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ASYMPTOTIC BEHAVIOR OF THIN FERROELECTRIC MATERIALS

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Abstract: We are dealing with the model of ferroelectric materials that has been introduced by J.M. Greenberg and Al in Physica D 134, 362–383 (1999). We suppose that the ferroelectric material occupies a thin cylinder with regular cross section and small thickness $\nu > 0$ and give the limit model as ν goes to 0. Linear and nonlinear potentials are considered. In both cases, one notices that the limit problem is sensitive to the choice of the boundary conditions. We observe that Silver–Müller boundary conditions induce new terms in the limit problems.

1 – Introduction

We shall discuss the model equations of ferroelectric materials introduced by Greenberg and Al. in [9] and discussed in [8]. The characteristic feature of ferroelectric crystal is the appearance of a spontaneous electric dipole. It can be reversed, with no net change in magnitude, by an applied electric field. The current density j of the ferroelectric domain Ω is driven by the difference between the electric equilibrium field $\widehat{E}(\mathbf{P})$ and the electric field $\mathbf{E}$ where $\mathbf{P}$ is the spontaneous electric polarization. If one denotes by m the internal magnetic field then the model equations introduced in [9] takes the form in $\mathbb{R}^+ \times \Omega$

$$(1) \qquad \begin{cases} \epsilon(\partial_t \mathbf{P} + \theta j) = \operatorname{curl} m \\ \mu(\partial_t m + \theta\, \alpha\, m) = -\operatorname{curl} \mathbf{P} \\ \partial_t j + \theta\, \alpha\, j = \gamma\, \theta\big(\widehat{E}(P) - E\big) \end{cases}$$

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the set of equations is completed by initial conditions $P(0) = P^0$, $m(0) = m^0$, $j(0) = j^0$. The boundary conditions will be discussed later. Eliminating the variables j and m in the previous system, we get the following Maxwell equation satisfied by $\mathbf{P}$

$$\partial_t^2 \mathbf{P} + (\epsilon\mu)^{-1} \operatorname{curl}^2 \mathbf{P} + a\,\partial_t \mathbf{P} \;=\; -\gamma\,\theta\big(\widehat{E}(\mathbf{P}) - \mathbf{E}\big) \tag{2}$$

where $\operatorname{curl}^2 \mathbf{P} = \operatorname{curl}(\operatorname{curl} \mathbf{P})$, $a = \theta\alpha$. The parameters $\epsilon > 0$ and $\mu > 0$ are the permittivity and the magnetic permeability of the vacuum and the other ones are some physical constants. The equilibrium field is given by $\widehat{E}(\mathbf{P}) = \mathbf{P}\phi'(|\mathbf{P}|^2)$ where ϕ is a two wells potential satisfying some hypotheses given later. The electric displacement D is linked to the electric and polarization field $\mathbf{E}$ and $\mathbf{P}$ by the law $D = \epsilon(\mathbf{E} + \mathbf{P})$. Hence the electromagnetic field $(\mathbf{H}, \mathbf{E})$ satisfies in $\mathbb{R}^+ \times \Omega$ the Maxwell's equations

$$\mu\,\partial_t \mathbf{H} - \operatorname{curl} \mathbf{E} = 0\ , \qquad \epsilon\,\partial_t(\mathbf{E} + \mathbf{P}) + \operatorname{curl} \mathbf{H} + \sigma \mathbf{E} = 0 \tag{3}$$

where $\sigma > 0$ is the conductivity constant. The initial conditions are $\mathbf{E}(0) = \mathbf{E}^0$, $\mathbf{H}(0) = \mathbf{H}^0$. The boundary conditions satisfied by $\mathbf{E}$ and $\mathbf{P}$ take an important place in the characterization of the limit of the problem as the thickness goes to zero. If m satisfies the boundary condition $m \times \mathbf{n} = 0$ on $\partial\Omega$, $\mathbf{n}$ being the outward unit normal to $\partial\Omega$, one deduces by using (1) that $\mathbf{P}$ satisfies the boundary condition

$$\operatorname{curl} \mathbf{P} \times \mathbf{n} = 0\ . \tag{4}$$

This condition was proposed in [9] and studied in [2]. If more generally m and $\mathbf{P}$ satisfy a Silver–Müller type boundary condition like $\mathbf{n} \times m + \rho\, \mathbf{n} \times (\mathbf{P} \times \mathbf{n}) = 0$ where $\rho \geq 0$ is a function defined on $\partial\Omega$ then we obtain directly from (1) the following boundary condition for $\mathbf{P}$

$$\operatorname{curl} \mathbf{P} \times \mathbf{n} + \rho\,\mu\; \mathbf{n} \times \big((\partial_t \mathbf{P} + a\,\mathbf{P}) \times \mathbf{n}\big) \;=\; 0\ . \tag{5}$$

This boundary condition will be used in our work only in the linear case when $\phi'(s) \equiv k$, k being a real constant. For the nonlinear case, we will use the boundary condition

$$\mathbf{P} \times \mathbf{n} = 0\ . \tag{6}$$

The reason we use (6) is that we can prove the $\mathbb{H}^1$ regularity of the polarization field $\mathbf{P}$ which allows to pass to the limit in the nonlinear equilibrium electric field

$\widehat{E}(\mathbf{P})$. We proved in [1] that the boundary condition (5) ensures an $\mathbb{H}^{\frac{1}{2}}$ regularity of the polarization field $\mathbf{P}$. As $\mathbb{H}^{\frac{1}{2}}$ is compactly imbedded in $\mathbb{L}^2$ then we can also pass to the limit in the nonlinear potential.

For the linear as well as the nonlinear case, we use for the electric field the following Silver–Müller boundary condition

$$\mathbf{H}\times\mathbf{n} + \beta\ \mathbf{n}\times(\mathbf{E}\times\mathbf{n}) = 0 \tag{7}$$

where $\beta \geq 0$ is some function defined on $\partial\Omega$.

The equilibrium electric field $\widehat{E}(\mathbf{P})$ is given by $\widehat{E}(\mathbf{P}) = \mathbf{P}\phi'(|\mathbf{P}|^2)$ where ϕ is the two wells potential function defined in [9]. Recall that $\phi\colon \mathbb{R}\to\mathbb{R}$ is of class $\mathcal{C}^2$ such that $\phi(0) = 0$, $r_0^2 > 0$ is the location of the unique minimum of $\phi(r^2)$ with $\phi(r_0^2) < 0$ and $\phi(r^2) > 0$ for $r^2 \geq r_1^2$. Moreover, ϕ satisfies the following hypotheses

$$\phi(s) \sim C_0\, s \quad \text{for} \quad s \to +\infty\ , \qquad |\phi'(s)| \leq C_1, \quad s\phi^{(2)}(s) \leq C_2 \quad \text{for} \quad s \geq 0 \tag{8}$$

where $C_0,\ C_1 > 0$ and $C_2 > 0$ are some constants depending only of ϕ. It follows that there exists $C_\star$ depending only of ϕ such that

$$\left|\left(s\,\phi'(s^2)\right)'\right| \leq C_\star \quad \text{for} \quad s \geq 0 \tag{9}$$

consequently we get the following useful inequality

$$\left|A\,\phi'(|A|^2) - B\,\phi'(|B|^2)\right| \leq C_\star|A - B|\ , \qquad \forall\,(A,B) \in \mathbb{R}^3\times\mathbb{R}^3\ . \tag{10}$$

Let us precise the models we shall discuss. To simplify the presentation we equate to 1 all the constants appearing in the model except $a > 0$ and $\sigma > 0$ to measure the dissipation process. Let $\nu > 0$ representing the thickness of the cylinder $\Omega^\nu = \widehat{\Omega}\times(0,\nu)$ with cross section $\widehat{\Omega} \subset \mathbb{R}^2$ assumed to be an open bounded, convex and regular domain. We denote by $\mathbf{n}$ the outward unit normal to $\partial\Omega^\nu$. The generic point $x \in \Omega^\nu$ is denoted by $x = (\widehat{x}, x_3)$ where $\widehat{x} = (x_1, x_2) \in \widehat{\Omega}$ and $0 < x_3 < \nu$. The electromagnetic field $(\mathbf{H}^\nu, \mathbf{E}^\nu)$ satisfies in $\mathbb{R}^+\times\Omega^\nu$ the problem

$$\begin{cases} \partial_t\mathbf{H}^\nu - \operatorname{curl}\mathbf{E}^\nu = 0, \quad \partial_t(\mathbf{E}^\nu + \mathbf{P}^\nu) + \sigma\mathbf{E}^\nu + \operatorname{curl}\mathbf{H}^\nu = 0 \quad \text{in} \ \ \mathbb{R}^+\times\Omega^\nu\ , \\ \mathbf{H}^\nu\times\mathbf{n} + \beta^\nu\,\mathbf{n}\times(\mathbf{E}^\nu\times\mathbf{n}) = 0 \quad \text{on} \ \ \mathbb{R}^+\times\partial\Omega^\nu\ , \\ \mathbf{H}_{|t=0} = \mathbf{H}^0 \ \ \text{and} \ \ \mathbf{E}_{|t=0} = \mathbf{E}^0 \quad \text{in} \ \ \Omega^\nu\ . \end{cases} \tag{11}$$

coupled to the polarization equation which writes in the nonlinear case

$$\begin{cases} \partial_t^2\mathbf{P}^\nu + a\,\partial_t\mathbf{P}^\nu + \operatorname{curl}^2\mathbf{P}^\nu + \mathbf{P}^\nu\phi'(|\mathbf{P}^\nu|^2) = \mathbf{E}^\nu \quad \text{in} \ \ \mathbb{R}^+\times\Omega^\nu\ , \\ \mathbf{P}^\nu\times\mathbf{n} = 0\ , \quad \mathbb{R}^+\times\partial\Omega^\nu\ , \\ \mathbf{P}^\nu(0) = \mathbf{P}^0 \ \ \text{and} \ \ \partial_t\mathbf{P}^\nu(0) = \mathbf{P}^1 \quad \text{in} \ \ \Omega^\nu\ . \end{cases} \tag{12}$$

For the linear case, the system (11) is coupled to

$$(13)\qquad \begin{cases} \partial_t^2 \mathbf{P}^\nu + a\,\partial_t \mathbf{P}^\nu + \mathrm{curl}^2\,\mathbf{P}^\nu + k\,\mathbf{P}^\nu = \mathbf{E}^\nu \quad \text{in } \ \mathbb{R}^+\times\Omega^\nu\,, \\ \mathrm{curl}\,\mathbf{P}^\nu\times\mathbf{n} + \rho^\nu\,\mathbf{n}\times\big((\partial_t\mathbf{P}^\nu + a\,\mathbf{P}^\nu)\times\mathbf{n}\big) = 0\,, \quad \mathbb{R}^+\times\partial\Omega^\nu\,, \\ \mathbf{P}^\nu(0) = \mathbf{P}^0 \ \text{ and } \ \partial_t\mathbf{P}^\nu(0) = \mathbf{P}^1 \quad \text{in } \ \Omega^\nu\,, \end{cases}$$

where k is a real constant and β^ν and ρ^ν are two functions defined on the boundary $\partial\Omega^\nu$ and depending only of the variable x_3 and the thickness parameter ν.

Before stating the existence, uniqueness and regularity results leading respectively with the systems (11)–(12) and (11)–(13), we first define the following spaces and the corresponding norms that will be used throughout this manuscript.

Let $\mathcal{O}$ be an open bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$. We denote by $\mathbb{L}^2(\mathcal{O})$ the Lebesgue space $(L^2(\mathcal{O}))^2$ or $(L^2(\mathcal{O}))^3$ constituted by integrable functions, equipped with the usual norm denoted by $|.|$ and the scalar product $(.,.)$. Let $\mathcal{H}(\mathrm{curl},\mathcal{O})$ be the usual Hilbert space used in the theory of Maxwell equations equipped with the norm $|.|_{\mathcal{H}}$. We also use the Banach space $L^p(\mathbb{R}^+;\mathbb{L}^2(\mathcal{O}))$ for $p\ge 1$, $p\ne 2$ and the Hilbert space $L^2(\mathbb{R}^+;\mathbb{L}^2(\mathcal{O}))$ provided respectively with the norms $\|.\|_p$ and $\|.\|$. Finally, the norm of the Sobolev space $\mathbb{H}^1(\mathcal{O})$ is denoted by $|.|_{\mathbb{H}^1}$.

The existence, uniqueness and regularity of solutions $(\mathbf{H}^\nu,\mathbf{E}^\nu,\mathbf{P}^\nu)$ to the problem (11)–(12) has been proved in [3] and [10] with the boundary conditions $\mathbf{E}^\nu\times\mathbf{n} = 0$ and either $\mathbf{P}^\nu\times\mathbf{n} = 0$ or $\mathrm{curl}\,\mathbf{P}^\nu\times\mathbf{n} = 0$. Following the lines of the proof given in [3], we may prove with minor changes the following results dealing with the Silver–Müller boundary conditions which are usual in the theory of Maxwell's equations.

Theorem 1.1 (The linear case). *Let $\rho^\nu,\beta^\nu\in L^\infty(0,\nu)$ such that $\rho^\nu(x_3)\ge 0$ and $\beta^\nu(x_3)\ge 0$ a.e.. We assume that*

$$(14)\qquad \mathbf{H}^0,\mathbf{E}^0,\mathbf{P}^1\in\mathbb{L}^2(\Omega^\nu)\,,\quad \mathbf{P}^0\in\mathcal{H}(\mathrm{curl},\Omega^\nu)\,,\quad \mathbf{P}^0\times\mathbf{n}\in\mathbb{L}^2(\partial\Omega^\nu)\,.$$

Then, there exists a unique weak solution $(\mathbf{H}^\nu,\mathbf{E}^\nu,\mathbf{P}^\nu)$ to problem (11)–(13) such that $\mathbf{H}^\nu,\mathbf{E}^\nu\in L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega^\nu))$ and $\mathbf{P}^\nu\in L^\infty(\mathbb{R}^+;\mathcal{H}(\mathrm{curl},\Omega^\nu))$. The tangential traces $\mathbf{H}^\nu\times\mathbf{n}$, $\mathbf{E}^\nu\times\mathbf{n}$, $\partial_t\mathbf{P}^\nu\times\mathbf{n}$ belong to $L^2(\mathbb{R}^+;\mathbb{L}^2(\partial\Omega^\nu))$ and $\mathbf{P}^\nu\times\mathbf{n}\in L^\infty(\mathbb{R}^+;\mathbb{L}^2(\partial\Omega^\nu))$. Moreover, for all $t\ge 0$, we have the energy inequality

$$(15)\qquad \mathcal{E}^\nu(t) + 2\int_0^t a\,|\partial_t\mathbf{P}^\nu(s)|^2 + \sigma\,|\mathbf{E}^\nu(s)|^2 + |\sqrt{\beta^\nu}\,\mathbf{E}^\nu\times\mathbf{n}|^2 + |\sqrt{\rho^\nu}\,\partial_t\mathbf{P}^\nu\times\mathbf{n}|^2\,ds \le \mathcal{E}_0^\nu$$

where the energy at time t is defined by

(16)

$$\mathcal{E}^\nu(t) \;=\; |\partial_t \mathbf{P}^\nu(t)|^2 + k\,|\mathbf{P}^\nu(t)|^2 + |\mathrm{curl}\,\mathbf{P}^\nu(t)|^2 + a\,|\sqrt{\rho^\nu}\,\mathbf{P}^\nu{\times}\mathbf{n}|^2 + |\mathbf{E}^\nu(t)|^2 + |\mathbf{H}^\nu(t)|^2$$

and the initial energy $\mathcal{E}_0^\nu$ is given by

$$\mathcal{E}_0^\nu \;=\; |\mathbf{P}^1|^2 + k\,|\mathbf{P}^0|^2 + |\mathrm{curl}\,\mathbf{P}^0|^2 + a\,|\sqrt{\rho^\nu}\,\mathbf{P}^0{\times}\mathbf{n}|^2 + |\mathbf{E}^0|^2 + |\mathbf{H}^0|^2\;. \tag{17}$$ ∎

Concerning the nonlinear problem (11)–(12), we have

Theorem 1.2 (The nonlinear case)**.** *Assume that ϕ satisfies hypotheses (8) and that β^ν is a positive function belonging to $L^\infty(0,\nu)$. If the initial data satisfy*

$$\mathbf{H}^0, \mathbf{E}^0, \mathbf{P}^1 \in \mathbb{L}^2(\Omega^\nu)\,, \quad \mathbf{P}^0 \in \mathcal{H}(\mathrm{curl}, \Omega^\nu)\,, \quad \mathbf{P}^0{\times}\mathbf{n} \in \mathbb{L}^2(\partial\Omega^\nu)\;. \tag{18}$$

Then, there exists a unique weak solution $(\mathbf{H}^\nu, \mathbf{E}^\nu, \mathbf{P}^\nu)$ to problem (11)–(12) such that $\mathbf{H}^\nu, \mathbf{E}^\nu \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega^\nu))$ and $\mathbf{P}^\nu \in L^\infty(\mathbb{R}^+; \mathcal{H}(\mathrm{curl}, \Omega^\nu))$. The tangential traces $\mathbf{H}^\nu{\times}\mathbf{n}$, $\mathbf{E}^\nu{\times}\mathbf{n}$ belong to $L^2(\mathbb{R}^+; \mathbb{L}^2(\partial\Omega^\nu))$. Moreover, for all $t \ge 0$, we have the energy inequality

$$\mathcal{E}^\nu(t) + 2\int_0^t a\,|\partial_t \mathbf{P}^\nu(s)|^2 + \sigma |\mathbf{E}^\nu(s)|^2 + |\sqrt{\beta^\nu}\,\mathbf{E}^\nu{\times}\mathbf{n}|^2 \; ds \;\le\; \mathcal{E}_0^\nu \tag{19}$$

where the energy at time t is defined by

$$\mathcal{E}^\nu(t) \;=\; |\partial_t \mathbf{P}^\nu(t)|^2 + \int_{\Omega^\nu} \phi(|\mathbf{P}^\nu|^2)\,dx \;+\; |\mathrm{curl}\,\mathbf{P}^\nu(t)|^2 + \; |\mathbf{E}^\nu(t)|^2 + \; |\mathbf{H}^\nu(t)|^2 \tag{20}$$

and the initial energy $\mathcal{E}_0^\nu$ is given by

$$\mathcal{E}_0^\nu \;=\; |\mathbf{P}^1|^2 + \int_{\Omega^\nu} \phi(|\mathbf{P}^0|^2)\,dx \;+\; |\mathrm{curl}\,\mathbf{P}^0|^2 + \; |\mathbf{E}^0|^2 + \; |\mathbf{H}^0|^2\;. \tag{21}$$ ∎

We assume that in both linear and nonlinear cases

$$\begin{cases} \beta^\nu(x_3) = \beta \quad \text{if} \;\; 0 < x_3 < \nu\,, \quad \beta^\nu(0) = \nu\,\beta_1\,, \quad \beta^\nu(\nu) = \nu\,\beta_2 \\[2mm] \rho^\nu(x_3) = \rho \quad \text{if} \;\; 0 < x_3 < \nu\,, \quad \rho^\nu(0) = \nu\,\rho_1\,, \quad \rho^\nu(\nu) = \nu\,\rho_2 \end{cases} \tag{22}$$

where $\beta,\ \beta_1,\ \beta_2,\ \rho,\ \rho_1,\ \rho_2$ are some strictly positive constants. Note that using the hypothesis (8) satisfied by ϕ, we get for all $t \ge 0$

$$|\mathbf{P}^\nu(t)|^2 \;\le\; C\Big(\int_{\Omega^\nu} \phi\big(|\mathbf{P}^\nu(t)|^2\big)\,dx \;+\; |\Omega^\nu|\Big) \tag{23}$$

for some constant $C > 0$ depending only of ϕ. Here $|\Omega^\nu| = \nu\,|\widehat{\Omega}|$ is the Lebesgue measure of Ω^ν. Finally, arguing like in [3], we have the following time regularity result

Proposition 1.1 (Time regularity). *Let* $(\mathbf{H}^\nu, \mathbf{E}^\nu, \mathbf{P}^\nu)$ *be a weak solution of either (11)–(13) or (11)–(12) problem. We assume in both cases that* $(\mathbf{H}^0, \mathbf{E}^0, \mathbf{P}^0, \mathbf{P}^1)$ *satisfies*

$$\mathbf{H}^0,\ \mathbf{E}^0,\ \mathbf{P}^0,\ \mathbf{P}^1,\ \operatorname{curl}\mathbf{P}^0 \ \in\ \mathcal{H}(\operatorname{curl};\Omega^\nu)\ , \tag{24}$$

we assume moreover for the linear case that $\mathbf{P}^0 \times \mathbf{n}, \mathbf{P}^1 \times \mathbf{n} \in \mathbb{L}^2(\partial\Omega^\nu)$. *Then*

$$\begin{aligned} &\partial_t\mathbf{H}^\nu,\ \partial_t\mathbf{E}^\nu,\ \partial_t^2\mathbf{P}^\nu \ \in\ L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega^\nu)) \\ &\mathbf{H}^\nu,\ \mathbf{E}^\nu,\ \mathbf{P}^\nu,\ \partial_t\mathbf{P}^\nu \ \in\ L^\infty(\mathbb{R}^+;\mathcal{H}(\operatorname{curl};\Omega^\nu))\ . \ \blacksquare \end{aligned} \tag{25}$$

2 – Scaling and main result

In the sequel, let $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be the canonical basis of $\mathbb{R}^3$ and let Ω be the cylinder $\widehat{\Omega}\times(0,1)$ where $\widehat{\Omega}$ is a regular bounded convex domain of $\mathbb{R}^2$. The generic point x of Ω is denoted by $x = (\widehat{x}, z)$ with $\widehat{x} = (x_1, x_2)$ and $0 < z < 1$. If $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is a vector function, we set

$$\begin{cases} \operatorname{curl}_\nu \mathbf{f} \ = \Big(\partial_2\mathbf{f}_3 - \frac{1}{\nu}\,\partial_z\mathbf{f}_2\,,\ \frac{1}{\nu}\,\partial_z\mathbf{f}_1 - \partial_1\mathbf{f}_3\,,\ \partial_1\mathbf{f}_2 - \partial_2\mathbf{f}_1\Big) \\ \widehat{\mathbf{f}} = (\mathbf{f}_1, \mathbf{f}_2)\ , \qquad \widehat{\operatorname{curl}}\,\widehat{\mathbf{f}} = \partial_1\mathbf{f}_2 - \partial_2\mathbf{f}_1 \\ \widehat{\operatorname{div}}\,\mathbf{f} = \partial_1\mathbf{f}_1 + \partial_2\mathbf{f}_2\ , \qquad \operatorname{div}_\nu\mathbf{f} = \widehat{\operatorname{div}}\,\mathbf{f} + \frac{1}{\nu}\,\partial_z\mathbf{f}_3\ . \end{cases} \tag{26}$$

If $\mathbf{f}$ is a scalar function, we set

$$\operatorname{Curl}\mathbf{f} = (\partial_2\mathbf{f}, -\partial_1\mathbf{f})\ , \qquad \widehat{\Delta}\mathbf{f} = \partial_1^2\mathbf{f} + \partial_2^2\mathbf{f}\ . \tag{27}$$

Let $(\mathbf{H}^\nu, \mathbf{E}^\nu, \mathbf{P}^\nu)$ be the global solution of (11)–(12) or (11)–(13). We consider the scaled solution defined in $\mathbb{R}^+\times\Omega$ associated with $(\mathbf{H}^\nu, \mathbf{E}^\nu, \mathbf{P}^\nu)$

$$\mathbf{h}^\nu(t,\widehat{x},z) = \mathbf{H}^\nu(t,\widehat{x},\nu\, z)\,,\ \ \mathbf{e}^\nu(t,\widehat{x},z) = \mathbf{E}^\nu(t,\widehat{x},\nu\, z)\,,\ \ \mathbf{p}^\nu(t,\widehat{x},z) = \mathbf{P}^\nu(t,\widehat{x},\nu\, z)\,. \tag{28}$$

It follows that

$$\operatorname{curl}\mathbf{H}^\nu = \operatorname{curl}_\nu\mathbf{h}^\nu\ , \quad \operatorname{curl}\mathbf{E}^\nu = \operatorname{curl}_\nu\mathbf{e}^\nu\ , \quad \operatorname{curl}\mathbf{P}^\nu = \operatorname{curl}_\nu\mathbf{p}^\nu\ . \tag{29}$$

Our main results are the following

Theorem 2.1 (The linear case). *Assume that the initial data are independent of the variable x_3 and satisfy the hypotheses given proposition 1.1 and consider β^ν, ρ^ν defined by (22). Let $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$ be the scaled solution associated with the global solution to problem (11)–(13). Then, there exists a subsequence still denoted $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$ converging weakly-$\star$ in $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ to $(\mathbf{h}, \mathbf{e}, \mathbf{p})$ which is independent of the variable z and such that $\widehat{h} = 0$. The weak-$\star$ limit $(\mathbf{h}_3, \widehat{\mathbf{e}}, \widehat{\mathbf{p}})$ satisfies in $\mathbb{R}^+ \times \widehat{\Omega}$ the problem*

$$
(30) \quad \begin{cases}
\partial_t \mathbf{h}_3 - \widehat{\operatorname{curl}}\, \widehat{\mathbf{e}} = 0\,, \quad \partial_t(\widehat{\mathbf{e}} + \widehat{\mathbf{p}}) + \operatorname{Curl} \mathbf{h}_3 + (\sigma + \beta_1 + \beta_2)\widehat{\mathbf{e}} = 0 & \text{a.e. in } \mathbb{R}^+ \times \widehat{\Omega}\,, \\
(\partial_t^2 + a\partial_t + k)\, \widehat{\mathbf{p}} + \operatorname{Curl} \widehat{\operatorname{curl}}\, \widehat{\mathbf{p}} + (\rho_1 + \rho_2)(\partial_t + a)\, \widehat{\mathbf{p}} = \widehat{\mathbf{e}} & \text{a.e. in } \mathbb{R}^+ \times \widehat{\Omega}\,, \\
\widehat{\mathbf{e}}(0) = \widehat{\mathbf{E}}^0, \quad \widehat{\mathbf{p}}(0) = \widehat{\mathbf{P}}^0, \quad \partial_t \widehat{\mathbf{p}}(0) = \widehat{\mathbf{P}}^1, \quad \mathbf{h}_3(0) = \mathbf{H}_3^0 & \text{a.e. in } \widehat{\Omega}\,, \\
\mathbf{h}_3 = \beta\,(\mathbf{e}_1 \mathbf{n}_2 - \mathbf{e}_2 \mathbf{n}_1), \quad \widehat{\operatorname{curl}}\, \widehat{\mathbf{p}} = \rho\,(\partial_t + a)\,(\mathbf{p}_1 \mathbf{n}_2 - \mathbf{p}_2 \mathbf{n}_1) & \text{a.e. on } \mathbb{R}^+ \times \partial\widehat{\Omega}\,.
\end{cases}
$$

The third components $(\mathbf{e}_3, \mathbf{p}_3)$ satisfy the system of o.d.e

$$
(31) \quad \begin{cases}
\partial_t(\mathbf{e}_3 + \mathbf{p}_3) + \sigma\, \mathbf{e}_3 = 0\,, \quad (\partial_t^2 + a\,\partial_t + k)\, \mathbf{p}_3 = \mathbf{e}_3 & \text{a.e. in } \mathbb{R}^+ \times \widehat{\Omega}\,, \\
\mathbf{e}_3(0) = \mathbf{E}_3^0\,, \quad \mathbf{p}_3(0) = \mathbf{P}_3^0\,, \quad \partial_t \mathbf{p}_3(0) = \mathbf{P}_3^1 & \text{a.e. in } \widehat{\Omega}\,.
\end{cases}
$$

Moreover, we have

$$
\begin{aligned}
&\mathbf{h}_3 \in L^\infty(\mathbb{R}^+; H^1(\widehat{\Omega}))\,, \quad \widehat{\mathbf{e}} \in L^\infty(\mathbb{R}^+; \mathcal{H}(\widehat{\operatorname{curl}}, \widehat{\Omega}))\,, \\
&\widehat{\mathbf{p}} \in L^\infty(\mathbb{R}^+; \mathcal{H}(\widehat{\operatorname{curl}}, \widehat{\Omega}))\,, \quad \partial_t \widehat{\mathbf{p}} \in L^\infty(\mathbb{R}^+; \mathcal{H}(\widehat{\operatorname{curl}}, \widehat{\Omega}))\,, \\
&\widehat{\operatorname{curl}}\, \widehat{\mathbf{p}} \in L^\infty(\mathbb{R}^+; H^1(\widehat{\Omega}))\,.
\end{aligned}
$$

For the nonlinear case, we assume that initial data are independent of the variable x_3 and are such that

$$
(32) \quad \begin{cases}
\mathbf{H}^0 = (0, 0, \mathbf{H}_3^0)\,, \quad \mathbf{E}^0 = (\widehat{\mathbf{E}}^0, \mathbf{E}_3^0)\,, \quad \mathbf{P}^0 = (0, 0, \mathbf{P}_3^0)\,, \quad \mathbf{P}^1 = (0, 0, \mathbf{P}_3^1) \\
\mathbf{H}_3^0, \mathbf{P}_3^0, \mathbf{P}_3^1 \in H^1(\widehat{\Omega})\,, \quad \widehat{\operatorname{div}}\, \mathbf{E}^0 \in L^2(\widehat{\Omega})\,.
\end{cases}
$$

The limit problem in the nonlinear case is given by

Theorem 2.2 (The nonlinear case). *Assume that the initial data are independent of the variable x_3 and satisfy (32). We assume moreover that β^ν is given by (22). Let $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$ be the scaled solution associated with the global solution to problem (11)–(12). Then, there exists a subsequence still denoted $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$ such that $\mathbf{h}^\nu \rightharpoonup (0, 0, \mathbf{h}_3)$, $\mathbf{e}^\nu \rightharpoonup \mathbf{e}$ weakly in $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ weak-$\star$,*

$\mathbf{p}^\nu \to (0,0,\mathbf{p}_3)$ *strongly in* $L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$. *The weak-*$\star$ *limit* $(\mathbf{h}_3,\mathbf{e},\mathbf{p}_3)$ *is independent of the variable* z *and is such that* $(\mathbf{h}_3,\widehat{\mathbf{e}})$ *is the solution in* $\mathbb{R}^+\times\widehat{\Omega}$ *of the problem*

$$(33)\quad \begin{cases} \partial_t\mathbf{h}_3 - \widehat{\operatorname{curl}}\,\widehat{\mathbf{e}} = 0\,, \quad \partial_t\widehat{\mathbf{e}} + \operatorname{Curl}\mathbf{h}_3 + (\sigma+\beta_1+\beta_2)\widehat{\mathbf{e}} = 0 & \text{a.e. in } \mathbb{R}^+\times\widehat{\Omega}\,,\\ \widehat{\mathbf{e}}(0) = \widehat{\mathbf{E}}^0, \quad \mathbf{h}_3(0) = \mathbf{H}_3^0 & \text{a.e. in } \widehat{\Omega}\,,\\ \mathbf{h}_3 = \beta\,(\mathbf{e}_1\mathbf{n}_2 - \mathbf{e}_2\mathbf{n}_1) & \text{a.e. on } \mathbb{R}^+\times\partial\widehat{\Omega}\,. \end{cases}$$

The third components $(\mathbf{e}_3,\mathbf{p}_3)$ *satisfy in* $\mathbb{R}^+\times\widehat{\Omega}$ *the system*

$$(34)\quad \begin{cases} \partial_t(\mathbf{e}_3+\mathbf{p}_3) + \sigma\,\mathbf{e}_3 = 0 & \text{a.e. in } \mathbb{R}^+\times\widehat{\Omega}\,,\\ \partial_t^2\mathbf{p}_3 + a\,\partial_t\mathbf{p}_3 - \widehat{\Delta}\mathbf{p}_3 + \mathbf{p}_3\,\phi'(|\mathbf{p}_3|^2) = \mathbf{e}_3 & \text{a.e. in } \mathbb{R}^+\times\widehat{\Omega}\,,\\ \mathbf{e}_3(0) = \mathbf{E}_3^0, \quad \mathbf{p}_3(0) = \mathbf{P}_3^0, \quad \partial_t\mathbf{p}_3(0) = \mathbf{P}_3^1 & \text{a.e. in } \widehat{\Omega}\,,\\ \mathbf{p}_3 = 0 & \text{a.e. on } \mathbb{R}^+\times\partial\widehat{\Omega}\,. \end{cases}$$

with $\mathbf{h}_3,\mathbf{p}_3 \in L^\infty(\mathbb{R}^+;H^1(\widehat{\Omega}))$ *and* $\widehat{\mathbf{e}} \in L^\infty(\mathbb{R}^+;\mathcal{H}(\widehat{\operatorname{curl}},\widehat{\Omega}))$.

Let us comment the limit problems obtained. We notice that in both linear and nonlinear cases, the limit magnetic field is orthogonal to the cross section and the limit system is decoupled into two independent systems settled in the cross section $\widehat{\Omega}$. On the one hand, the first system consists of the Maxwell's equations satisfied by $(\widehat{\mathbf{e}},\mathbf{h}_3)$ (respectively $(\widehat{\mathbf{e}},\mathbf{h}_3,\widehat{\mathbf{p}})$) for the nonlinear (respectively linear) case. In this system, in comparison with the systems (11)–(12) and (11)–(13), there are additional terms in the Maxwell's equations representing the contribution of the boundary conditions. On the other hand, the second system describes the dynamic of the third components of the electric and polarization limit field. However, the effect of the boundary condition is not observed in this system.

3 – Uniform estimates and weak convergences

As we assumed that the initial data are independent of the variable x_3 and β^ν, ρ^ν are given by (22) then the initial energy defined in Theorem 1.1 and Theorem 1.2 satisfies $\mathcal{E}_0^\nu \le \nu\,C$ hence $\mathcal{E}^\nu(t) \le \nu\,C$ for some constant $C>0$ independent of ν. Consequently, setting $\theta^\nu = \big(\partial_2\mathbf{p}_3^\nu - \frac{1}{\nu}\,\partial_z\mathbf{p}_2^\nu\,,\ \frac{1}{\nu}\,\partial_z\mathbf{p}_1^\nu - \partial_1\mathbf{p}_3^\nu\big)$, the scaled solution associated to the solution to problem (11)–(13) or (11)–(12) satisfies the following uniform estimates

3.1. Uniform estimates

Lemma 3.1. *There exists a constant $C > 0$ independent of ν such that, if $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$ is the scaled solution associated with the global solution of (11)–(13) or (11)–(12), we have*

$$\left\{\begin{array}{l} \|\mathbf{e}^\nu\|_\infty^2 + \|\mathbf{h}^\nu\|_\infty^2 + \|\mathbf{p}^\nu\|_\infty^2 + \|\mathbf{e}^\nu\|^2 + \|\partial_t \mathbf{p}^\nu\|^2 \,\le\, C \\ \|\partial_t \mathbf{p}^\nu\|_\infty^2 + \|\partial_1 \mathbf{p}_2^\nu - \partial_2 \mathbf{p}_1^\nu\|_\infty + \|\theta^\nu\|_\infty^2 \,\le\, C \;. \end{array}\right. \tag{35}$$

Moreover we have

$$\left\{\begin{array}{l} \big\|(\mathbf{e}^\nu \times \mathbf{n})_{|z=0,1}\big\|^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} + \|\mathbf{e}^\nu \times \mathbf{n}\|_{L^2(\mathbb{R}^+;L^2(\partial\widehat{\Omega}\times(0,1)))} \,\le\, C \\ \big\|(\mathbf{h}^\nu \times \mathbf{n})_{|z=0,1}\big\|^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} \,\le\, \nu\, C \;. \end{array}\right. \tag{36}$$

and, for the linear case, we have

$$\left\{\begin{array}{l} \big\|(\mathbf{p}^\nu \times \mathbf{n})_{|z=0,1}\big\|^2_{L^\infty(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} + \|\mathbf{p}^\nu \times \mathbf{n}\|_{L^\infty(\mathbb{R}^+;L^2(\partial\widehat{\Omega}\times(0,1)))} \,\le\, C \\ \big\|(\partial_t \mathbf{p}^\nu \times \mathbf{n})_{|z=0,1}\big\|^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} + \|\partial_t \mathbf{p}^\nu \times \mathbf{n}\|_{L^2(\mathbb{R}^+;L^2(\partial\widehat{\Omega}\times(0,1)))} \,\le\, C \;. \end{array}\right. \tag{37}$$

Notice that from the boundary condition satisfied by $\mathbf{p}^\nu$ and the previous estimates, we deduce in the linear case, that we have

$$\big\|(\operatorname{curl}_\nu \mathbf{p}^\nu \times \mathbf{n})_{|z=0,1}\big\|_{L^2_{loc}(\mathbb{R}^+;\mathbb{L}^2(\widehat{\Omega}))} \,\le\, \nu\, C \;. \;\blacksquare \tag{38}$$

In a similar way, we get the following estimates for the time partial derivatives of the solution

Lemma 3.2. *There exists a constant $C > 0$ independent of ν such that, if $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$ is the scaled solution associated with the global solution of (11)–(13) or (11)–(12), we have*

$$\left\{\begin{array}{l} \|\partial_t \mathbf{e}^\nu\|_\infty^2 + \|\partial_t \mathbf{h}^\nu\|_\infty^2 + \|\partial_t^2 \mathbf{p}^\nu\|_\infty^2 + \|\operatorname{curl}_\nu \mathbf{h}^\nu\|_\infty^2 + \|\operatorname{curl}_\nu \partial_t \mathbf{p}^\nu\|_\infty^2 \,\le\, C \\ \|\operatorname{curl}_\nu \mathbf{p}^\nu\|_\infty^2 + \|\operatorname{curl}_\nu \mathbf{e}^\nu\|_\infty^2 + \|\operatorname{curl}_\nu^2 \mathbf{p}^\nu\|_\infty^2 + \|\operatorname{curl}_\nu^2 \partial_t \mathbf{p}^\nu\|_\infty^2 \,\le\, C \;. \;\blacksquare \end{array}\right. \tag{39}$$

3.2. Weak convergences

For a subsequence still denoted by $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$, where $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$ is either the solution to the linear problem (11)–(13) or to the nonlinear problem (11)–(12), the following convergences hold

$$
\left\{
\begin{array}{ll}
(\mathbf{h}^\nu,\, \mathbf{e}^\nu,\, \mathbf{p}^\nu) \rightharpoonup (\mathbf{h},\, \mathbf{e},\, \mathbf{p}) & \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \text{ weakly-}\star \\
(\mathbf{e}^\nu,\, \partial_t \mathbf{p}^\nu) \rightharpoonup (\mathbf{e},\, \partial_t \mathbf{p}) & \text{in } L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \text{ weakly} \\
\partial_t \mathbf{p}^\nu \rightharpoonup \partial_t \mathbf{p}\,, \quad (\theta^\nu,\, \widehat{\mathrm{curl}}\widehat{\mathbf{p}}^\nu) \rightharpoonup (\theta,\, \widehat{\mathrm{curl}}\widehat{\mathbf{p}}) & \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \text{ weakly-}\star \\
(\partial_t \mathbf{h}^\nu,\, \partial_t \mathbf{e}^\nu,\, \partial_t^2 \mathbf{p}^\nu) \rightharpoonup (\partial_t \mathbf{h},\, \partial_t \mathbf{e},\, \partial_t^2 \mathbf{p}) & \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \text{ weakly-}\star \\
\widehat{\mathrm{curl}}\, \partial_t \widehat{\mathbf{p}}^\nu \rightharpoonup \widehat{\mathrm{curl}}\, \partial_t \widehat{\mathbf{p}} & \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \text{ weakly-}\star
\end{array}
\right. \tag{40}
$$

where $\theta \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ is some function which will be identified later.

3.3. Convergence of the boundary terms

We denote by $\mathcal{H}(\mathrm{curl}_\nu, \Omega) = \{u \in \mathbb{L}^2(\Omega),\ \mathrm{curl}_\nu\, u \in \mathbb{L}^2(\Omega)\}$ and $\mathcal{H}(\widehat{\mathrm{curl}}, \Omega) = \{u \in \mathbb{L}^2(\Omega),\ \partial_1 u_2 - \partial_2 u_1 \in \mathbb{L}^2(\Omega)\}$. For $T > 0$ fixed, set $\Omega_T = (0,T)\times\Omega$ and $\widehat{\Omega}_T = (0,T)\times\widehat{\Omega}$. In order to pass to the limit in the boundary terms and to characterise their limit, we first establish the following result which will applied to $\mathbf{e}^\nu$, $\mathbf{h}^\nu$, $\mathbf{p}^\nu$ and $\mathrm{curl}_\nu\, \mathbf{p}^\nu$.

Proposition 3.1. *Let $\Omega = \widehat{\Omega}\times(0,1)$ be a bounded cylinder of $\mathbb{R}^3$ and let v^ν be a uniformly bounded sequence in $L^\infty(\mathbb{R}^+; \mathcal{H}(\mathrm{curl}_\nu; \Omega))$ such that the tangential trace $v^\nu\times\mathbf{n}$ is uniformly bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{L}^2(\partial\Omega))$ (or in $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\partial\Omega))$). Then $v^\nu \rightharpoonup v$ in $L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ weak-$\star$ such that*

$$
\left\{
\begin{array}{ll}
\widehat{v} \text{ is independent of } z, \quad \widehat{v} \in L^\infty(\mathbb{R}^+; \mathcal{H}(\widehat{\mathrm{curl}}, \widehat{\Omega})) & \\
(v^\nu\times\mathbf{n})_{|z=1} \rightharpoonup (v_2, -v_1, 0) & \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega})) \text{ weakly} \\
\big(\text{or} \quad (v^\nu\times\mathbf{n})_{|z=1} \rightharpoonup (v_2, -v_1, 0) & \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega})) \text{ weakly-}\star\big) \\
(v^\nu\times\mathbf{n})_{|z=0} \rightharpoonup (-v_2, v_1, 0) & \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega})) \text{ weakly} \\
\big(\text{or} \quad (v^\nu\times\mathbf{n})_{|z=0} \rightharpoonup (-v_2, v_1, 0) & \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega})) \text{ weakly-}\star) \\
\displaystyle\int_0^1 (v^\nu\times\mathbf{n}).\mathbf{u}_3\, dz \rightharpoonup v_1\mathbf{n}_2 - v_2\mathbf{n}_1 & \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{L}^2(\partial\widehat{\Omega})) \text{ weakly} \\
\big(\text{or} \quad \displaystyle\int_0^1 (v^\nu\times\mathbf{n}).\mathbf{u}_3\, dz \rightharpoonup v_1\mathbf{n}_2 - v_2\mathbf{n}_1 & \text{in } L^\infty(\mathbb{R}^+; \mathbb{L}^2(\partial\widehat{\Omega})) \text{ weakly-}\star)\,.
\end{array}
\right. \tag{41}
$$

Proof: We first prove that $\widehat{v}$ is independent of the variable z. Indeed, as $\mathrm{curl}_\nu\, v^\nu$ is bounded in $L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ then $\frac{1}{\nu}\,\partial_z v_i^\nu - \partial_i v_3^\nu$ converges weakly-$\star$ for $i=1,2$ and hence $\partial_z v_i^\nu - \nu\,\partial_i v_3^\nu \rightharpoonup 0$ in the sense of distributions. Since $\nu\,\partial_i v_3^\nu \rightharpoonup 0$ in the sense of distributions then $\partial_z v_i^\nu \rightharpoonup 0$ the sense of distributions and hence $\partial_z v_i = 0$ in the sense of distributions for $i=1,2$. Next, it derives from the fact that $v^\nu \rightharpoonup v$ and $\widehat{\mathrm{curl}}\, v^\nu \rightharpoonup \widehat{\mathrm{curl}}\, v$ in $L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ weak-$\star$ that $\widehat{v} \in L^\infty(\mathbb{R}^+;\mathcal{H}(\widehat{\mathrm{curl}},\widehat{\Omega}))$ which means that the tangential trace $v_1\mathbf{n}_2 - v_2\mathbf{n}_1$ is well defined in $L^\infty(\mathbb{R}^+;H^{-\frac{1}{2}}(\partial\widehat{\Omega}))$. Finally, the convergence of the traces can be deduced from the following Green's formula

$$
\begin{aligned}
\int_{\Omega_T} (\mathrm{curl}_\nu\, v^\nu)\cdot\varphi\; dx\,dt \;&= \int_{\Omega_T} (\mathrm{curl}_\nu\, \varphi)\cdot v^\nu\; dx\,dt \\
&\quad - \frac{1}{\nu}\int_{\widehat{\Omega}_T} (v^\nu\times\mathbf{n})_{|z=0,1}\cdot\varphi\; dt\,d\widehat{x} \\
&\quad - \int_{(0,T)\times\partial\widehat{\Omega}\times(0,1)} (v^\nu\times\mathbf{n})\cdot\varphi\; dt\,d\alpha\,dz
\end{aligned}
\tag{42}
$$

by using successively in (42) the test functions $\varphi(t,\widehat{x},z)=\big(\nu z\varphi_1(t,\widehat{x}),\nu z\varphi_2(t,\widehat{x}),0\big)$, $\varphi(t,\widehat{x},z)=\big(\nu(1-z)\,\varphi_1(t,\widehat{x}),\ \nu(1-z)\,\varphi_2(t,\widehat{x}),\ 0\big)$ and $\varphi(t,\widehat{x},z)=(0,0,\varphi_3(t,\widehat{x}))$ with $\varphi_i \in \mathcal{D}((0,T)\times\overline{\widehat{\Omega}})$ and by letting $\nu\to 0$. ∎

Corollary 3.1. *In both cases, the limit functions* $\widehat{\mathbf{e}}$, $\widehat{\mathbf{h}}$, $\widehat{\mathbf{p}}$, θ *are independent of the variable* z *and we have*

$$
\begin{cases}
(\mathbf{e}^\nu\times\mathbf{n})_{|z=0} \rightharpoonup (-\mathbf{e}_2,\mathbf{e}_1,0)\,, \quad (\mathbf{e}^\nu\times\mathbf{n})_{|z=1} \rightharpoonup (\mathbf{e}_2,-\mathbf{e}_1,0) & \text{in } L^2(\mathbb{R}^+,\mathbb{L}^2(\widehat{\Omega})) \\
\widehat{\mathbf{h}} \equiv 0 & \text{in } \mathbb{R}^+\times\Omega\ .
\end{cases}
\tag{43}
$$

Moreover, for the linear case, we have

$$
\begin{cases}
(\mathbf{p}^\nu\times\mathbf{n})_{|z=0} \rightharpoonup (-\mathbf{p}_2,\mathbf{p}_1,0) & \text{in } L^\infty(\mathbb{R}^+,\mathbb{L}^2(\widehat{\Omega})) \text{ weak-}\star \\
(\mathbf{p}^\nu\times\mathbf{n})_{|z=1} \rightharpoonup (\mathbf{p}_2,-\mathbf{p}_1,0) & \text{in } L^\infty(\mathbb{R}^+,\mathbb{L}^2(\widehat{\Omega})) \text{ weak-}\star \\
(\partial_t\mathbf{p}^\nu\times\mathbf{n})_{|z=0} \rightharpoonup (-\partial_t\mathbf{p}_2,\partial_t\mathbf{p}_1,0) & \text{in } L^2(\mathbb{R}^+,\mathbb{L}^2(\widehat{\Omega})) \text{ weakly} \\
(\partial_t\mathbf{p}^\nu\times\mathbf{n})_{|z=1} \rightharpoonup (\partial_t\mathbf{p}_2,-\partial_t\mathbf{p}_1,0) & \text{in } L^2(\mathbb{R}^+,\mathbb{L}^2(\widehat{\Omega})) \text{ weakly} \\
\theta \equiv 0 & \text{in } \mathbb{R}^+\times\Omega\ .
\end{cases}
\tag{44}
$$

Furthermore, for the nonlinear case, we have

$$\text{(45)}\qquad \begin{cases} \widehat{\mathbf{p}} \equiv 0\,, \quad \theta = \mathrm{Curl}\Big(\displaystyle\int_0^1 \mathbf{p}_3\,dz\Big) & \textit{in } \mathbb{R}^+{\times}\Omega\,,\\ \displaystyle\int_0^1 \mathbf{p}_3\,dz = 0 & \textit{on } \mathbb{R}^+{\times}\partial\widehat{\Omega}\,. \end{cases}$$

Proof: It is clear that the weak convergences and the independency with respect to the variable z can be deduced directly from the previous proposition. Next, in both cases, we have $\widehat{\mathbf{h}} \equiv 0$. Indeed, on the one hand, by virtue of the previous proposition $(\mathbf{h}^\nu{\times}\mathbf{n})_{|z=0} \rightharpoonup (-\mathbf{h}_2, \mathbf{h}_1, 0)$ weakly in $L^2(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega}))$. On the other hand, thanks to (36) we have $(\mathbf{h}^\nu{\times}\mathbf{n})_{|z=0} \to 0$ strongly in $L^2(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega}))$. Hence, we have $\widehat{\mathbf{h}} \equiv 0$. Moreover, using (38), we may proceed similarly to prove $\theta \equiv 0$ in the linear case. Furthermore, we get in a similar way $\widehat{\mathbf{p}} \equiv 0$ in the nonlinear case thanks to the boundary condition $\mathbf{p}^\nu{\times}\mathbf{n} = 0$. Finally, since we have $\mathbf{p}^\nu{\times}\mathbf{n} = 0$ in the nonlinear case, then using the Green's formula

$$\text{(46)}\qquad \int_{\mathbb{R}^+\times\Omega} (\mathrm{curl}_\nu\, \mathbf{p}^\nu)\cdot\varphi\; dx\,dt \;=\int_{\mathbb{R}^+\times\Omega} (\mathrm{curl}_\nu\, \varphi)\cdot \mathbf{p}^\nu\; dx\,dt$$

with the test function $\varphi(t,\widehat{x},z) = \big(\varphi_1(t,\widehat{x}),\, \varphi_2(t,\widehat{x}),\, 0\big)$, $\varphi_i(t,\widehat{x}) \in \mathcal{D}(\mathbb{R}^+{\times}\overline{\widehat{\Omega}})$ and letting $\nu\to 0$ we get $\theta = \mathrm{Curl}\Big(\int_0^1 \mathbf{p}_3\; dz\Big)$ and hence $\int_0^1 \mathbf{p}_3\; dz$ belongs to $L^\infty(\mathbb{R}^+; H^1(\widehat{\Omega}))$. Moreover using the Green's formula in $H^1(\widehat{\Omega})$ we get $\int_0^1 \mathbf{p}_3\; dz = 0$ on $\mathbb{R}^+{\times}\partial\widehat{\Omega}$. ∎

Lemma 3.3 (Initial Data). *The traces at $t=0$ of $\mathbf{h}$, $\mathbf{e}$, $\mathbf{p}$, $\partial_t\mathbf{p}$ make sense in $\mathbb{L}^2(\Omega)$ and we have*

$$\text{(47)}\qquad \mathbf{h}(0) = \mathbf{H}^0,\quad \mathbf{e}(0) = \mathbf{E}^0,\quad \mathbf{p}(0) = \mathbf{P}^0,\quad \partial_t\mathbf{p}(0) = \mathbf{P}^1 \qquad \textit{a.e. in } \Omega\,.$$

Proof: We will prove the lemma for $\mathbf{e}$, the proof is similar for the other functions. Thanks to Lemma 3.1 and Lemma 3.2, we have $\mathbf{e}^\nu$, $\partial_t\mathbf{e}^\nu$, $\mathbf{e}$, $\partial_t\mathbf{e} \in L^\infty(\mathbb{R}^+, \mathbb{L}^2(\Omega))$ and hence $\mathbf{e}^\nu, \mathbf{e} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{L}^2(\Omega))$. It follows that $\mathbf{e}^\nu, \mathbf{e} \in \mathcal{C}^0(\mathbb{R}^+, \mathbb{L}^2(\Omega))$ so $\mathbf{e}(0)$ makes sense in $\mathbb{L}^2(\Omega)$. Next, as $\mathbf{e}^\nu \in W^{1,\infty}(\mathbb{R}^+, \mathbb{L}^2(\Omega))$, we have

$$\text{(48)}\qquad \int_{\Omega_T} \partial_t(\mathbf{e}^\nu\cdot\varphi)\; dx\,dt \;= \int_{\Omega_T} (\partial_t\mathbf{e}^\nu)\cdot\varphi\; dx\,dt \;+ \int_{\Omega_T} \mathbf{e}^\nu\cdot\partial_t\varphi\; dx\,dt$$

then

$$(49) \qquad \int_\Omega \mathbf{e}^\nu(T)\cdot\varphi(T) - \mathbf{E}^0\cdot\varphi(0)\; dx \;=\; \int_{\Omega_T} (\partial_t \mathbf{e}^\nu)\cdot\varphi\; dx\,dt \;+\; \int_{\Omega_T} \mathbf{e}^\nu\cdot\partial_t\varphi\; dx\,dt\;.$$

Let $\phi \in (\mathcal{D}(\Omega))^3$, taking $\varphi(t,x) = \frac{t-T}{T}\,\phi(x)$ in the last equality and letting $\nu\to 0$ we get

$$(50) \qquad \int_\Omega \mathbf{E}^0\cdot\phi\; dx \;=\; \int_{\Omega_T} \partial_t(\mathbf{e}\cdot\varphi)\; dx\,dt \;=\; \int_\Omega \mathbf{e}(0)\cdot\phi\; dx\;, \qquad \forall\phi\in(\mathcal{D}(\Omega))^3$$

hence $\mathbf{e}(0) = \mathbf{E}^0$ a.e. in Ω. ∎

4 – Proof of Theorem 2.1

The scaled solution $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$, associated with the solution $(\mathbf{H}^\nu, \mathbf{E}^\nu, \mathbf{P}^\nu)$ to problem (11)–(13), satisfies in $\mathbb{R}^+\times\Omega$ the problem

$$(51) \qquad \begin{cases} \partial_t \mathbf{h}^\nu - \operatorname{curl}_\nu \mathbf{e}^\nu = 0\,, \quad \partial_t(\mathbf{e}^\nu + \mathbf{h}^\nu) + \operatorname{curl}_\nu \mathbf{h}^\nu + \sigma\,\mathbf{e}^\nu = 0 \\ \mathbf{h}^\nu(0) = \mathbf{H}^0(\widehat{x})\,, \quad \mathbf{e}^\nu(0) = \mathbf{E}^0(\widehat{x}) \\ \mathbf{h}^\nu\times\mathbf{n} + \beta^\nu\,\mathbf{n}\times(\mathbf{e}^\nu\times\mathbf{n}) = 0 \end{cases}$$

coupled to the polarization equation

$$(52) \qquad \begin{cases} \partial_t^2 \mathbf{p}^\nu + a\,\partial_t \mathbf{p}^\nu + \operatorname{curl}_\nu^2 \mathbf{p}^\nu + k\,\mathbf{p}^\nu \;=\; \mathbf{e}^\nu \\ \mathbf{p}^\nu(0) = \mathbf{P}^0(\widehat{x})\,, \quad \partial_t \mathbf{p}^\nu(0) = \mathbf{P}^1(\widehat{x}) \\ \operatorname{curl}_\nu \mathbf{p}^\nu\times\mathbf{n} + \rho^\nu\,\mathbf{n}\times\big((\partial_t + a)\mathbf{p}^\nu\times\mathbf{n}\big) \;=\; 0\;. \end{cases}$$

Setting $Q = \mathbb{R}^+\times\Omega$, the weak formulation of this problem writes as

$$(53) \qquad -\int_Q \mathbf{h}^\nu\cdot\partial_t\varphi\; dx\,dt \;-\; \int_Q \mathbf{e}^\nu\cdot\operatorname{curl}_\nu\varphi\; dx\,dt \;=\; 0$$

with

$$(54) \qquad \begin{cases} \displaystyle\int_Q -(\mathbf{e}^\nu + \mathbf{p}^\nu)\cdot\partial_t\eta\; dx\,dt \;+\; \int_Q \mathbf{h}^\nu\cdot\operatorname{curl}_\nu\eta\; dx\,dt \;+\; \sigma\int_Q \mathbf{e}^\nu\cdot\eta\; dx\,dt \;+ \\ \displaystyle +\;\beta\int_{\mathbb{R}^+\times\partial\widehat{\Omega}\times(0,1)} (\mathbf{e}^\nu\times\mathbf{n})\cdot(\eta\times\mathbf{n})\; dt\,d\alpha\,dz \\ \displaystyle +\;\beta_2\int_{\mathbb{R}^+\times\widehat{\Omega}} (\mathbf{e}^\nu\times\mathbf{n})_{|z=1}\cdot(\eta\times\mathbf{n})_{|z=1}\; dt\,d\widehat{x} \\ \displaystyle +\;\beta_1\int_{\mathbb{R}^+\times\widehat{\Omega}} (\mathbf{e}^\nu\times\mathbf{n})_{|z=0}\cdot(\eta\times\mathbf{n})_{|z=0}\; dt\,d\widehat{x} \;=\; 0 \end{cases}$$

for all regular test functions $\varphi \in \mathcal{D}\big(]0,\infty[\times\Omega\big)$, $\eta \in \mathcal{D}\big(]0,\infty[\times\overline{\Omega}\big)$. Here we used the boundary condition $\mathbf{h}^\nu\times\mathbf{n} + \beta^\nu\,\mathbf{n}\times(\mathbf{e}^\nu\times\mathbf{n}) = 0$. The polarization field $\mathbf{p}^\nu$ satisfies

$$(55)\quad \begin{cases} \displaystyle\int_Q (\partial_t^2 + a\partial_t + k)\mathbf{p}^\nu\cdot\psi\;\,dx\,dt \;+\; \int_Q (\theta^\nu, \widehat{\operatorname{curl}}\,\widehat{\mathbf{p}}^\nu)\cdot\operatorname{curl}_\nu\psi\;\,dx\,dt \;- \\ \displaystyle -\int_Q \mathbf{e}^\nu\cdot\psi\;\,dx\,dt \;+\; \rho\int_{\mathbb{R}^+\times\partial\widehat{\Omega}\times(0,1)} \big((\partial_t\mathbf{p}^\nu + a\,\mathbf{p}^\nu)\times\mathbf{n}\big)\cdot(\psi\times\mathbf{n})\;\,dt\,d\alpha\,dz \\ \displaystyle +\;\rho_2\int_{\mathbb{R}^+\times\widehat{\Omega}} \big((\partial_t\mathbf{p}^\nu + a\,\mathbf{p}^\nu)\times\mathbf{n}\big)_{|z=1}\cdot(\psi\times\mathbf{n})_{|z=1}\;\,dt\,d\widehat{x} \\ \displaystyle +\;\rho_1\int_{\mathbb{R}^+\times\widehat{\Omega}} \big((\partial_t\mathbf{p}^\nu + a\,\mathbf{p}^\nu)\times\mathbf{n}\big)_{|z=0}\cdot(\psi\times\mathbf{n})_{|z=0}\;\,dtd\,\widehat{x} \;=\; 0 \end{cases}$$

for all test function ψ defined in $\overline{Q}$. Here, we used the boundary condition $\operatorname{curl}_\nu\mathbf{p}^\nu\times\mathbf{n} + \rho^\nu\mathbf{n}\times((\partial_t\mathbf{p}^\nu + a\,\mathbf{p}^\nu)\times\mathbf{n}) = 0$. Before passing to the limit in the weak formulation, we first prove

Lemma 4.1. *For both cases, the functions* $\mathbf{e}_3$, $\mathbf{h}_3$, $\mathbf{p}_3$ *are independent of the variable* z.

Proof: We prove the lemma for the linear case. The proof in the nonlinear case is similar. The compatibility conditions for problem (51)–(52) (obtained by using $\operatorname{div}_\nu(\operatorname{curl}_\nu) = 0$) can be written in the sense of distributions as

$$(56)\quad \begin{cases} \partial_t\big(\widehat{\operatorname{div}}\,\widehat{\mathbf{h}^\nu} + \frac{1}{\nu}\,\partial_z\mathbf{h}_3^\nu\big) \;=\; 0\,, \\ \partial_t\big(\widehat{\operatorname{div}}\,(\widehat{\mathbf{e}}^\nu + \widehat{\mathbf{p}}^\nu) + \frac{1}{\nu}\,\partial_z(\mathbf{e}_3^\nu + \mathbf{p}_3^\nu)\big) + \sigma\,\big(\widehat{\operatorname{div}}\,\widehat{\mathbf{e}}^\nu + \frac{1}{\nu}\,\partial_z\mathbf{e}_3^\nu\big) \;=\; 0\,, \\ (\partial_t^2 + a\partial_t + k)\,\big(\widehat{\operatorname{div}}\,\widehat{\mathbf{p}}^\nu + \frac{1}{\nu}\,\partial_z\mathbf{p}_3^\nu\big) - \big(\widehat{\operatorname{div}}\,\widehat{\mathbf{e}^\nu} + \frac{1}{\nu}\,\partial_z\mathbf{e}_3^\nu\big) \;=\; 0\;. \end{cases}$$

Using the test function $\nu\phi$ and letting $\nu\to 0$, we get

$$(57)\quad \begin{cases} \partial_t(\partial_z\mathbf{h}_3) = 0\,, \\ \partial_t(\partial_z\mathbf{e}_3 + \partial_z\mathbf{p}_3) + \sigma\,\partial_z\mathbf{e}_3 = 0\,, \\ (\partial_t^2 + a\partial_t + k)\,\partial_z\mathbf{p}_3 - \partial_z\mathbf{e}_3 = 0 \end{cases}$$

in the sense of distributions. By virtue of Lemma 3.3 and the independency of the initial data with respect to the third variable, we have $\partial_z\mathbf{h}_3(0) = \partial_z\mathbf{e}_3(0) = \partial_z\mathbf{p}_3(0) = 0$ and $\partial_z\partial_t\mathbf{p}_3(0) = 0$. Consequently, we have $\partial_z\mathbf{h}_3 = \partial_z\mathbf{e}_3 = \partial_z\mathbf{p}_3 = 0$ in the sense of distributions and the lemma is proved. ∎

End of proof: To end the proof of Theorem 2.1, we can pass easily to the limit in the weak formulation (53)–(54)–(55) by using (40) and Corollary 3.1. ∎

5 – Proof of Theorem 2.2

Let $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$ be the scaled solution associated with the solution to problem (11)–(12). Then $(\mathbf{h}^\nu, \mathbf{e}^\nu, \mathbf{p}^\nu)$ satisfies (51) coupled to the polarisation equation

$$\tag{58} \begin{cases} \partial_t^2 \mathbf{p}^\nu + a\,\partial_t \mathbf{p}^\nu + \mathrm{curl}_\nu^2\, \mathbf{p}^\nu + \phi'(|\mathbf{p}^\nu|^2)\mathbf{p}^\nu = \mathbf{e}^\nu\,, & \mathbb{R}^+\times\Omega \\ \mathbf{p}^\nu(0) = \mathbf{P}^0(\widehat{x})\,, \quad \partial_t \mathbf{p}^\nu(0) = \mathbf{P}^1(\widehat{x})\,, & \Omega \\ \mathbf{p}^\nu\times\mathbf{n} = 0\,, & \mathbb{R}^+\times\partial\Omega\,. \end{cases}$$

The weak formulation is given by (53)–(54) and is coupled to

$$\tag{59} \begin{cases} \displaystyle\int_Q (\partial_t^2 \mathbf{p}^\nu + a\,\partial_t \mathbf{p}^\nu)\cdot\psi\; dx\,dt + \int_Q (\theta^\nu, \widehat{\mathrm{curl}}\,\widehat{\mathbf{p}}^\nu)\cdot \mathrm{curl}_\nu\,\psi\; dx\,dt - \int_Q \mathbf{e}^\nu\cdot\psi\; dx\,dt = \\ \displaystyle = -\int_Q \phi'(|\mathbf{p}^\nu|^2)\,\mathbf{p}^\nu\cdot\psi\; dx\,dt \end{cases}$$

for all test functions ψ satisfying the boundary condition $\psi\times\mathbf{n} = 0$. As stated in the introduction, the boundary condition $\mathbf{P}^\nu\times\mathbf{n} = 0$ ensures the $\mathbb{H}^1$ regularity of the polarization field and hence allows us to pass to the limit in the nonlinear equilibrium electric field. This is the aim of the following proposition

Proposition 5.1 (Space regularity)**.** *Assume that the open and bounded domain $\widehat{\Omega}$ is convex. We assume moreover that the data $(\mathbf{H}^0, \mathbf{E}^0, \mathbf{P}^0)$ satisfy (32) and are independent of the variable x_3. Then, for all $T>0$, there exists $C_T>0$ (which is independent of ν) such that for all $\nu>0$, the solution $(\mathbf{H}^\nu, \mathbf{E}^\nu, \mathbf{P}^\nu)$ to problem (11)–(12) satisfies for all $T>0$ the uniform bound*

$$\tag{60} \|\mathbf{P}^\nu\|^2_{L^\infty(0,T;\mathbb{H}^1(\Omega^\nu))} + \|\mathrm{div}\,\mathbf{H}^\nu\|^2_\infty + \|\mathrm{div}\,\mathbf{E}^\nu\|^2_\infty \le \nu\, C_T\,.$$

Proof: Let $(\mathbf{H}^\nu, \mathbf{E}^\nu, \mathbf{P}^\nu)$ be the solution to problem (11)–(12). With the same notations used in Theorem 1.2, taking into account the assumptions on the initial data, we have $\mathcal{E}_0^\nu \le \nu\, C$ where C is some constant independent of ν then $\mathcal{E}^\nu(t) \le \nu\, C$ for all $t\ge 0$ and consequently by virtue of (23)

$$\tag{61} \|\mathbf{P}^\nu\|^2_\infty + \|\mathrm{curl}\,\mathbf{P}^\nu\|^2_\infty + \|\mathbf{H}^\nu\|^2_\infty + \|\mathbf{E}^\nu\|^2_\infty + \|\partial_t \mathbf{P}^\nu\|^2_\infty \le \nu\, C\,, \qquad \forall t\ge 0$$

for some constant C independent of ν.

First, we will suppose that $\mathbf{P}^\nu$ is smooth and we set $W^\nu = (\mathrm{div}\,\mathbf{E}^\nu, \mathrm{div}\,\mathbf{P}^\nu, \partial_t(\mathrm{div}\,\mathbf{P}^\nu))$. The compatibility system satisfied by $(\mathbf{E}^\nu, \mathbf{P}^\nu)$ writes

$$\tag{62} \begin{cases} \dfrac{dW^\nu}{dt} + \mathcal{C}W^\nu = \mathcal{S}(\mathbf{P}^\nu) \\ W^\nu(0) = W_0 \end{cases}$$

with

$$\mathcal{C} = \begin{pmatrix} \sigma & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & a \end{pmatrix}, \qquad \mathcal{S}(\mathrm{P}^\nu) = \begin{pmatrix} 0 \\ 0 \\ -\operatorname{div}\big(\mathbf{P}^\nu \phi(|\mathbf{P}^\nu|^2)\big) \end{pmatrix}. \tag{63}$$

Notice that, as the data are independent of the variable x_3, we have

$$|W_0|^2 = \nu\, |W_0|^2_{\mathbb{L}^2(\widehat{\Omega})}\ . \tag{64}$$

Moreover, assuming that $\mathbf{P}^\nu$ is smooth, we get

$$\operatorname{div}\big(\mathbf{P}^\nu\, \phi'(|\mathbf{P}^\nu|^2)\big) \;=\; \phi'(|\mathbf{P}^\nu|^2)\ \operatorname{div}\mathbf{P}^\nu + 2\,\phi^{(2)}(|\mathbf{P}^\nu|^2) \sum_{k,l} \mathbf{P}^\nu_k\ \mathbf{P}^\nu_j\ \partial_k \mathbf{P}^\nu_j \tag{65}$$

then using hypothesis (8), there exists $C > 0$ which independent of ν, depending only of ϕ such that

$$|\mathcal{S}(\mathbf{P}^\nu)| \;\le\; C\,\big(|\nabla \mathbf{P}^\nu| + |\operatorname{div}\mathbf{P}^\nu|\big) \tag{66}$$

hence there exists $C > 0$ and $\delta > 0$ independent of ν, depending only of σ, a and ϕ such that for all $t \ge 0$

$$|W^\nu|^2(t) \;\le\; e^{\delta\, t}\left(|W_0|^2 \;+\; C\int_0^t |\nabla \mathbf{P}^\nu(s)|^2 + |\operatorname{div}\mathbf{P}^\nu|^2\ ds\right). \tag{67}$$

Next, as $\mathbf{P}^\nu \times \mathbf{n} = 0$ and Ω^ν is a convex cylinder (because we assumed that $\widehat{\Omega}$ is convex) then thanks to [5] or [4, lemma 2.17]

$$|\nabla \mathbf{P}^\nu(s)|^2 \;\le\; |\operatorname{curl}\mathbf{P}^\nu(s)|^2 + |\operatorname{div}\mathbf{P}^\nu(s)|^2\ , \qquad \forall s \ge 0 \tag{68}$$

consequently, by virtue of (67), (68), (61), (64), for fixed $T > 0$ there exists $C_T > 0$ independent of ν such that for all $t \in [0, T]$

$$|W^\nu|^2(t) \;\le\; \nu\, C_T + C_T \int_0^t |\operatorname{div}\mathbf{P}^\nu(s)|^2\ ds \;\le\; \nu\, C_T + C_T \int_0^t |W^\nu(s)|^2\ ds \tag{69}$$

then thanks to the Gronwall's lemma, we deduce that for all $T > 0$, there exists $C_T > 0$ independent of ν such that for all $t \in [0, T]$

$$|\operatorname{div}\mathbf{P}^\nu(t)|^2 + |\operatorname{div}\partial_t \mathbf{P}^\nu(t)|^2 + |\operatorname{div}\mathbf{E}^\nu(t)|^2 \;\le\; \nu\, C_T\ . \tag{70}$$

This implies, by using (68), (61) that $\|\mathbf{P}^\nu\|_{L^\infty(0,T;\mathbb{H}^1(\Omega^\nu))} \le \nu\, C_T$ with some constant C_T independent of ν. To end the proof of the proposition, we may justify the previous formal calculus by regularizing the system (11)–(12) as in [3] or [10] by replacing the potential $\mathbf{P}^\nu\phi'(|\mathbf{P}^\nu|^2)$ by $(\mathbf{P}^\nu\star\rho^\varepsilon)\,\phi'(|\mathbf{P}^\nu\star\rho^\varepsilon|^2)$ and by passing to the limit as $\varepsilon\to 0$ where ρ^ε is a regularizing sequence with unit mass. ∎

It follows that

Corollary 5.1. *For every $T>0$, there exists $C_T>0$ independent of ν such that*

$$\|\mathbf{p}^\nu\|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} \le C_T\ . \quad (71)$$ ∎

End of proof: Thanks to Lemma 3.1, the previous corollary and Aubin's compacity theorem, then for a subsequence we have, $\mathbf{p}^\nu\to\mathbf{p}$ in $L^\infty(0,T;\mathbb{L}^2(\Omega))$. Hence by using (10) we have $\mathbf{p}^\nu\phi'(|\mathbf{p}^\nu|^2)\to\mathbf{p}\phi'(|\mathbf{p}|^2)$ in $L^\infty(0,T;\mathbb{L}^2(\Omega))$. Since $\widehat{\mathbf{p}}=0$ and $\mathbf{p}_3$ is independent of the variable z then we use test functions of the form $\psi=(0,0,\psi_3(t,\widehat{x}))$ where $\psi_3\in\mathcal{D}((0,T)\times\widehat{\Omega})$. Then previous strong convergence and (40) allow us to pass to the limit in (59) and we get the result stated in Theorem 2.1 by using (45) and the independence of $\mathbf{p}_3$ with respect to the variable z. ∎

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