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THE SET OF PERIODIC SOLUTIONS OF A NEUTRAL DIFFERENTIAL EQUATION WITH CONSTANT DELAY AND PIECEWISE CONSTANT ARGUMENT

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Abstract: The complete set of ω-periodic solutions is found for a neutral differential equation with constant delay, piecewise constant argument and ω-periodic coefficients.

Considerable attention has been given to delay differential equations with piecewise constant arguments by several authors including Cooke and Wiener [1], Shah and Wiener [2], Aftabizaded et al. [3], and others. This class of differential equations has useful applications in biomedical models of disease that has been developed by Busenerg and Cooke [4] and in stabilization of hybrid control systems with feedback delay, where a hybrid system is one with a continuous plant with a discrete (sampled) controller.

In [3], the set of all periodic solutions of the following linear differential equation with constant coefficient and piecewise constant deviating argument

$$y'(t) + ay(t) + by([t-1]) = 0\,, \qquad t \geq 0\,,$$

are found, where $[\cdot]$ is the greatest integer function. In [4], the set of all periodic solutions of a more general equation

$$y'(t) + a(t)\,y(t) + b(t)\,y([t-1]) = 0\,, \qquad t \geq 0\,, \tag{1}$$

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is identified with the set of solutions of a linear system. It then follows that necessary and sufficient conditions for the existence of nontrivial periodic solutions of (1) can be found by considering a circulant matrix.

In this note, we proceed one step further and consider the following nonhomogeneous equation

$$\Big(y(t) + cy(t-1)\Big)' + a(t)\Big(y(t) + cy(t-1)\Big) + b(t)\,y([t-1]) = p(t)\ , \tag{2}$$

where c is a real constant such that $|c| \neq 1$, and $a(t)$, $b(t)$ as well as $p(t)$ are real, continuous functions with positive integer period ω. Note that we allow $c = 0$, in which case, (2) reduces to the following nonhomogeneous equation with piecewise constant argument.

$$y'(t) + a(t)\,y(t) + b(t)\,y([t-1]) = p(t)\ .$$

We will be interested in finding all the ω-periodic solutions of (2). To this end, let R be the set of real numbers and Z the set of all integers. By a solution $y = y(t)$ of (2), we mean a continuous function on R such that $(y(t) + cy(t-1))'$ exists at each point $t \in R$, with the possible exception of the points $[t] \in R$ where one-sided derivatives exist, and equation (2) is satisfied on each interval $[n, n+1) \subset R$ with integral endpoints.

Lemma 1. *Let c be a real constant such that $|c| \neq 1$. If $u(t)$ is a real, continuous and ω-periodic function on R, then there is unique ω-periodic continuous function $x(t)$ which is defined on R and*

$$u(t) = x(t) + cx(t-1)\ , \qquad t \in R\ . \tag{3}$$

Furthermore, if $|c| < 1$, then

$$x(t) = \sum_{i=0}^{\infty} (-1)^i c^i u(t-i)\ , \qquad t \in R\ , \tag{4}$$

while if $|c| > 1$, then

$$x(t) = \sum_{i=0}^{\infty} (-1)^i \left(\frac{1}{c}\right)^{i+1} u(t+i+1)\ , \qquad t \in R\ . \tag{5}$$

Proof: In case $|c| < 1$, it is easy to see that $\sum_{i=0}^{\infty} (-1)^i c^i u(t-i)$ is uniform convergent on compact intervals of R. If we define $x(t)$ by (4), then $x(t)$ is a real, continuous and ω-periodic function on R. Furthermore, it is not difficult to check that $x(t)$ satisfies (3). Similarly, in case $|c| > 1$, if we definite $x(t)$ as (5), then $x(t)$ is real, continuous and ω-periodic, and (3) holds.

Suppose $y(t)$ is a real, continuous and ω-periodic function defined on R which satisfies

$$u(t) = y(t) + cy(t-1)\,, \qquad t \in R\,. \tag{6}$$

From (3), (6) and the fact that $x(t)$ and $y(t)$ are ω-periodic, we see that for any $t \in R$,

$$|x(t) - y(t)| = |c|\,|x(t-1) - y(t-1)|\,. \tag{7}$$

By (7) and the fact that $x(t)$ and $y(t)$ are ω-periodic, we have

$$\begin{aligned} \max_{0 \le t \le \omega} |x(t) - y(t)| &= \sup_{t \in R} |x(t) - y(t)| \\ &= |c| \sup_{t \in R} |x(t-1) - y(t-1)| \\ &= |c|\,|\max_{0 \le t \le \omega} |x(t) - y(t)|\,. \end{aligned} \tag{8}$$

Since $|c| \neq 1$, (8) implies $x(t) = y(t)$ for $t \in R$. The proof is complete. ∎

The set of all ω-periodic solutions of (2) will be denoted by Ω_ω. Note that when $p(t) = 0$ for $t \in R$, Ω_ω, endowed with the usual addition and (real) scalar multiplication, is a linear space. In order to determine Ω_ω, we set

$$\alpha_n = \exp\Big(-\int_{n-1}^{n} a(u)\,du\Big) - c\,, \qquad n \in Z\,, \tag{9}$$

$$\beta_n = c\exp\Big(-\int_{n-1}^{n} a(u)\,du\Big) - \int_{n-1}^{n} b(s)\exp\Big(-\int_{s}^{n} a(u)\,du\Big)\,ds\,, \qquad n \in Z\,, \tag{10}$$

and

$$\gamma_n = -\int_{n-1}^{n} p(s)\exp\Big(-\int_{s}^{n} a(u)\,du\Big)\,ds\,, \qquad n \in Z\,. \tag{11}$$

Since $a(t)$, $b(t)$ and $c(t)$ are ω-periodic, it is easy to see that $\{\alpha_n\}_{n=-\infty}^{\infty}$, $\{\beta_n\}_{n=-\infty}^{\infty}$ and $\{\gamma_n\}_{n=-\infty}^{\infty}$ are ω-periodic sequences.

Let Ψ_ω be the set of all solutions of the following system of ω linear equations

$$\begin{cases} \beta_1 z_\omega + \alpha_1 z_1 - z_2 = \gamma_1\,, \\ \beta_2 z_1 + \alpha_2 z_2 - z_3 = \gamma_2\,, \\ \ldots = \ldots \\ \beta_\omega z_{\omega-1} + \alpha_\omega z_w - z_1 = \gamma_\omega\,. \end{cases} \tag{12}$$

In case $\gamma_1 = \ldots = \gamma_\omega = 0$, note that Ψ_ω is a linear subspace of R^ω.

Note that the system (12) can be written as

$$A_\omega z = \gamma \ ,$$

where $z = (z_1, z_2, ..., z_\omega)^\dagger$, $\gamma = (\gamma_1, ..., \gamma_\omega)^\dagger$, and

$$A_\omega = \begin{pmatrix} \alpha_1 & -1 & 0 & 0 & ... & ... & ... & \beta_1 \\ \beta_2 & \alpha_2 & -1 & 0 & ... & ... & ... & 0 \\ 0 & \beta_3 & \alpha_3 & -1 & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & \beta_{\omega-1} & \alpha_{\omega-1} & -1 \\ -1 & 0 & 0 & 0 & ... & 0 & \beta_\omega & \alpha_\omega \end{pmatrix} \tag{13}$$

when $\omega \geq 3$,

$$A_\omega = \begin{pmatrix} \alpha_1 & \beta_1 - 1 \\ \beta_2 - 1 & \alpha_2 \end{pmatrix} \tag{14}$$

when $\omega = 2$ and

$$A_\omega = (\alpha_1 + \beta_1 - 1) \tag{15}$$

when $\omega = 1$.

Theorem 1. *There is a one to one and onto mapping from Ω_ω to Ψ_ω. Furthermore, if $p(t) = 0$ for $t \in R$, then Ω_ω and Ψ_ω are isomorphic.*

Proof: Let $y(t)$ be an ω-periodic solution of (2). Then for $n \in Z$,

$$\Big(y(t)+cy(t-1)\Big)' + a(t)\Big(y(t)+cy(t-1)\Big) + b(t)\,y(n-1) = p(t), \quad n \leq t < n+1,$$

so that

$$\frac{d}{dt}\left(\Big(y(t)+cy(t-1)\Big)\exp\left(\int_n^t a(u)\,du\right)\right) + b(t)\exp\left(\int_n^t a(u)\,du\right)y(n-1)$$
$$= p(t)\exp\left(\int_n^t a(u)\,du\right),$$

and

$$\Big(y(t)+cy(t-1)\Big)\exp\left(\int_n^t a(u)\,du\right) - \Big(y(n)+cy(n-1)\Big)$$
$$+ y(n-1)\int_n^t b(s)\exp\left(\int_n^s a(u)\,du\right)ds$$
$$= \int_n^t p(s)\exp\left(\int_n^s a(u)\,du\right)ds \ ,$$

for $n \le t < n+1$. Thus

$$\begin{aligned} y(t) + cy(t-1) \;=\; & \Big(y(n) + cy(n-1)\Big) \exp\Big(-\int_n^t a(u)\, du\Big) \\ & - y(n-1) \int_n^t b(s) \exp\Big(-\int_s^t a(u)\, du\Big)\, ds \\ & + \int_n^t p(s) \exp\Big(-\int_s^t a(u)\, du\Big)\, ds \end{aligned} \tag{16}$$

for $n \le t < n+1$. Since $\lim_{t \to (n+1)^-} y(t) = y(n+1)$, we see further that for $n \in Z$,

$$\begin{aligned} y(n+1) + cy(n) \;=\; & \Big(y(n) + cy(n-1)\Big) \exp\Big(-\int_n^{n+1} a(u)\, du\Big) \\ & - y(n-1) \int_n^{n+1} b(s) \exp\Big(-\int_s^{n+1} a(u)\, du\Big)\, ds \\ & + \int_n^{n+1} p(s) \exp\Big(-\int_s^{n+1} a(u)\, du\Big)\, ds\ , \end{aligned}$$

so that

$$\begin{aligned} y(n{+}1) \;=\; & \left(\exp\Big(-\int_n^{n+1} a(u)\, du\Big) - c\right) y(n) \\ & + \left\{ c \exp\Big(-\int_n^{n+1} a(u)\, du\Big) - \int_n^{n+1} b(s) \exp\Big(-\int_s^{n+1} a(u)\, du\Big)\, ds \right\} y(n{-}1) \\ & + \int_n^{n+1} p(s) \exp\Big(-\int_s^{n+1} a(u)\, du\Big)\, ds\ . \end{aligned}$$

In terms of α_n, β_n and γ_n defined by (9), (10) and (11),

$$y(n+1) \;=\; \alpha_{n+1} y(n) + \beta_{n+1} y(n{-}1) - \gamma_{n+1}\ . \tag{17}$$

If we now let $z_k = y(k-1)$ for $k \in Z$, then $\{z_k\}_{k=-\infty}^{\infty}$ is a periodic sequence and from (17) we see that the column vector $(z_1, z_2, ..., z_\omega)^\dagger$ is a solution of (12), that is, $(z_1, z_2, ... z_\omega)^\dagger \in \Psi_\omega$.

Conversely, let $(z_1, z_2, ... z_\omega)^\dagger \in \Psi_\omega$. Define $z_0 = z_\omega$ and extend the finite sequence $\{z_0, z_1, ..., z_\omega\}$ to the unique ω-periodic sequence $\{z_n\}_{n=-\infty}^{\infty}$. Let $y_n = z_{n+1}$ for $n \in Z$, and let the function $u(t)$ on each interval $[n, n+1) \subset R$ be defined by

$$\begin{aligned} u(t) \;=\; & (y_n + cy_{n-1}) \exp\Big(-\int_n^t a(u)\, du\Big) - y_{n-1} \int_n^t b(s) \exp\Big(-\int_s^t a(u)\, du\Big)\, ds \\ & + \int_n^t p(s) \exp\Big(-\int_s^t a(u)\, du\Big)\ . \end{aligned} \tag{18}$$

and

$$u\,(n+1) = (y_n + cy_{n-1})\exp\Big(-\int_n^{n+1} a\,(u)\,du\Big) - y_{n-1}\int_n^{n+1} b\,(s)\exp\Big(-\int_s^{n+1} a\,(u)\,du\Big)\,ds + \int_n^{n+1} p\,(s)\exp\Big(-\int_s^{n+1} a\,(u)\,du\Big)\,. \tag{19}$$

Noting that $\{\alpha_n\}_{n=-\infty}^{\infty}$, $\{\beta_n\}_{n=-\infty}^{\infty}$, $\{\gamma_n\}_{n=-\infty}^{\infty}$ and $\{z_n\}_{n=-\infty}^{\infty}$ are ω-periodic sequences and $y_n = z_{n+1}$ for $n \in Z$, from (12) and (19), we see that

$$\begin{aligned} u(n{+}1) &= \Big\{\exp\Big(-\int_n^{n+1} a(u)\,du\Big) - c\Big\}\,y_n \\ &\quad + \Big\{c\,\exp\Big(-\int_n^{n+1} a(u)\,du\Big) - \int_n^{n+1} b(s)\,\exp\Big(-\int_s^{n+1} a(u)\,du\Big)\,ds\Big\}\,y_{n-1} \\ &\quad - \Big\{-\int_{n-1}^{n+1} p(s)\,\exp\Big(-\int_s^{n+1} a(u)\,du\Big)\,ds\Big\} + c\,y_n \\ &= \alpha_{n+1}\,y_n + \beta_{n+1}\,y_{n-1} - \gamma_{n+1} + c\,y_n \\ &= y_{n+1} + c\,y_n\ , \end{aligned}$$

that is

$$u\,(n{+}1) = y_{n+1} + cy_n\,, \qquad n \in Z\ . \tag{20}$$

By (18), (19) and (20), we see that $u\,(t)$ is a real, continuous and ω-periodic function on R, furthermore, from Lemma 1, there is a unique ω-periodic continuous function $y\,(t)$ which is defined on R and

$$u\,(t) = y\,(t) + cy\,(t-1)\ , \qquad t \in R\ . \tag{21}$$

By (18) and (21), we have

$$y\,(t) + cy\,(t-1) = (y_n + cy_{n-1})\exp\Big(-\int_n^t a\,(u)\,du\Big) - y_{n-1}\int_n^t b\,(s)\exp\Big(-\int_s^t a\,(u)\,du\Big)\,ds + \int_n^t p\,(s)\exp\Big(-\int_s^t a\,(u)\,du\Big)\,ds \tag{22}$$

for $t \in R$. Now, we prove that $y\,(n) = y_n$ for $n \in Z$. From (22), we have

$$y\,(n) + cy\,(n{-}1) = y_n + cy_{n-1}\,, \qquad n \in Z\ . \tag{23}$$

Noting that $\{y(n)\}_{n=-\infty}^{\infty}$ and $\{y_n\}_{n=-\infty}^{\infty}$ are ω-periodic sequences, (23) implies

$$\begin{aligned}\max_{0\le n\le\omega-1}|y(n)-y_n| &= \sup_{n\in Z}|y(n)-y_n| = |c|\sup_{n\in Z}|y(n-1)-y_{n-1}| \\ &= |c|\sup_{n\in Z}|y(n)-y_n| = |c|\max_{0\le n\le\omega-1}|y(n)-y_n| .\end{aligned} \tag{24}$$

Because $|c|\neq 1$, $y(n)=y_n$ for $n\in Z$. Furthermore, in view of (22), we know that $y(t)$ satisfies (16), it is therefore not difficult to check that the function $y(t)$ is an ω-periodic solution of (2). In other words, we have found a one to one and onto mapping from Ω_ω to Ψ_ω.

Note that in case $p(t)\equiv 0$, we have $\gamma_1=\gamma_2=...=\gamma_\omega=0$. Thus the solution sets Ω_ω and Ψ_ω are linear spaces. It is easily seen that the mapping found in the proof of Theorem 1 is linear. We may thus conclude that the solution spaces Ω_ω and Ψ_ω are isomorphic. The proof is complete. ∎

In view of the above identification theorem, we can apply standard results in linear algebra to yield the nature of the solutions of (2).

Theorem 2. *Suppose $p(t)\equiv 0$. Then the dimension of Ω_ω is $\omega-\mathrm{Rank}(A_\omega)$.*

In particular, when $p(t)\equiv 0$, (2) has a nontrivial ω-periodic solution if, and only if, $\det A_\omega=0$. In case $\omega=1$, $\det A_1=0$ if, and only if, $\beta_1+\alpha_1=1$; in case $\omega=2$, $\det A_2=0$ if, and only if, $(\beta_1-1)(\beta_2-1)=\alpha_1\alpha_2$; and in case $\omega\ge 3$, $\det A_\omega=0$ if $\beta_k+\alpha_k-1=0$ for $k=1,...,\omega$ (since 0 is an eigenvalue and $(1,1,...,1)^\dagger$ is the corresponding eigenvector of A_ω).

Theorem 3. *Equation (2) has an ω-periodic solution if, and only if, $\mathrm{Rank}(A_\omega)=\mathrm{Rank}([A_\omega\ \gamma])$, and has infinitely many ω-periodic solutions if, and only if, $\mathrm{Rank}(A_\omega)=\mathrm{Rank}([A_\omega\ \gamma])<\omega$. Here $\gamma=(\gamma_1,...,\gamma_\omega)^\dagger$ is defined by (11) and $[A_\omega\ \gamma]$ is the augmented matrix formed from A_ω and γ.*

In particular, equation (2) has a unique ω-periodic solution if, and only if, $\det A_\omega\neq 0$. As a consequence, when $\omega=1$, (2) has a unique 1-periodic solution if and only if $\beta_1+\alpha_1\neq 1$; when $\omega=2$, (2) has a unique ω-periodic solution if, and only if, $(\beta_1-1)(\beta_2-1)\neq\alpha_1\alpha_2$.

Example 1. Consider the following equation

$$\begin{aligned}(y(t)-3y(t-1))'+(\sin\pi t)(y(t)-3y(t-1))+\left(\exp\left(\pi^{-1}\cos\pi t\right)\right)y([t-1]) \\ = (\cos\pi t)\exp\left(\sin\left(2^{-1}\pi t\right)\right).\end{aligned} \tag{25}$$

Let $c=-3$, $a(t)=\sin \pi t$, $b(t)=\exp(\pi^{-1}\cos \pi t)$ and $p(t) = (\cos \pi t)\exp(\sin(2^{-1}\pi t))$. It is easy to verify that $a(t)$, $b(t)$ and $p(t)$ are continuous real functions with period 2. Then $\alpha_1 = 3+\exp\left(-2\pi^{-1}\right)$, $\alpha_2 = 3+\exp\left(2\pi^{-1}\right)$, $\beta_1 = -3\exp\left(-2\pi^{-1}\right) - \exp\left(-\pi^{-1}\right)$ and $\beta_2 = -3\exp\left(2\pi^{-1}\right) - \exp\left(\pi^{-1}\right)$. Thus $(\beta_1-1)(\beta_2-1) \neq \alpha_1\alpha_2$. By Theorem 3, our equation (25) has a unique 2-periodic solution. Furthermore, since the trivial function is not a solution of our equation, this 2-periodic solution is not trivial. □

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