PORTUGALIAE MATHEMATICA
Vol. 62 Fasc. 1 – 2005
Nova Série

ON A COMPLETION OF PREHILBERTIAN SPACES

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Abstract: We define and study a completion of a prehilbertian space, associated to a family of linear subspaces endowed with linear topologies such that the inclusions are continuous. This provides in particular an orthogonal complement of paraclosed subspaces and an adjoint of paraclosed linear relations.

1 – Introduction

In this work we introduce a completion $\tilde{H}$ of a prehilbertian space H with respect to a family $\mathcal{P}$ of linear subspaces P endowed with linear topologies making the inclusions $P \hookrightarrow H$ continuous, so that all the algebraic and topological duals P^* (for $P \in \mathcal{P}$) are contained in $\tilde{H}$, see Definition 2.4 and Theorem 3.2. In particular, this provides an orthogonal complement of paraclosed subspaces and an adjoint of paraclosed linear relations.

We remind that a linear subspace P of a Hilbert space H is called *paraclosed* if it can be endowed with a hilbertian norm making the inclusion $P \hookrightarrow H$ continuous. Such a norm is unique up to equivalence, by the closed graph theorem. This notion was evidenced in [4, 8], then it was studied in more general contexts, including Banach spaces. Paraclosed subspaces are called also *operator ranges*, because a linear subspace of H is paraclosed if and only if it is the range $R(T) = \{Tx : x \in H\}$ of a bounded operator $T : H \to H$. These spaces appear in various cases where it is not sufficient to consider only closed subspaces and

Received: September 16, 2003; *Revised*: February 25, 2004.
AMS Subject Classification: Primary 46C07, 46C50; Secondary 47A05, 47B99.
Keywords and Phrases: paraclosed subspace; linear relation; operator.

it is helpful to use operator ranges [5]. We refer also to [10, 11, 13, 14] for their general properties and various applications.

The linear relations $G \subset X \times X$, where X is a Hilbert or a Banach space, are considered as a generalization of the notion of graphs $G(T)$ of operators $T : D(T)\,(\subset X) \to X$, see [1, 2, 3]. Here $D(T)$ is the domain of T and $G(T) = \{(x, Tx) : x \in D(T)\}$. We remind that an operator T is called *closed* (resp. *paraclosed*) if its graph $G(T)$ is closed (resp. paraclosed). The class of the paraclosed operators is the minimal one that contains the closed operators and is stable under addition and product [4], see also [11] for its properties. Assume X is a (real or complex) Hilbert space and set $H = X \oplus X$. Let $\mathcal{CR}(X)$ (resp. $\mathcal{PR}(X)$) be the set of all closed (resp. paraclosed) linear subspaces $G \subset H$ of infinite dimension and codimension. An element $G \in \mathcal{CR}(X)$ (resp. $\mathcal{PR}(X)$) is called a *closed* (resp. *paraclosed*) *linear relation* [2]. The set $\mathcal{CR}(X)$ of all closed linear relations is a complete metric space endowed with an algebraic structure [2, 12] consistent with the usual one for closed densely defined operators T. That is, the notions of sum, composition, adjoint, etc. of operators have natural extensions to $\mathcal{CR}(X)$, enabling the study of various classes of closed linear relations [2]. In particular, the *adjoint* $G^\star$ of a closed linear relation $G \in \mathcal{CR}(X)$ is defined as

$$G^\star = \left\{(-y, x) \in H : \ (x, y) \in G^\perp\right\}, \tag{1}$$

by analogy with the equality $G(T^\star) = G'(-T)^\perp$ where $T^\star$ is the Hilbert space adjoint of T and $G'(T) = \{(v, u) : (u, v) \in G(T)\}$. The symbol $\sigma^\perp := \{h \in H : \langle h|s\rangle = 0 \text{ for all } s \in \sigma\}$ denotes as usual the orthogonal complement of a subset σ of a Hilbert space H, while $\langle\cdot|\cdot\rangle$ stands for the inner product of H.

The structure of $\mathcal{CR}(X)$ partially has a counterpart on $\mathcal{PR}(X)$, too. The starting point of this paper is an attempt to extend the adjoint $G \mapsto G^\star$ to $\mathcal{PR}(X)$ so that $(G^\star)^\star = G$ and $F \subset G \Rightarrow G^\star \subset F^\star$ for $F, G \in \mathcal{PR}(X)$. By virtue of (1) for $G \in \mathcal{CR}(X)$, a related question is then to find a corresponding notion of orthogonal complement of paraclosed subspaces.

We mention that for a paraclosed subspace $P \subset H$ endowed with a fixed hilbertian norm $\|\cdot\|_P$ defining its topology, there exists the notion of the de Branges complement, that is, a map taking $P = (P, \|\cdot\|_P)$ into a normed paraclosed subspace $P' = (P', \|\cdot\|_{P'})$ such that $(P')' = P$ and $P \subset Q \Rightarrow Q' \subset P'$, see [5] for details. Our present questions require to find a norm-independent notion of orthogonal complement for paraclosed subspaces.

Let $\mathcal{P}(H)$ denote the set of all paraclosed linear subspaces $P \subset H$ of an arbitrary Hilbert space H.

In what follows, we will consider the subspaces $P \in \mathcal{P}(H)$ to be endowed only with the linear topologies making $P \subset H$ continuous. If all P are Hilbert (or, more generally, Fréchét) spaces, these own topologies are uniquely determined whenever they exist.

The completion $\tilde{H} = \tilde{H}(\mathcal{P}) \supset H$ of H is an abstract analog of the Sobolev scale of distributions $\bigcup_{n\in\mathbb{Z}} H^n$ contained in the Schwartz space $\mathcal{D}'$ of all distributions, associated to a space $H := L^2(= H^0)$ on a smooth manifold [7] with $\mathcal{P} := \{H^n : n \geq 0\}$, see Example 4.1. We construct $\tilde{H}$ by means of the duals P^* of the spaces $P \in \mathcal{P}$, which resembles the definition of the rigged Hilbert spaces [6, Section I.3.1], see Example 4.3. Our definition is more general, providing for instance $\mathcal{P}$-completions that — unlike in Examples 4.1 and 4.3 — are not necessarily contained in the dual S^* of a single subspace S of $\bigcap_{P\in\mathcal{P}} P$, see Example 4.4. Also we take into account only the topologies of the spaces $P \in \mathcal{P}$ without requiring that a particular norm be fixed on each P. The space $\tilde{H}(\mathcal{P})$ is the (unique) solution of a corresponding universal problem. It provides us, for $\mathcal{P} := \mathcal{P}(H)$ (with H a Hilbert space), a suitable notion of orthogonal complement $P \mapsto P^\perp$ with $P^\perp \subset \tilde{H}(\mathcal{P}(H))$. Then we can define also the adjoint $G^\star$ of a paraclosed linear relation $G \subset X \oplus X$ on a Hilbert space X as $G^\star = \{(-y, x) : (x, y) \in G^\perp\}$, see Remark 3.9.

2 – $\mathcal{P}$-completions

Let H be an arbitrary Hilbert space. We prove that the orthogonal complement $P \mapsto P^\perp$ (for P closed) cannot be extended to a map $P \mapsto P^c$ on $\mathcal{P}(H)$ with the properties $(P^c)^c = P$ and $M \subset N \Rightarrow N^c \subset M^c$, where $M, N, P \in \mathcal{P}(H)$. Indeed, Proposition 2.1 gives $(M^c)^c = \overline{M}$ (the closure of M) and so $(M^c)^c \neq M$, in general. In order to have an extension of the orthogonal complement with such properties as above, we should let then the $P^\perp$'s be contained in a larger space $\tilde{H} \supset H$, endowed with an inner product so that all $P^\perp$ make sense. Proposition 2.2 shows that even in this case, the map $\mathcal{P} \ni P \mapsto P^\perp \subset \tilde{H}$ cannot satisfy $(P^\perp)^\perp = P$ if the inner product of $\tilde{H}$ is globally defined on $\tilde{H} \times \tilde{H}$. This leads us to Definition 2.3. We introduce then, by Definition 2.4, a class of such completions $\tilde{H}$ associated to families $\mathcal{P}$ of topological linear subspaces of H.

Proposition 2.1. *Consider an arbitrary function on $\mathcal{P}(H)$, denoted by $M \mapsto M^c$, such that for every $M, N \in \mathcal{P}(H)$ the following implications hold:*

$$M \subset N \implies N^c \subset M^c\ ; \tag{2}$$

$$N = \overline{N} \implies N^c = N^\perp \ . \tag{3}$$

Then for any $M \in \mathcal{P}(H)$ *we have* $M^c = M^\perp = \overline{M}^\perp$ *and* $(M^c)^c = (M^\perp)^\perp = \overline{M}$.

Proof: Let $M \in \mathcal{P}(H)$ be arbitrary. By [5, Theorem 1.1], there exists a sequence H_n $(n \geq 0)$ of closed mutually orthogonal subspaces of H such that

$$M = \left\{ x = \sum_{n\geq 0} x_n : \ x_n \in H_n, \ \sum_{n\geq 0} (2^n \|x_n\|)^2 < \infty \right\} .$$

For $n \geq 0$, set $M_n := \{x \in M : x_k = 0 \text{ for } k > n\}$. Thus $M_n = \bigoplus_{k=0}^n H_k$ is closed (and so paraclosed). The trace of the topology of M on M_n coincides with the topology induced by H, because the first one can be defined by the norm $\|x\|_M^2 := \sum_{k\geq 0} (2^k \|x_k\|)^2$ and we have

$$\|x\|^2 \leq \|x\|_{M_n}^2 = \sum_{k=0}^n (2^k \|x_k\|)^2 \ \leq \ 4^n \|x\|^2 \qquad (x \in M_n) \ .$$

Since all H_k are complete, each M_n is closed in H. Then $(M_n)^c = (M_n)^\perp$, by (3). Also $M_n \subset M$ implies $M^c \subset (M_n)^c$, by (2). Hence $M^c \subset \bigcap_{n\geq 0} M_n^\perp$. Since $\overline{M} = \overline{\bigoplus_n H_n} = \overline{\bigcup_n M_n}$, we have $\bigcap_n M_n^\perp \subset (\overline{M})^\perp$. Therefore $M^c \subset M^\perp$ $(= (\overline{M})^\perp)$. Due to the closedness of $\overline{M}$, we have $(\overline{M})^\perp = (\overline{M})^c$, by (3). Hence $M^\perp = (\overline{M})^c$. Obviously, $M \subset \overline{M}$ and $\overline{M}$ is closed, then (2) gives $(\overline{M})^c \subset M^c$. It follows that $M^\perp \subset M^c$. Since the opposite inclusion was proved earlier, we have $M^c = M^\perp$ for every $M \in \mathcal{P}(H)$. Replacing in this equality the space M by M^c ($\in \mathcal{P}(H)$ by the hypothesis of the proposition), we obtain $(M^c)^c = (M^c)^\perp = (M^\perp)^\perp = \overline{M}$. ∎

Let X' denote the algebraic dual of a vector space X. We remind that a linear subspace $Y \subset X'$ is called *total* on X if $y(x) = 0$ for all $y \in Y$ implies $x = 0$. In this case $\langle X, Y\rangle$ is said to be a *dual pair*. Then for any $E \subset X$ the sets

$$E^\circ = \Big\{ y \in Y : \ |y(x)| \leq 1 \ \text{ for every } \ x \in E \Big\}$$

and

$$E^\perp = \Big\{ y \in Y : \ y(x) = 0 \ \text{ for every } \ x \in E \Big\}$$

are called the *polar* and the *annihilator* of E, respectively (see, e.g., [9, Section III.3.2]). The *bipolar* and *biannihilator* of E are then $(E^\circ)^\circ \subset X$ and $(E^\perp)^\perp \subset X$, respectively. Using the same notation $\perp$ for the polar and the orthogonal complement is convenient since whenever X is a Hilbert space and $Y := X^*$ $(\subset X')$, the polar $E^{\perp} \subset Y$ of E can be identified with its orthogonal complement $E^{\perp} \subset X$ via the antilinear (that is, conjugate-linear) isometric isomorphism $X^* \ni y \mapsto \tilde{y} \in X$: $y(x) = \langle x|\tilde{y}\rangle$ $(x \in X)$ given by Riesz' lemma.

Proposition 2.2. *Let $P \subset H$ be a linear subspace of a topological vector space H. Let $\tilde{H}$ be a topological vector space endowed with a continuous bilinear form $\langle\cdot,\cdot\rangle$ defined on $\tilde{H}\times\tilde{H}$, such that $\langle x, y\rangle = 0$ for all $x \in \tilde{H}$ (resp. for all $y \in \tilde{H}$) implies $y = 0$ (resp. $x = 0$). Let $i : H \to \tilde{H}$ be an injective and continuous linear map. Set $\tilde{P} = i(P)$. Define*

$$\tilde{P}^{\perp} = \left\{x \in \tilde{H} : \ \langle x, z\rangle = 0 \ \text{ for all } \ z \in \tilde{P}\right\}$$

and

$$\tilde{P}^{\perp\perp} = \left\{y \in \tilde{H} : \ \langle x, y\rangle = 0 \ \text{ for all } \ x \in \tilde{P}^{\perp}\right\} .$$

If $\tilde{P}^{\perp\perp} = \tilde{P}$, then the subspace P must be closed in H.

Proof: Let $\tilde{H}^*$ be dual to $\tilde{H}$. The space $Y := \{\langle x,\cdot\rangle : x \in \tilde{H}\} \subset \tilde{H}^*$ is total on $\tilde{H}$. Thus $\langle\tilde{H}, Y\rangle$ is a dual pair. Also $\tilde{H}$ is embedded into the dual space of Y by the mapping $\tilde{H} \ni z \mapsto f_z$ where $f_z(\langle x,\cdot\rangle) = \langle x, z\rangle$ for $x \in \tilde{H}$. Then $\tilde{H}$ is total on Y. Since $\tilde{P}$ is a linear subspace of $\tilde{H}$, the (bi)polar of $\tilde{P}$ coincides with the (bi)annihilator of $\tilde{P}$, see [9, Section III.3, Lemma 2(4)]. Thus $(\tilde{P}^{\circ})^{\circ} = (\tilde{P}^{\perp})^{\perp} = \tilde{P}^{\perp\perp}$ $(= \tilde{P}$ by the hypothesis). By the bipolar theorem (see, e.g., [9, Section III.3, Theorem 4]), $(\tilde{P}^{\circ})^{\circ}$ is the closure of $\tilde{P}$ with respect to the topology $\sigma(\tilde{H}, Y)$. Now if $h \in H$ is the limit of a generalized sequence $(m_\nu)_\nu$ with $m_\nu \in P$, then $im_\nu \to ih$ in $\tilde{H}$. We have $\langle x, im_\nu\rangle \to \langle x, ih\rangle$ for any $x \in \tilde{H}$, due to the conitnuity of $\langle\cdot,\cdot\rangle$. Since $\tilde{P}$ is $\sigma(\tilde{H}, Y)$-closed, it follows that $ih \in \tilde{P}(= iP)$. In view of the injectivity of i, we infer that $h \in P$. Thus P is closed. ∎

In what follows, we state a context in which the questions raised in the introduction can get positive answers.

An antilinear map $x \mapsto \overline{x}$ on a complex vector space X is called an *involution* if $\overline{\overline{x}} = x$ for all $x \in X$. An involution on a prehilbertian space H is called *unitary* if $\langle h|k\rangle = \langle\overline{k}|\overline{h}\rangle$ for all $h, k \in H$.

Given a prehilbertian space H, a linear subspace L of H endowed with a linear topology making the inclusion $L \subset H$ continuous will be called a *topological linear subspace* of H.

Definition 2.3. A *space with inner product* is a real or complex linear space $\mathcal{H}$ endowed with a scalar-valued map $\langle\cdot|\cdot\rangle$ defined on a subset D of $\mathcal{H} \times \mathcal{H}$ and an involution $x \mapsto \overline{x}$ (= the identity in the real case) such that

– for any $x \in \mathcal{H}$, the set of all $y \in \mathcal{H}$ such that (y,x) (resp. (x,y)) belongs to D is a linear subspace, on which the functional $\langle \cdot | x\rangle$ (resp. $\langle x|\cdot\rangle$) is linear (resp. antilinear); this functional is null only if $x = 0$;

– if both $(x,y),(y,x) \in D$, then $\langle x|y\rangle = \overline{\langle y|x\rangle}$;

– if $(x,y) \in D$, then $(\overline{x},\overline{y}) \in D$ and $\langle \overline{x}|\overline{y}\rangle = \overline{\langle x|y\rangle}$;

– the set $\{x \in \mathcal{H} : (x,x) \in D\}$ is a linear subspace, prehilbertian when endowed with $\langle\cdot|\cdot\rangle$. ◻

Any vectors $x, y \in \mathcal{H}$ are said to be *orthogonal* if either $(x,y) \in D$ and $\langle x|y\rangle = 0$, or $(y,x) \in D$ and $\langle y|x\rangle = 0$. The *orthogonal complement* $\sigma^\perp$ of a subset $\sigma \subset \mathcal{H}$ is then defined as the set of those vectors $x \in \mathcal{H}$ that are orthogonal to all $y \in \sigma$. A linear map f between spaces with inner product $(\mathcal{H}, D, \langle\cdot|\cdot\rangle)$ and $(\mathcal{H}', D', \langle\cdot|\cdot\rangle)$ is said to be *isometric* if for any $(x,y) \in D$ we have $(fx, fy) \in D'$ and $\langle fx|fy\rangle = \langle x|y\rangle$. Whenever $(x,\overline{y}) \in D$, we set $\langle x, y\rangle := \langle x|\overline{y}\rangle$.

Hypotheses. We shall consider real or complex prehilbertian spaces $(H, \langle\cdot|\cdot\rangle)$ together with sets $\mathcal{P}$ of topological linear subspaces $P \subset H$. We always suppose that $H \in \mathcal{P}$. All the prehilbertian spaces $(H, \mathcal{P})$ under consideration are assumed to be endowed with a unitary involution. Moreover all $P \in \mathcal{P}$ are supposed to be invariant under this involution. For every $L \in \mathcal{P}$, let L^* denote its algebraic and topological dual with respect to the uniform convergence on the bounded subsets of L. For $M, P \in \mathcal{P}$ with $M \subset P$, we say that M is P*-dense* (resp. P*-closed*) in P if it is dense (resp. closed) with respect to the own topology of P. The symbol $sp\{\sigma_i : i \in I\}$ will denote the linear space generated by a family of subsets σ_i. We denote by $R(T)$ the range of a linear map T. For any $h, k \in H$, we set $\langle h, k\rangle := \langle h|\overline{k}\rangle$.

Under the hypotheses from above, we give the following definition.

Definition 2.4. Let $\mathcal{P}$ be a set of topological linear subspaces of a prehilbertian space H such that $H \in \mathcal{P}$. For any $L \in \mathcal{P}$, we define the *inclusion* of L into L^* by

$$i_{LL^*}\colon L \to L^*\,, \qquad i_{LL^*}l := \langle \cdot, l\rangle|_L \ \ (= \langle\cdot|\overline{l}\rangle \ \text{ for } \ l \in L)\ .$$

Let $K, L, M, P \in \mathcal{P}$. A $\mathcal{P}$*-completion* of H is a space with inner product $\tilde{H}$ together with the linear maps

$$i\colon H \to \tilde{H}\,, \quad i_{L^*}\colon L^* \to \tilde{H}$$

called the *inclusions* and the linear maps

$$r_L\colon \tilde{H}_L \to L^*, \quad \text{where } \ \tilde{H}_L := sp\Big\{R(i_{P^*})\colon P \in \mathcal{P},\ P \supset L\Big\}\ ,$$

called the *restrictions* such that

$$\begin{aligned}
&i_{L^*} i_{LL^*} = i|_L\,, \qquad \langle il, i_{L^*} u\rangle = ul \quad \text{for all} \quad l \in L, \;\; u \in L^*\;;\\
&r_K i_{L^*} u = u|_K \quad \text{for all} \quad u \in L^*, \quad K \subset L\;;\\
&i_{P^*} u = i_{M^*}(u|_M) \quad \text{if} \quad u \in P^* \quad \text{and} \quad M \text{ is } P\text{-dense in } P\;.
\end{aligned}$$

We will denote the above defined $\mathcal{P}$-completion by $(\tilde{H}, i, (i_{L^*})_{L\in\mathcal{P}}, (r_L)_{L\in\mathcal{P}})$. A *morphism* of $\mathcal{P}$-completions

$$f\colon \Big(\tilde{H}, i, (i_{L^*})_{L\in\mathcal{P}}, (r_L)_{L\in\mathcal{P}}\Big) \to \Big(\mathcal{H}, j, (j_{L^*})_{L\in\mathcal{P}}, (\rho_L)_{L\in\mathcal{P}}\Big)$$

is a linear map $f : \tilde{H} \to \mathcal{H}$ such that $f i_{L^*} = j_{L^*}$ for all $L \in \mathcal{P}$. □

3 – Main results

We will establish now the existence and main properties of a $\mathcal{P}$-completion as defined in Section 2. This completion will turn also to be unique in a certain sense.

Proposition 3.1. *Let H be a prehilbertian space and $\mathcal{P}$ be a set of topological linear subspaces satisfying the hypotheses stated in Section 2. If $(\tilde{H}, i, (i_{L^*})_{L\in\mathcal{P}}, (r_L)_{L\in\mathcal{P}})$ is a $\mathcal{P}$-completion of H, then the inclusion $i : H \to \tilde{H}$ is isometric and for every $L \in \mathcal{P}$ the inclusions $i_{LL^*} : L \to L^*$, $i_{L^*} : L^* \to \tilde{H}$ are injective.*

If $f : (\tilde{H}, i, (i_{L^*})_{L\in\mathcal{P}}, (r_L)_{L\in\mathcal{P}}) \to (\mathcal{H}, j, (j_{L^*})_{L\in\mathcal{P}}, (\rho_L)_{L\in\mathcal{P}})$ is a morphism of $\mathcal{P}$-completions, then it is isometric, $f|_H = 1_H$ (that is, $fi = j$) and f commutes with the restrictions, namely $\rho_L f|\tilde{H}_L = r_L$ whenever $L \in \mathcal{P}$.

Proof: For all $L \in \mathcal{P}$, the mappings i_{LL^*} are injective, see Definition 2.4. Since $r_L i_{L^*} = 1_{L^*}$, all i_{L^*} are injective, too. Then $i = i_{H^*} i_{HH^*}$ is also injective. For any $h, h' \in H$ we have

$$\langle ih, ih'\rangle = \langle ih, i_{H^*}(i_{HH^*} h')\rangle = (i_{HH^*} h')(h) = \langle h, h'\rangle\;.$$

Thus i is isometric. We have $fi = f i_{H^*} i_{HH^*} = j_{H^*} i_{HH^*} = j$. Fix $K \in \mathcal{P}$. Take an arbitrary finitely supported set $\{u^L \in L^* : L \in \mathcal{P}, L \supset K\}$, that is, all the functionals u^L are null except for a finite number of them. For any $L \in \mathcal{P}$ with $L \supset K$ and any $u \in L^*$, we have

$$\rho_K f i_{L^*} u = \rho_K j_{L^*} u = u|_K = r_K i_{L^*} u \ .$$

Take $u = u^L$ and sum over $L \in \mathcal{P}$. It follows that $\rho_K f = r_K$ on $\tilde{H}_K$. Now f is isometric. Indeed, if $D \subset \tilde{H} \times \tilde{H}$ denotes the domain of the inner product $\langle \cdot | \cdot \rangle$ of $\tilde{H}$, then for any $(ik, \sum_L i_{L^*} u^L) \in D$ we have

$$\left\langle fik, f \sum_L i_{L^*} u^L \right\rangle = \left\langle jk, \sum_L j_{L^*} u^L \right\rangle = \sum_L u^L k = \left\langle ik, \sum_L i_{L^*} u^L \right\rangle \ . \ \blacksquare$$

Theorem 3.2. *Let H be a prehilbertian space and $\mathcal{P}$ be a set of topological linear subspaces of H satisfying the hypotheses stated in Section 2. Then there exists a $\mathcal{P}$-completion $\tilde{H}$ of H such that for any other $\mathcal{P}$-completion $\mathcal{H}$ of H there is a unique morphism from $\tilde{H}$ to $\mathcal{H}$.*

Proof: Let $S := \bigoplus_{L \in \mathcal{P}} L^*$ denote the algebraic direct sum of all duals L^* of spaces L from $\mathcal{P}$. Thus S consists of all the formal sums $\bigoplus_{L \in \mathcal{P}} u^L$ of functionals $u^L \in L^*$ on various domains $L \in \mathcal{P}$ with the family $(u^L)_{L \in \mathcal{P}}$ of finite support. In what follows, whenever the symbol L will be used as an index, it will be assumed to run the whole set $\mathcal{P}$ if not otherwise specified.

For every $P \in \mathcal{P}$, let $s_P : P^* \to S$ be the canonical injection. That is, for any $u \in P^*$ we have $s_P u = \bigoplus_L u^L$ with $u^P := u$ and $u^L := 0$ for $L \neq P$. Define a linear subspace S_1 of S by

$$S_1 = \left\{ \bigoplus_L i_{LL^*} l_L \in S : \ l_L \in L \ \text{ for every } L, \ \sum_L l_L = 0 \ \text{ in } H \right\} . \tag{4}$$

Remind that $i_{LL^*} : L \to L^*$ is the injective map defined by $i_{LL^*} l = \langle \cdot, l \rangle$. Set

$$\delta := \left\{ (M, P) \in \mathcal{P}^2 : \ M \text{ is } P\text{-dense in } P \right\} .$$

Let $S_2 \subset S$ be the linear span of all the vectors of the form $s_P u - s_M(u|_M)$ with $(M, P) \in \delta$ and $u \in P^*$. Let $\tilde{H} = \tilde{H}(\mathcal{P})$ be the quotient space

$$\tilde{H}(\mathcal{P}) := S/(S_1 + S_2) \tag{5}$$

where $S_1 + S_2 := \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$. Let $p : S \to \tilde{H}$ denote the linear canonical map of factorization through the linear subspace $S_1 + S_2$ of S. The involution

$$\overline{u} l := \overline{u \overline{l}} \qquad (l \in L \in \mathcal{P}, \ \ u \in L^*)$$

induces an involution on $\tilde{H}$ by factorization through the subspace $S_1 + S_2 \subset S$. For any $L \in \mathcal{P}$, let

$$i_{L^*} := ps_L \ .$$

Set also

$$i := i_{H^*} i_{HH^*} \ .$$

For every $K \in \mathcal{P}$, define the subspace $\tilde{H}_K$ of $\tilde{H}$ by

$$\tilde{H}_K = sp\Big\{R(i_{L^*}):\ L \in \mathcal{P},\ L \supset K\Big\} \ .$$

Define the set $D \subset \tilde{H}(\mathcal{P}) \times \tilde{H}(\mathcal{P})$ as

$$D = \bigcup_{K \in \mathcal{P}} \Big(i(K) \times \tilde{H}_K\Big) \ . \tag{6}$$

We verify now that the conditions of Definitions 2.3, 2.4 are satisfied. For every $s_2 \in S_2$ there exists a family $\{u^{MP} \in P^* : (M,P) \in \delta\}$ of finite support such that

$$s_2 = \sum_{(M,P)\in\delta} \Big(s_P u^{MP} - s_M(u^{MP}|_M)\Big) \ .$$

For every $(M,P) \in \delta$, represent $s_P u^{MP} \in S$ as $s_P u^{MP} = \bigoplus_L u^L$ by a finitely supported set $\{u^L \in L^* : L \in \mathcal{P}\}$, where $u^P = u^{MP}$ while $u^L = 0$ if $L \neq P$. Thus $u^L = \delta_{PL} u^{ML}$, where δ_{PL} is Kronecker's symbol. Then

$$\sum_{(M,P)\in\delta} s_P u^{MP} = \sum_{(M,P)\in\delta} \bigoplus_L \delta_{PL} u^{ML} = \bigoplus_L \sum_{M:(M,P)\in\delta} \delta_{PL} u^{ML} = \bigoplus_L \sum_{M:(M,L)\in\delta} u^{ML} \ .$$

Similarly, we obtain the equality

$$s_M(u^{MP}|_M) = \bigoplus_L \sum_{P:(L,P)\in\delta} u^{LP}|_L \ .$$

Then any vector $s_2 \in S_2$ has the following form

$$s_2 = \sum_{(M,P)\in\delta} \Big(s_P u^{MP} - s_M(u^{MP}|_M)\Big) = \bigoplus_L \sum_{M:(M,L)\in\delta} u^{ML} - \bigoplus_L \sum_{P:(L,P)\in\delta} u^{LP}|_L \ .$$

Given any finitely supported set $\{u^L \in L^* : L \in \mathcal{P}\}$, we have then the implication

$$\sum_L i_{L^*} u^L = 0 \implies u^L = \langle \cdot, l_L\rangle|_L + \sum_{M:(M,L)\in\delta} u^{ML} - \sum_{P:(L,P)\in\delta} u^{LP}|_L \tag{7}$$

for some sets of finite support $\{u^{MP} \in P^* : (M,P) \in \delta\}$ and

$$\left\{ l_L \in L : \ \ L \in \mathcal{P}, \ \sum_{L \in \mathcal{P}} l_L = 0 \right\} ,$$

see the equalities (4) and (5).

This shows that r_K is well-defined on $\tilde{H}_K$ by

$$r_K \sum_L i_{L^*} u^L := \sum_L u^L|_K \ .$$

Indeed, if $\sum_L i_{L^*} u^L = 0$, then we infer that $\sum_L u^L|_K = 0$, by summing in the equality (7) over all $L \in \mathcal{P}$ with $L \supset K$ and using the equalities $\sum_L l_L = 0$ and

$$\sum_L \sum_{M:(M,L)\in\delta} u^{ML}|_K = \sum_L \sum_{P:(L,P)\in\delta} u^{LP}|_K \ .$$

Since $r_L i_{L^*} = 1_{L^*}$, all i_{L^*} are injective. Hence i $(= i_{H^*} i_{HH^*})$ is injective too.

Now if $L \in \mathcal{P}$ and $l \in L$ are arbitrary, then the vector $s = s_l \in S$ given by $s := s_L i_{LL^*} l - s_H i_{HH^*} l$ belongs to S_1, see (4). Indeed, if $L \neq H$ (the nontrivial case), then s has the form $s = \bigoplus_P i_{PP^*} l_P$, where $l_L = l, l_H = -l$ and $l_P = 0$ for all $P \neq L, H$. Hence $\sum_P l_P = 0$. Then $ps = 0$. Therefore,

$$i_{L^*} \, i_{LL^*} \, l - i \, l \; = \; ps_L \, i_{LL^*} \, l - p \, s_H \, i_{HH^*} \, l \; = \; p \, s \, l \; = \; 0 \ .$$

Thus $i_{L^*} i_{LL^*} = i|_L$.

To define the inner product, let $d := (ik, \sum_L i_{L^*} u^L) \in D$ be arbitrary. That is, we fix a space $K \in \mathcal{P}$, a vector $k \in K$, and a finitely supported set $\{u^L \in L^* : \ L \in \mathcal{P}\}$ with the property that $L \supset K$ whenever $u^L \neq 0$, see the definition (6) of D. Set

$$\left\langle ik, \sum_L i_{L^*} u^L \right\rangle := \sum_L u^L \, k \ .$$

To prove that $\langle \cdot, \cdot \rangle$ is well-defined above, represent $d \in D$ in a similar form, $d = (ik', \sum_L i_{L^*} v^L)$. More precisely, $k' \in K' \in \mathcal{P}$ and the set $\{v^L \in L^* : L \in \mathcal{P}\}$ has finite support and satisfies $L \supset K'$ whenever $v^L \neq 0$. Then $ik = ik'$ and

$$\sum_L i_{L^*} (u^L - v^L) = 0 \ .$$

Since i is injective, we have $k = k'$. Moreover, $u^L - v^L$ can be represented as in (7). By summing over L, it follows that $\sum_L u^L k - \sum_L v^L k' = 0$.

We let

$$f\sum_L i_{L^*}u^L := \sum_L j_{L^*}u^L \ .$$

To show that f is well-defined, suppose that we have $\sum_L i_{L^*}u^L = 0$. Hence u^L can be represented as in the equality (7), that we use as follows. Remind that $\langle\cdot, l_L\rangle|_L = i_{LL^*}l_L$, apply j_{L^*} to (7) and use the equality $j_{L^*}i_{LL^*} = j$. Finally, sum over L and use the equalities $\sum_L l_L = 0$ and $j_{L^*}(u^{LP}|_L) = j_{P^*}u^{LP}$ to derive, after canceling the terms in the right-hand side, that $\sum_L j_{L^*}u^L = 0$. Since $\tilde{H} = sp\{R(i_{L^*}) : L \in \mathcal{P}\}$, it follows that f is also uniquely determined. ∎

Given H and $\mathcal{P}$, we have established, by Theorem 3.2, the existence of an initial object $\tilde{H} = (\tilde{H}, i, (i_{L^*})_{L\in\mathcal{P}}, (r_L)_{L\in\mathcal{P}})$ in the category of completions (see Definition 2.4). This object is then uniquely determined modulo an isomorphism in this category. We will call $\tilde{H} = \tilde{H}(\mathcal{P})$ *the $\mathcal{P}$-completion* of H. As will follow by Remark 3.8, the completion does not essentially change if we replace the inner product of H by an equivalent one.

Remark 3.3. If H is a prehilbertian space and all $L \in \mathcal{P}$ are endowed with the induced topology, then $\tilde{H}(\mathcal{P})$ can be identified with the usual completion $H^{\tilde{}}$ of H and $i : H \hookrightarrow H^{\tilde{}}$ becomes the inclusion. In particular, this holds if all $L \in \mathcal{P}$ are closed in a Hilbert space H. Indeed, in this case $S_2 = \{0\}$ and, by (4), the map $i_{H^*} : H^* \to \tilde{H}(\mathcal{P}) = S/S_1$ is an isomorphism. Then use $H^* \equiv (H^{\tilde{}})^*$ and Riesz' isomorphism $H^{\tilde{}} \equiv (H^{\tilde{}})^*$ taking x into $\langle\cdot|\overline{x}\rangle$. □

Proposition 3.4. *Let H, K be prehilbertian spaces and $\mathcal{P}, \mathcal{Q}$ be sets of topological linear subspaces of H, K, respectively. Let $f : H \to K$ be isometric and such that $\mathcal{Q} = \{f(P) : P \in \mathcal{P}\}$ and for every $L \in \mathcal{P}$ the map $f|_L : L \to fL$ be bicontinuous with respect to the own topologies of L and fL. Then f has a unique isometric extension $\tilde{f} : \tilde{H}(\mathcal{P}) \to \tilde{K}(\mathcal{Q})$.*

Proof: Let $(\tilde{H}(\mathcal{P}), i, (i_{L^*})_{L\in\mathcal{P}}, (r_L)_{L\in\mathcal{P}})$ and $(\tilde{K}(\mathcal{Q}), k, (k_{(fL)^*})_{L\in\mathcal{P}}, (t_{fL})_{L\in\mathcal{P}})$ denote the corresponding completions. We define $j : H \to \tilde{K}(\mathcal{Q})$, $j_{L^*} : L^* \to (fL)^*$, and $\rho_{L^*} : \tilde{K}_{fL} \to (fL)^*$ as follows. Set $j := kf$. For $L \in \mathcal{P}$ and $u \in L^*$, set $j_{L^*}u := k_{(fL)^*}(uf^{-1})$. For $\xi \in sp\{R(j_{P^*}) : P \in \mathcal{P}, P \supset L\}$, set $\rho_L\xi := (t_{fL}\xi)f|_L$. We obtain thus another $\mathcal{P}$-completion $(\tilde{K}(\mathcal{Q}), j, (j_{L^*})_{L\in\mathcal{P}}, (\rho_L)_{L\in\mathcal{P}})$ of H. The conclusion follows then by Theorem 3.2 and Proposition 3.1. ∎

Note that if all $P \in \mathcal{P}$ are Fréchét spaces, then any isometric map f with $\mathcal{Q} = f(\mathcal{P})$ as in Proposition 3.4 is automatically bicontinuous from P to fP whenever $P \in \mathcal{P}$, by the closed graph and the open map theorems.

The completion $H \mapsto \tilde{H}$ is also monotonic, namely if $K \subset H$ then $\tilde{K} \subset \tilde{H}$ in the following sense.

Corollary 3.5. *Let $\mathcal{P}$ be a set of topological linear subspaces of a prehilbertian space H. Let $K \in \mathcal{P}$ have the induced topology. Endow K with the restriction of the norm of H. Set $\mathcal{P}_K = \{L \in \mathcal{P} : L \subset K\}$. Then $\tilde{K}(\mathcal{P}_K) \subset \tilde{H}(\mathcal{P})$.*

Proof: Denote by $(\tilde{H}(\mathcal{P}), i, (i_{L^*})_{L\in\mathcal{P}}, (r_L)_{L\in\mathcal{P}})$ the $\mathcal{P}$-completion of H. Hence the $\mathcal{P}_K$-completion of K is $(\tilde{K}(\mathcal{P}_K), i|_K, (i_{L^*})_{L\in\mathcal{P}_K}, (t_L)_{L\in\mathcal{P}_K})$, where, for every $L \in \mathcal{P}_K$, t_L is the restriction of r_L to the linear span of all $R(i_{P^*})$ with $P \in \mathcal{P}$ and $P \supset L$. Let $f : K \hookrightarrow H$ be the inclusion of K into H. By Proposition 3.4, there exists a unique isometric extension $\tilde{f}$ of f taking $\tilde{K}(\mathcal{P}_K)$ into $\tilde{H}(\mathcal{P})$ such that $\tilde{f} i_{L^*} = f i_{L^*}$ for $L \in \mathcal{P}_K$. We use also Proposition 3.1 to derive $fi = i|_K$. Hence the desired conclusion follows. ∎

Proposition 3.6. *Let $\mathcal{P}$ be a set of topological linear subspaces of a Hilbert space H. Assume each $L \in \mathcal{P}$ to be a separated locally convex space. Suppose that for any $L \in \mathcal{P}$ and $x \in H$ there exists $P \in \mathcal{P}$ with $L \subset P$ and $x \in P$ such that L is P-closed in P. Then $i(L)^{\perp\perp} = i(L)$ for every $L \in \mathcal{P}$ and $(iN)^{\perp} \subset (iM)^{\perp}$ for any $M, N \in \mathcal{P}(H)$ with $M \subset N$.*

Proof: The inclusions $iL \subset ((iL)^{\perp})^{\perp}$ and $(iN)^{\perp} \subset (iM)^{\perp}$ hold by the definition of orthogonality. Let $\eta \in ((iL)^{\perp})^{\perp}$ be arbitrary. Then $\eta \in \tilde{H}(\mathcal{P})$ is orthogonal to $(iL)^{\perp}$. From (6) it follows that $\eta \in iH$. Hence $\eta = ix$ for some $x \in H$. Suppose that $\eta \notin iL$. Then $x \notin L$. By the hypothesis, there exists a subspace $P \in \mathcal{P}$ such that $L \subset P$, $x \in P$ and L is P-closed in P. By the Hahn–Banach theorem, there exists a functional $u \in P^*$ such that $u|_L = 0$ and $u(x) \neq 0$. It follows that $\xi := i_{L^*}u \in (iL)^{\perp}$ and $\langle \eta, \xi\rangle = \langle ix, i_{L^*}u\rangle = u(x) \neq 0$. Then η is not orthogonal to $(iL)^{\perp}$, which is false. This contradiction shows that $\eta \in iL$. ∎

Remark 3.7. Let $\mathcal{P}$ be a set of topological linear subspaces of a Hilbert space H. For any $L \in \mathcal{P}$, factorize $i_{LL^*} : L \to L^*$ as $L \hookrightarrow H \equiv H^* \xrightarrow{\rho_L} L^*$ where H, H^* are identified, by Riesz' lemma, via $h \mapsto \langle \cdot | \overline{h}\rangle$, while ρ_L is the map of restriction to L. Taking adjoints provides a factorization of $i^*_{LL^*}$ as $L^{**} \xrightarrow{\iota_L} H \to L^*$ where $\iota_L = \rho_L^*$ is the adjoint of ρ_L, namely for $\xi \in (L^*)^*$, $\iota_L(\xi) = \xi \circ \rho_L \in H^{**} \equiv H$. Then we can complete H by starting as well with the family $\mathcal{P}^{**} := \{\iota_L(L^{**}) : L \in \mathcal{P}\}$. If each $L \in \mathcal{P}$ is a reflexive Banach space, then we obtain a $\mathcal{P}$-completion $\tilde{H}(\mathcal{P}^{**})$

isomorphic to $\tilde{H}(\mathcal{P})$. This holds using the canonical embedding $J_L : L \to L^{**}$ of L into its bidual L^{**} and the equalities $i^*_{LL^*} J_L = i_{LL^*}$ for $L \in \mathcal{P}$. ▫

Remark 3.8. Let $\mathcal{P}$ be a set of topological linear subspaces of the Hilbert space $(H, \langle\cdot|\cdot\rangle)$. Let A be a strictly positive bounded operator on H. Set $(x|y) := \langle Ax|Ay\rangle$ for $x, y \in H$. Let K denote the Hilbert space H endowed with the inner product $(\cdot|\cdot)$. Set $\mathcal{Q} = \{A^{-1}P : P \in \mathcal{P}\}$. Then, by Proposition 3.4, there exists a bijective linear isometric map $\tilde{A} : \tilde{K}(\mathcal{Q}) \to \tilde{H}(\mathcal{P})$ such that $(x|y)_{\tilde{K}} = \langle \tilde{A}x|\tilde{A}y\rangle_{\tilde{H}}$. ▫

Remark 3.9. Let X be a Hilbert space. Set $H = X \oplus X$ and take

$$\mathcal{P} = \left\{P \oplus Q:\ P, Q \in \mathcal{P}(X)\right\} .$$

Define the *adjoint* $G^\star \subset \tilde{H}(\mathcal{P})$ of a paraclosed linear relation $G \in \mathcal{PR}(X)$ as

$$G^\star := \left\{(-y, x):\ (x, y) \in G^\perp\right\} .$$

Then we have $(G^\star)^\star = G$ and $F \subset G \Rightarrow G^\star \subset F^\star$ for any $F, G \in \mathcal{PR}(X)$. These properties follow easily from Proposition 3.6, using the fact that $\mathcal{P}(X)$ is the set of all paraclosed linear spaces of X. ▫

4 – Examples

We give below concrete examples of $\mathcal{P}$-completions $\tilde{H}(\mathcal{P})$.

Example 4.1. Let V be a compact smooth manifold without boundary. Let H be $L^2(V, m)$ with respect to an absolutely continuous measure m on V with continuous positive density. Thus $\langle f|g\rangle = \int_V f\overline{g}dm$ for $f,\ g \in H$. Let $\mathcal{P}$ be the set of all hilbertian Sobolev spaces $H^n(V) \subset H$ of positive integer order $n \geq 0$, each of them endowed with the usual hilbertian topology [7]. Then $\tilde{H}(\mathcal{P})$ is the space $\mathcal{D}'(V) = \bigcup_{n\in\mathbb{Z}} H^n(V)$ of all distributions on V, where $H^{-n}(V) \equiv (H^n(V))^*$ for $n \geq 0$. The mapping i is the inclusion $L^2(V, m) \hookrightarrow \mathcal{D}'(V)$. The bilinear map $\langle f, g\rangle = \langle f|\overline{g}\rangle$ on H is extended by the duality $\langle \varphi, u\rangle$ between test functions $\varphi \in \mathcal{D}(V)$ and distributions $u \in \mathcal{D}'(V)$. The domain D of $\langle\cdot,\cdot\rangle$ is the union $\bigcup_{n\geq 0} H^n(V) \times H^{-n}(V)$. For any $L := H^n(V)$ $(n \geq 0)$ the map i_{L^*} can be identified with the inclusion of $H^{-n}(V)$ into $\mathcal{D}'(V)$. ▫

Definition 4.2. [6, Section I.3.2]. Let S be a linear space endowed with a set of prehilbertian norms $\|\cdot\|_n$ $(n \geq 1)$ such that if a $\|\cdot\|_n$-null sequence is $\|\cdot\|_m$-Cauchy, then it is $\|\cdot\|_m$-null, too $(n, m \geq 0)$. Assume S is complete when endowed with the topology whose basis of neighborhoods of 0 is given by $\|s\|_n < \varepsilon$ $(n \geq 1,\ \varepsilon > 0)$. We may assume the sequence of the norms is increasing. Let L_n be the $\|\cdot\|_n$-completion of S. We call S a *nuclear space* if for any m there is $n \geq m$ such that the inclusion $i_{nm} : L_n \hookrightarrow L_m$ can be represented as

$$i_{nm}\, s \;= \sum_{k\geq 1} t_k\, \langle s | s'_k \rangle_n\, s''_k$$

with $t_k \geq 0$ and $\sum_k t_k < \infty$ for orthonormal systems $(s'_k)_k \subset L_n$, $(s''_k)_k \subset L_m$. □

Example 4.3. Let H be the completion of a nuclear space $(S, (\|\cdot\|_n)_{n\geq 0})$ with respect to a separately continuous inner product $\langle\cdot|\cdot\rangle$. Set $\mathcal{P} := \{S, L_0, L_1, ...\}$ where L_n is the $\|\cdot\|_n$-completion of S, see Definition 4.2. Then $\tilde{H}(\mathcal{P}) = S^*$ and $ih = \langle\cdot, h\rangle|_S$ for $h \in H$. □

The spaces from above have good properties with respect to certain operator theoretic problems. For instance, any selfadjoint operator A on S has a complete system of *generalized eigenvectors* [6, Section I.4.5], namely vectors $u \in S^*$, $u \neq 0$ such that there is a scalar λ with $uA = \lambda u$ on S.

Example 4.4. Let H be $L^2(\mathbb{R})$ with respect to the Lebesgue measure. Let $L_1 = H^1(\mathbb{R})$ be the Sobolev space of order 1, namely the space of all $f \in H$ with generalized derivative $f' \in H$, endowed with the usual hilbertian topology. Then L_1 consists of continuous functions. Let $L_0 = \{f \in L_1 : f(0) = 0\}$ have the topology induced by L_1. Let $H^1(0,\infty)$ be the Sobolev space of order 1, defined as the space of all $u \in \mathcal{D}'(0,\infty)$ with $u, u' \in L^2(0,\infty)$. Then $H^1(0,\infty)$ is continuously contained in the space of the continuous functions on $[0,\infty)$ [7]. The limit $u(0+)$ exists for any $u \in H^1(0,\infty)$. Extend u to $\tilde{u} \in H$ by 0 on $(-\infty, 0)$. Let $S_+ = \{\tilde{u} : u \in H^1(0,\infty)\}$ and $S_- = \{f(-x) : f \in S_+\}$ have the topology induced by $H^1(0,\infty)$. Let $L_2 = S_+ + S_-$ have the topology of sum of paraclosed subspaces [5] and $L_3 = H$. Thus $L_0 \subset \ldots \subset L_3$. Set $\mathcal{P} = \{L_j\}_{j=0}^3$. Using the density of L_j in H we obtain in this case

$$\tilde{H}(\mathcal{P}) = \Big(\bigoplus_j L_j^*\Big)/\mathcal{N}$$

where

$$\mathcal{N} = \left\{ \bigoplus_j h_j \in H^4 \colon\ \sum_j h_j = 0 \right\} .$$

The duals L_j^* have known concrete descriptions: $L_1^* = H^{-1}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ is the dual of $H^1(\mathbb{R})$, L_0^* is the quotient space $H^{-1}(\mathbb{R})/L_0^\perp$, and $L_2^* \equiv (H^{-1}(0,\infty))^2$. The inclusions $i_{L_j^*} : L_j^* \subset \tilde{H}(\mathcal{P})$ are obvious. With the notation from the proof of Theorem 3.2, we have $s_{L_j} u = u$ for $u \in L_j^*$ and $ps = s$ for $s \in S$, see (5). Dirac's functional δ belongs to L_1^*. The elements $\delta_+, \delta_- \in L_2^*$ defined by $\delta_\pm u = u(0\pm)$ do not belong to L_0^*, since $\delta, \delta_\pm \in L_0^\perp$, but $\delta, \delta_\pm \neq 0$ (one shows easily that they cannot be represented as L^2-functions $\sum_j h_j \in \mathcal{N}$). We have $\delta_+ - \delta_- \in L_1^\perp$ since $\delta_+ = \delta_-$ on L_1. □

ACKNOWLEDGEMENTS – This paper was started while visiting Laboratoire J.A. Dieudonnée at the University of Nice and is based to a large extent on a joint work with professor J.-Ph. Labrousse.
The author is indebted to the referee for many remarks and suggestions that have considerably improved the presentation of the paper.
This research was supported by grant 201/03/0041 of GA CR.

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