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NORMAL FAMILIES OF HOLOMORPHIC FUNCTIONS ON INFINITE DIMENSIONAL SPACES

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Abstract: The purpose of the present work is to extend some classical results of holomorphic functions of one complex variable to holomorphic functions defined on infinite dimensional spaces. Montel-type and other classical theorems regarding normal families and exceptional values are established for holomorphic functions of infinitely many complex variables. This text culminates in a generalization of the classical Schottky Theorem, from which we derive Montel's fundamental criterion for normal families with exceptional values.

1 – Introduction

If E is a locally convex space, always assumed complex and Hausdorff, and U is a nonvoid open subset of E, let $\mathcal{H}(U)$ denote the vector space of all holomorphic functions $f\colon U \to \mathbb{C}$. Throughout this text, $\mathcal{H}(U)$ is endowed with the topology of uniform convergence on all compact subsets of U, which will be designated by $(\mathcal{H}(U), \tau_c)$.

In this paper we establish infinite dimensional versions of several classical theorems from the theory of normal families of holomorphic functions of one complex variable.

In Section 2 we establish an infinite dimensional version of a classical theorem of Montel on normal families (Montel [7], Section 10). Our result improves earlier results of Hue and Yue [5] and Kim and Krantz [6]. Several related theorems are also included in this section.

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Section 3 is devoted to the study of normal families of holomorphic functions with exceptional values. After giving infinite dimensional versions of the classical Schottky Theorem and Hurwitz Theorem, we prove an infinite dimensional version of a classical theorem of Montel for normal families of holomorphic functions with exceptional values (Montel [7], Section 32). As an additional application of our Schottky-type theorem, we prove an infinite dimensional version of the Little Picard Theorem.

We refer to the books of Dineen [4] or Mujica [8] for background information on infinite dimensional complex analysis.

2 – Normal families

We designate by $\mathbb{C}_\infty$ the extended complex plane, that is, $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Definition 1. **a**) A family $\mathcal{F} \subset \mathcal{H}(U)$ is said to be *normal* if each sequence in $\mathcal{F}$ has a subsequence which converges in $(\mathcal{H}(U), \tau_c)$.

b) The family $\mathcal{F}$ is called $\mathbb{C}_\infty$-*normal* if each sequence in $\mathcal{F}$ either admits a subsequence which converges in $(\mathcal{H}(U), \tau_c)$ or admits a subsequence which diverges to infinity uniformly on each compact subset of U. □

Remark 2. Clearly both definitions above coincide when $\mathcal{F}$ is bounded at a fixed point of U. □

Next we give a sufficient condition for a family $\mathcal{F} \subset \mathcal{H}(U)$ to be normal. It extends the classical *Montel Theorem* (Montel [7], Section 10 or Conway [3], Theorem VII.2.9) to holomorphic functions on separable locally convex spaces.

Theorem 3. *Let E be a separable locally convex space and $U \subset E$ a nonvoid open set. Then every locally bounded family $\mathcal{F} \subset \mathcal{H}(U)$ is normal.*

Proof: Since E is separable, there is a subset $D = \{x_1, x_2, ...\} \subset U$ dense in U. For each n, let $X_n := \overline{\mathcal{F}(x_n)}$. By hypothesis, each X_n is a closed and bounded subset of $\mathbb{C}$, therefore compact, and according to Tychonoff's Theorem, $X := \prod_{n=1}^{\infty} X_n$ is a compact metric space.

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{F}$. For each k, we denote by x^{f_k} the element in X given by $(f_k(x_1), f_k(x_2), ...)$. Thus $\{x^{f_k}\}_{k=1}^{\infty}$ is a sequence in X, and therefore it admits a convergent subsequence $\{x^{f_{k_i}}\}_{i=1}^{\infty}$. It will be shown that $\{f_{k_i}\}_{i=1}^{\infty}$ converges in $(\mathcal{H}(U), \tau_c)$.

Since $\mathcal{F}$ is locally bounded and thus equicontinuous, it suffices to prove that $\{f_{k_i}\}_{i=1}^{\infty}$ converges pointwise on all of U. To accomplish this, let $y=(y_1, y_2, ...) \in X$ be the limit of $\{x^{f_{k_i}}\}_{i=1}^{\infty}$, that is, $y_n = \lim\limits_{i\to\infty} f_{k_i}(x_n)$. In other words, $\{f_{k_i}\}_{i=1}^{\infty}$ converges pointwise at each x_n. Finally, let $x \in U$ and $\varepsilon > 0$. Since $\{f_{k_i}\}_{i=1}^{\infty}$ constitutes an equicontinuous family, there is a neighborhood $V \subset U$ of x such that:

$$\left|f_{k_i}(x') - f_{k_i}(x)\right| < \frac{\varepsilon}{3}\,, \qquad \forall\, x' \in V, \ \ \forall\, i\ .$$

Since D is dense in U, there is some $x_n \in V$. Moreover, $\{f_{k_i}(x_n)\}$ is convergent, so that there exists I such that:

$$\left|f_{k_i}(x_n) - f_{k_j}(x_n)\right| < \frac{\varepsilon}{3}\,, \qquad \forall\, i,j \geq I\ .$$

Hence, for $i, j \geq I$, we have:

$$\begin{aligned}\left|f_{k_i}(x) - f_{k_j}(x)\right| \;&\leq\; \left|f_{k_i}(x) - f_{k_i}(x_n)\right| + \left|f_{k_i}(x_n) - f_{k_j}(x_n)\right| + \left|f_{k_j}(x_n) - f_{k_j}(x)\right| \\ &<\; \varepsilon\ .\end{aligned}$$

Thus $\{f_{k_i}(x)\} \subset \mathbb{C}$ is a Cauchy sequence, and therefore convergent. The limit function f is G-holomorphic. Since $\{f_{k_i}\}$ is locally bounded, f is locally bounded as well, and thus holomorphic. ∎

Separability hypothesis is indeed essential in Theorem 3, as we can see in the following example, which can be found in [6]:

Example 4. Consider the non-separable Banach space $E = \ell^{\infty}$ and let $\mathcal{F} = \overline{B}_{E'}$ be the closed unit ball in E'. Clearly $\mathcal{F}$ is a locally bounded subset of $\mathcal{H}(E)$, but we claim that $\mathcal{F}$ is not a normal family.

In fact, let $\{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{F}$ be the sequence of the canonical linear functionals defined on E by $\varphi_j\big(\{\xi_n\}_{n=1}^{\infty}\big) = \xi_j$. If $\mathcal{F}$ were normal, then it would admit a subsequence which converges in $(\mathcal{H}(E), \tau_c)$. However, if $\{\varphi_{j_k}\}_{k=1}^{\infty}$ is any subsequence, let $x = (\xi_j)_{j=1}^{\infty} \in E$ be given by $\xi_{j_k} = (-1)^k$ for every k and $\xi_j = 0$ for $j \neq j_k$. Then $\{\varphi_{j_k}(x)\}_{k=1}^{\infty} = \{(-1)^k\}_{k=1}^{\infty}$ is the alternating sequence of scalars that does not converge, contradiction. □

When E is metrizable, Theorem 3 has been stated by Hu and Yue ([5], Theorem 2.1), but their proof has a gap. The authors claim that if D is a dense subset of an open set U then $D \cap K$ is a dense subset of K for each compact set $K \subset U$. It is very easy to give examples where $D \cap K$ is empty.

If U is a nonvoid open subset of a locally convex space E, then every locally bounded family $\mathcal{F} \subset \mathcal{H}(U)$ is relatively compact in $(\mathcal{H}(U), \tau_c)$ (see Dineen [4], Lemma 3.25). We could also prove Theorem 3 by combining the preceding result with the fact that on $\mathcal{F}$ the topology τ_c coincides with the topology τ_p of pointwise convergence at each point of U and also with the topology of pointwise convergence at each point of the dense sequence $D = \{x_n\}$. Indeed this shows that $\overline{\mathcal{F}}^{\tau_p}$ is compact and metrizable for τ_c.

Corollary 5. *Let E be a separable metrizable locally convex space and $U \subset E$ a nonvoid open set. Then a family $\mathcal{F} \subset \mathcal{H}(U)$ is normal iff $\mathcal{F}$ is bounded in $(\mathcal{H}(U), \tau_c)$.*

Proof: If $\mathcal{F}$ is bounded in $(\mathcal{H}(U), \tau_c)$ and E is metrizable, then $\mathcal{F}$ is locally bounded (see Dineen [4], Example 3.20) and Theorem 3 applies.

Conversely suppose $\mathcal{F}$ is normal but fails to be bounded in $(\mathcal{H}(U), \tau_c)$. Then there is a compact set $K \subset U$ and a sequence $\{f_n\}$ in $\mathcal{F}$ such that $\sup_{x \in K} |f_n(x)| > n$, $\forall\, n$. Since $\mathcal{F}$ is normal, $\{f_n\}$ admits a subsequence $\{f_{n_i}\}$ which converges in $(\mathcal{H}(U), \tau_c)$, say to f. But $f(K)$ is compact, and thus bounded. So there exists $M < \infty$ such that $\sup_{x \in K} |f(x)| < M$. This gives that:

$$n_i < \sup_{x \in K} \left|f_{n_i}(x)\right| \le \sup_{x \in K} \left|f_{n_i}(x) - f(x)\right| + \sup_{x \in K} \left|f(x)\right| , \qquad \text{for each } \ i \,.$$

By letting $i \to \infty$, we obtain a contradiction. ∎

When E is a Banach space, Corollary 5 has been stated by Kim and Krantz ([6], Theorem 1.8). Corollary 5 follows also from a remark of Boyd and Dineen ([1], p. 34).

The next result on convergence of holomorphic functions extends the *Stieltjès Theorem* (Montel [7], Théorème 15).

By a *domain* in a locally convex space E we mean a connected open subset $U \subset E$.

Theorem 6. *Let E be a separable locally convex space, $U \subset E$ a domain, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence which is locally bounded. If $\{f_n\}_{n=1}^{\infty}$ converges pointwise on some nonvoid open set $V \subset U$ then $\{f_n\}_{n=1}^{\infty}$ converges in $(\mathcal{H}(U), \tau_c)$.*

Proof: Let K be a compact subset of U and suppose for the moment that there exist $\varepsilon_0 > 0$, two strictly increasing sequences $\{m_k\}_{k=1}^\infty, \{n_k\}_{k=1}^\infty \subset \mathbb{N}$ and a sequence $\{x_k\}_{k=1}^\infty \subset K$ such that:

$$\left|f_{m_k}(x_k) - f_{n_k}(x_k)\right| > \varepsilon_0 , \quad \forall\, k .$$

For each k, consider the holomorphic function:

$$g_k := f_{m_k} - f_{n_k} .$$

Since the sequence $\{f_n\}_{n=1}^\infty$ is locally bounded, so is the family:

$$\mathcal{F} = \left\{g_k \colon k \in \mathbb{N}\right\} ,$$

and therefore $\mathcal{F}$ is normal. Hence, there exists a subsequence $\{g_{k_i}\}_{i=1}^\infty$ which converges in $(\mathcal{H}(U), \tau_c)$, say to a function g. Since the values of g_k converge to zero at each point of V, g is identically zero on V. According to the Identity Principle (Mujica [8], Proposition 5.7), g is identically zero on all of U. It means in particular that $\{g_{k_i}(x)\}_{i=1}^\infty$ converges uniformly to zero on all of K. But this is a contradiction, for $|g_k(x_k)| > \varepsilon_0$, $\forall\, k$. ∎

An examination of the proof of the Identity Principle established in [8] for holomorphic mappings defined on Banach spaces shows that it is still true for holomorphic functions defined on locally convex spaces.

Before extending the following Montel-type theorems (Montel [7], Section 19), some definitions are needed:

Definition 7. Let $\mathcal{F} \subset \mathcal{H}(U)$ and let $V \subset U$ be an open set. The family $\mathcal{F}$ is said to be *normal on V* if each sequence in $\mathcal{F}$ has a subsequence which converges uniformly on each compact subset of V. In this case, we say that each sequence in $\mathcal{F}$ has a subsequence which converges in $(\mathcal{H}(V), \tau_c)$. □

When $V = U$, this definition simply coincides with the definition of a normal family.

Definition 8. A family $\mathcal{F} \subset \mathcal{H}(U)$ is said to be *normal at a point* $x_0 \in U$ if there exists an open neighborhood $V \subset U$ of x_0 where the family is normal. □

A family which is normal on an open set U is evidently normal at each of its points. It will be shown that the converse is also true for separable metrizable locally convex spaces. Before this theorem is proved, it is necessary to obtain a further result.

Theorem 9. *Let U be an open subset of a locally convex space E and let $\{V_n\}_{n=1}^{\infty}$ be a sequence of open sets in U. If a family $\mathcal{F} \subset \mathcal{H}(U)$ is normal on each V_n then $\mathcal{F}$ is normal on $\bigcup_{n=1}^{\infty} V_n$.*

Proof: Consider a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$. Since $\mathcal{F}$ is normal on each V_n, then in particular $\{f_n\}$ has a subsequence $\{f_n^{(1)}\}$ which converges uniformly on each compact subset of V_1. On the one hand $\{f_n^{(1)}\}$ has in particular a subsequence $\{f_n^{(2)}\}$ which is uniformly convergent on each compact subset of V_2, and on the other hand, being a subsequence of $\{f_n^{(1)}\}$, $\{f_n^{(2)}\}$ is also uniformly convergent on each compact subset of V_1. Thus $\{f_n^{(2)}\}$ converges in $(\mathcal{H}(V_1 \cup V_2), \tau_c)$. Analogously, $\{f_n^{(2)}\}$ admits a subsequence $\{f_n^{(3)}\}$ which converges in $(\mathcal{H}(V_1 \cup V_2 \cup V_3), \tau_c)$. Proceeding this way, we obtain for each k a subsequence $\{f_n^{(k)}\}$ of $\{f_n^{(k-1)}\}$ which converges in $(\mathcal{H}(V_1 \cup \cdots \cup V_k), \tau_c)$. It will be shown that the diagonal sequence $\{f_n^{(n)}\}$ converges in $\big(\mathcal{H}(\bigcup_{n=1}^{\infty} V_n), \tau_c\big)$.

Let K be a compact subset of $\bigcup_{n=1}^{\infty} V_n$. Since $\bigcup_{n=1}^{\infty} V_n$ forms an open cover of K, we can extract a finite subcover and thus we can write $K \subset V_1 \cup \cdots \cup V_k$, for some k. But $\{f_n^{(n)}\}$ is, except for the $k-1$ first terms $f_1^{(1)}, f_2^{(2)}, ..., f_{k-1}^{(k-1)}$, a subsequence of $\{f_n^{(k)}\}$, which converges uniformly on the compact subsets of $V_1 \cup \cdots \cup V_k$, whence $\{f_n^{(n)}\}$ is uniformly convergent in K. By hypothesis the limit function f is holomorphic on each V_n. Thus f is holomorphic on $\bigcup_{n=1}^{\infty} V_n$. ∎

Theorem 10. *Let U be an open subset of a separable metrizable locally convex space E. Then a family $\mathcal{F} \subset \mathcal{H}(U)$ is normal iff it is normal at each point of U.*

Proof: To prove the nontrivial implication assume $\mathcal{F}$ is normal at each point of U. It means that for each $x \in U$ there is an open neighborhood $V_x \subset U$ of x such that $\mathcal{F}$ is normal on V_x. Since $U = \bigcup_{x \in U} V_x$ and every separable metric space is a Lindelöf space, there exists a countable subcover of $\bigcup_{x \in U} V_x$, and thus $U = \bigcup_{n=1}^{\infty} V_{x_n}$, and the desired conclusion follows immediately from Theorem 9. ∎

3 – Holomorphic functions with exceptional values

This section is devoted to the study of normal families of holomorphic functions with exceptional values. To prove two of the theorems we will need the following extension of the classical *Hurwitz Theorem*, which result is of interest in itself.

Theorem 11. *Let E be a locally convex space, $U \subset E$ a domain, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}(U)$ that converges to f. If each f_n never vanishes on U then either $f \equiv 0$ or f never vanishes on U.*

Proof:

First Case: First assume U convex.

Suppose that there exists $x_0 \in U$ such that $f(x_0) = 0$. We must show that $f(x) = 0$, for all $x \in U$.

For this purpose, let $x \in U$. Define:

$$\Lambda := \Big\{ \lambda \in \mathbb{C} \colon \; x_0 + \lambda(x - x_0) \in U \Big\} \; .$$

Since U is convex, the open set Λ is convex as well, and in particular connected, and then $\Lambda \subset \mathbb{C}$ is a domain. For each n, the function:

$$g_n(\lambda) \; := \; f_n\Big(x_0 + \lambda(x - x_0)\Big)$$

is holomorphic in Λ and never vanishes. By defining:

$$g(\lambda) \; := \; f\Big(x_0 + \lambda(x - x_0)\Big) \; ,$$

we have $g_n \to g$ uniformly on compact subsets of Λ. Hence, Hurwitz Theorem for holomorphic functions of one complex variable (Conway [3], Corollary VII.2.6) implies that $g \equiv 0$ or g never vanishes on Λ. But $0 \in \Lambda$ and $g(0) = f(x_0) = 0$, and therefore $g \equiv 0$. In particular, $0 = g(1) = f(x)$. Since $x \in U$ was arbitrary it follows that $f \equiv 0$.

General Case: Consider $A := \big\{ x \in U \colon \; f(x) = 0 \big\}$.

A is obviously closed. We claim that A is also open. In fact, let $a \in A$ and let $V \subset U$ be a convex open neighborhood of a. In particular, $\{f_n\}$ is a sequence of holomorphic functions on V that converge to f uniformly on compact subsets of V and that never vanish on V. Since V is convex, it follows from the first case that either $f \equiv 0$ on V or f never vanishes on V. But $f(a) = 0$. Hence $f \equiv 0$ on V, that is, $V \subset A$, and A is open.

Since U is connected and $A \subset U$ is open and closed, either $A = U$ or $A = \varnothing$. In other words, either $f \equiv 0$ or f never vanishes on U. ∎

Definition 12. When a function f omits a value a, we say that a is an *exceptional value* of f. ◻

To begin with, we extend a theorem on functions with an exceptional region (Montel [7], Section 17). However, Theorem 16 will substantially improve this result.

Theorem 13. *Let E be a separable locally convex space, $U \subset E$ a domain and let $\mathcal{F} \subset \mathcal{H}(U)$. If there exist $a \in \mathbb{C}$ and $m > 0$ such that $|f(x) - a| > m$ for all $f \in \mathcal{F}$ and for all $x \in U$, then $\mathcal{F}$ is $\mathbb{C}_\infty$-normal.*

Proof: Consider the family $\mathcal{G}$ of the following functions:

$$g(x) := \frac{1}{f(x) - a}, \quad \text{with} \ f \in \mathcal{F} .$$

Each $g \in \mathcal{G}$ is holomorphic and their values are bounded by the constant $1/m$ on all of U. In particular, $\mathcal{G}$ is locally bounded and therefore $\mathcal{G}$ is normal.

Now let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathcal{F}$. Then the corresponding sequence $\{g_n\}_{n=1}^\infty \subset \mathcal{G}$ admits a convergent subsequence $\{g_{n_k}\}_{k=1}^\infty$, say to a function g. Since f_{n_k} is always finite, each g_{n_k} never vanishes. Thus, according to Theorem 11, either $g \equiv 0$ or g never vanishes on U. If $g \equiv 0$ then it is easy to see that $f_{n_k}(x) \to \infty$ uniformly on each compact subset of U; if g never vanishes then the function:

$$f(x) := a + \frac{1}{g(x)}$$

is holomorphic on U and $f_{n_k} \to f$ in $(\mathcal{H}(U), \tau_c)$. ∎

The main tool used in the proof of the next results is the following extension of the classical *Schottky Theorem*. If we denote the open disc and the closed disc in $\mathbb{C}$ respectively by:

$$\Delta(z_0, R) = \Big\{ z \in \mathbb{C} \colon |z - z_0| < R \Big\} ,$$

$$\overline{\Delta}(z_0, R) = \Big\{ z \in \mathbb{C} \colon |z - z_0| \le R \Big\} ,$$

then the classical Schottky theorem asserts that, for each $0 < \alpha < \infty$ and $0 < \beta < 1$, there is a constant $c(\alpha, \beta) > 0$ such that, if $f \in \mathcal{H}(\Delta(0,1))$ is a function that omits the values 0 and 1, and such that $|f(0)| \le \alpha$, then $|f(z)| \le c(\alpha, \beta)$ for every $z \in \overline{\Delta}(0, \beta)$. We refer to the books of Carathodory ([2], p. 201) or Saks–Zygmund ([9], p. 348) for this version of the Schottky Theorem.

We remark that there is a misprint in the version of the Schottky Theorem that appears in the book of Conway ([3], p. 298). The constant β there should be strictly less than one. Indeed the functions $f_\rho(z) = (z-\rho)^{-1}$, with $\rho > 1$, give a counterexample to the statement for $\beta = 1$.

Before proving the generalization for locally convex spaces we recall that a set $U \subset E$ is said to be *balanced* if $\mu x \in U$ for each $x \in U$ and each $\mu \in \overline{\Delta}(0,1)$.

Theorem 14. *For each $0 < \alpha < \infty$ and $0 < \beta < 1$, there is a constant $c(\alpha,\beta) > 0$ such that, given a locally convex space E and a balanced open set $U \subset E$, if $f \in \mathcal{H}(U)$ is a function that omits the values 0 and 1, and such that $|f(0)| \le \alpha$, then:*

$$|f(x)| \le c(\alpha,\beta)\,, \qquad \forall\, x \in \beta U\ .$$

Proof: Fix α, β and f in the above conditions. For each $x \in U$, define:

$$\Lambda_x := \Big\{\lambda \in \mathbb{C} \colon\ \lambda x \in U\Big\}\ .$$

Since U is balanced each $\Lambda_x \supset \Delta(0,1)$, and the functions:

$$g_x(\lambda) := f(\lambda x)$$

are holomorphic on Λ_x (and therefore on $\Delta(0,1)$), omit the values 0 and 1, and $|g_x(0)| = |f(0)| \le \alpha$ for all $x \in U$. Hence the classical Schottky theorem can be applied to yield a constant $c(\alpha,\beta) > 0$ such that $|g_x(\lambda)| \le c(\alpha,\beta)$, for all $\lambda \in \overline{\Delta}(0,\beta)$ and $x \in U$. In particular,

$$|f(\beta x)| = |g_x(\beta)| \le c(\alpha,\beta),$$

for all $x \in U$, and the theorem is proved. ∎

We can also state the more general form:

Corollary 15. *For each $0<\alpha<\infty$ and $0<\beta<1$, there is a constant $c(\alpha,\beta)$ such that, given a locally convex space E and a balanced open set $U \subset E$, if $f \in \mathcal{H}(x_0+U)$ is a function that omits the values 0 and 1, and such that $|f(x_0)| \le \alpha$, then:*

$$|f(x)| \le c(\alpha,\beta)\,, \qquad \forall\ x \in x_0 + \beta U\ .$$

Proof: It suffices to consider the function $g(x) := f(x_0 + x)$, $x \in U$. ∎

We shall next apply this result to extend a classical theorem of Montel for normal families with exceptional values (Montel [7], Section 32).

Theorem 16. *Let E be a separable locally convex space and $U \subset E$ a domain. Then every family of holomorphic functions on U which have two distinct exceptional values a and b is $\mathbb{C}_\infty$-normal.*

Proof: Let $\mathcal{F}$ be a family of holomorphic functions on U that omit the values a and b (we can always assume the exceptional values of the functions in $\mathcal{F}$ are 0 and 1, for replacing, if necessary, each $f \in \mathcal{F}$ by the function φ given by $\varphi(x) := \frac{f(x)-a}{b-a}$, we obtain a family of holomorphic functions that do not assume the values 0 and 1 which will be equally $\mathbb{C}_\infty$-normal or not).

Fix a point $x_0 \in U$ and define the following families:

$$\mathcal{G} := \Big\{ f \in \mathcal{F} \colon\ |f(x_0)| \leq 1 \Big\} \quad \text{and} \quad \mathcal{H} := \Big\{ f \in \mathcal{F} \colon\ |f(x_0)| \geq 1 \Big\}\ .$$

It is clear that $\mathcal{F} = \mathcal{G} \cup \mathcal{H}$. The proof will be accomplished by showing that $\mathcal{G}$ is normal and that $\mathcal{H}$ is $\mathbb{C}_\infty$-normal.

Since E is separable, Theorem 3 is applied to show the normality of $\mathcal{G}$, so that it is sufficient to show that $\mathcal{G}$ is locally bounded. Thus if a is any point of U, let γ be a curve in U from x_0 to a and let $V_0, V_1, ..., V_n \subset U$ be open neighborhoods of $x_0, x_1, ..., x_n = a$, where each x_k lies on the trace of γ and $V_k = x_k + U_k$, U_k being balanced and such that $x_k + 2\,U_k \subset U$ for $0 \leq k \leq n$, and such that x_{k-1} and x_k are in $V_{k-1} \cap V_k$ for $1 \leq k \leq n$. It will be shown that $\mathcal{G}$ is uniformly bounded on V_n.

Notice that each function in $\mathcal{G}$ is in particular holomorphic on $x_0 + 2\,U_0$, $2\,U_0$ being balanced. Then applying Corollary 15 for each function in $\mathcal{G}$ and for $\alpha = 1$ and $\beta = 1/2$ we obtain a constant $c_0 := c(\alpha, \beta)$ such that $|f(x)| \leq c_0$, for all $x \in V_0$ and for all $f \in \mathcal{G}$. That is, $\mathcal{G}$ is uniformly bounded by the constant c_0 on V_0. In particular $x_1 \in V_0$, so that $|f(x_1)| \leq c_0$, for all $f \in \mathcal{G}$. Another application of Corollary 15 for each function in $\mathcal{G}$ and for $\alpha = c_0$ and $\beta = 1/2$ yields a constant c_1 such that $\mathcal{G}$ is uniformly bounded on V_1 by c_1. Proceeding this way, we get that $\mathcal{G}$ is uniformly bounded by a constant c_n on V_n, as asserted.

Now to show that $\mathcal{H}$ is $\mathbb{C}_\infty$-normal, we consider the family:

$$\widetilde{\mathcal{H}} := \Big\{ 1/f \colon\ f \in \mathcal{H} \Big\}\ .$$

Note that $\widetilde{\mathcal{H}} \subset \mathcal{G}$, and therefore $\widetilde{\mathcal{H}}$ is normal. Hence, if $\{f_n\}_{n=1}^\infty$ is a sequence in $\mathcal{H}$, there is a subsequence $\{f_{n_k}\}_{k=1}^\infty$ and a function $f \in \mathcal{H}(U)$ such that $\{1/f_{n_k}\}_{k=1}^\infty$ converges to f. Since each f_{n_k} is always finite, the functions $1/f_{n_k}$ never vanish. Thus, according to Theorem 11, either $f \equiv 0$ or f never vanishes. If $f \equiv 0$ then $f_{n_k}(x) \to \infty$ uniformly on compact subsets of U; if f never vanishes then $1/f$ is holomorphic and $f_{n_k} \to 1/f$ in $(\mathcal{H}(U), \tau_c)$. This completes the proof. ∎

This section concludes by extending the classical *Picard Theorem* (Conway [3], Theorem XII.2.3). It should be mentioned that it is easy to derive the next theorem from the *Little Picard Theorem* for entire functions of one complex variable. The proof presented here uses the concept of families of holomorphic functions with exceptional values as another application of Schottky Theorem.

Theorem 17. *If f is an entire function on a locally convex space that do not reduces to a constant then f assumes each complex number, with one possible exception.*

Proof: Suppose f omits two distinct values a and b. We must show that f is a constant. (Again, we may assume the exceptional values are 0 and 1, for replacing, if necessary, the function f by the function φ given by $\varphi(x) := \frac{f(x)-a}{b-a}$, we obtain an entire function that does not assume the values 0 and 1, and is equally constant or not.)

Fix a balanced open neighborhood $U \subset E$ of the origin and for each n define $U_n := 2^n U$ and $f_n(x) := f(2^n x)$.

Each f_n is an entire function, and in particular holomorphic on $2U$. Moreover, 0 and 1 are exceptional values of f_n on $2U$ and $f_n(0) = f(0)$, for each n. Thus applying Theorem 14 for each f_n and for $\alpha = |f(0)|$ and $\beta = 1/2$, we obtain a constant $C := c(\alpha, \beta)$ such that:

$$|f_n(x)| \leq C \ ,$$

for all $x \in U$ and for all n.

Finally, each $x \in E$ is in some U_n, and f assumes on each U_n the same values that f_n assumes on U, these last ones being bounded by C. That is, f is bounded in all of E, so that Liouville Theorem (Mujica [8], Proposition 5.10) implies that f is a constant. ∎

A glance at the proof of Liouville Theorem established in [8] for holomorphic mappings defined on Banach spaces shows that it is equally valid for holomorphic functions defined on locally convex spaces.

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