{N-1}{2}}\left(|x|^2 - h(0)\right)^{\frac{N-1}{2}} - \left(\frac{h'(0)}{2h(0)}\right)^{\frac{N-1}{2}}\left(r^2 - h(0)\right)^{\frac{N-1}{2}}\,, \tag{51}$$

which achieves the proof of our theorem.

We remark that the statement of Theorem 7 is also valid if we have $gg' = A.|\nabla u|^{2N}$, where $A$ is a positive constant. In the special case when $A = N = 1$, we obtain the result of Ma [11] in $\mathbb{R}^2$.

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