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Comparing TL-Moments, L-Moments and Conventional Moments of Dagum Distribution by Simulated data

Comparación de momentos TL, momentos L y momentos convencionales de la distribución Dagum mediante datos simulados

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Abstract

Modeling income, wage, wealth, expenditure and various other social variables have always been an issue of great concern. The Dagum distribution is considered quite handy to model such type of variables. Our focus in this study is to derive the L-moments and TL-moments of this distribution in closed form. Using L & TL-moments estimators we estimate the scale parameter which represents the inequality of the income distribution from the mean income. Comparing L-moments, TL-moments and conventional moments, we observe that the TL-moment estimator has lessbias and root mean square errors than those of L and conventional estimators considered in this study. We also find that the TL-moments have smaller root mean square errors for the coefficients of variation, skewness and kurtosis. These results hold for all sample sizes we have considered in our Monte Carlo simulation study.

***Key words*:** Dagum distribution, L-moments, Method of moments, Parameter estimation, TL-moments.

Resumen

La modelación de ingresos, salarios, riqueza, gastos y muchas otras variables de tipo social han sido siempre un tema de gran interés. La distribución Dagum es considerada para modelar este tipo de variables. Nos centraremos en este artículo en la derivación de los momentos L y los momentos TL de esta distribución de manera cerrada. Mediante el uso de los estimadores de momentos L y TL, estimamos el parámetro de escala que representa la desigualdad de la distribución de ingresos a partir de la media. Comparando los

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momentos L, los momentos TL y los momentos convencionales, concluimos que los momentos TL tienen menor sesgo y errores cuadráticos medios. También concluimos que los momentos TL tiene la menor error cuadrático medio para los coeficientes de variación, sesgo y curtosis. Estas conclusiones son igualmente aplicables para todos los tamaños de muestras considerados en nuestro estudio de simulación de Monte Carlo.

***Palabras clave*:** distribución Dagum, estimacón de parámetros, momentos TL, momentos L, método de momentos.

1. Introduction

Dagum (1977*a*, 1977*b*) studied the income, wage and wealth distribution using the Dagum Distributions. Dagum Distribution (DD) belongs to the family of Beta distributions. Kleiber (1996) showed that this family models income distribution at the univariate level. Dagum (1690) considered DD to model income data of several countries and found that it provides superior fit over the whole range of data. Perez & Alaiz (2011) studied personal income data of Spain using DD and found this model to be adequate. Quintano & Dagostino (2006) analyzed the single-person household income distribution for four European countries, and concluded that DD provide a better fit for all four countries. Bandourian, McDonald & Turley (2003) showed that DD provide the best fit in the case of two or three parameter distributions for data from 23 countries. Various other studies also support the use of DD as model for income data.

Identifying the pattern of income distribution is very important because the trend provides a guide for the assessment of living standards and level of income inequality in the population of a country. Recently, there has been an increasing interest in the exploration of parametric models for income distribution and DD has proved to be quite useful in modeling such data. But this distribution has yet not been studied and estimated assuming the L-moment and TL moment. It is has been demonstrated that L & TL- moments provide accurate fit and more exact parameter estimation compared to the other techniques. The method of L-moments and TL-moments were introduced Hosking (1990) and Elamir & Seheult (2003), respectively. TL-moments have some merits over L-moments because the former can be calculated, even if mean data does not exist.

This paper seeks to derive the first four L & TL-moments of DD and coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) estimators. To our knowledge, these moments for DD has not been derived and evaluated. We estimate the scale parameter of DD assuming L & TL-moments estimators and compare these with conventional moments. To achieve this objective, we measure the biasedness and RMSEs to recommend an efficient method of estimation. We also estimate the CV, CS & CK with the central, L & TL-moments estimators. e set up a Monte Carlo simulation study assuming different sample sizes and parametric values.

TL-moment estimators (TLMEs), L-moments estimators (LMEs) are derived and compared with the conventional method of moment estimators (MMEs) for

DD. The rest of the study is organized as follows: Section two is about the introduction of the population and sample TL-moments and L-moments. In Section 3, probability density function (pdf), distribution function, conventional moments and some other details of DD are presented. The derivations of the first four L & TL-moments is given in Section 4 and the coefficients are also presented. In Section 5, we setup the Monte Carlo simulation study to compare the properties of the TLMEs, LMEs and MMEs of DD. Finally we conclude our study in the final section.

2. L-Moment and TL-Moments

Hosking (1990) introduced L-moments and showed that these moments provide superior fit, parameter estimation, hypothesis testing and empirical description of data. Bílková (2012) used the L-moment of lognormal distribution to model the income distribution data of the Czech Republic in 1992–2007 and obtained consistent results as compared to the other methods of estimation. Due to the advantages of L-moments over the convention moments, many distributions are analyzed by these moments. Linear combinations of the ordered data values are used to compute L-moments. Furthermore, these moments are less sensitive in the case of outlier (Vogel & Fennessey 1993). Hosking (1990) defined the rth population L-moments (λ_r)as the linear combinations of probability weighted moments of an ordered sample data($\mathrm{Y}_{1:n} \leq \mathrm{Y}_{2:n} \leq ... \leq \mathrm{Y}_{n:n}$), that is

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} E\left(Y_{r-k:r}\right), \qquad r = 1, 2, 3, \ldots \tag{1}$$

For the real-numbered random variable Y with cumulative distribution function (cdf) $F(y)$; let $y(F)$ denote the quantile function of the ordered statistics of the sample of size n; then $E(Y_{j:r})$is given by

$$\begin{aligned} E(Y_{j:r}) &= \frac{r!}{(1-r)!(r-j)!}\int_0^1 y(F)^{j-1}(1-F)^{r-j}dF \\ = \frac{r!}{(1-r)!(r-j)!} &\int_{-\infty}^{\infty} yf(y)\left[F(y)\right]^{j-1}\left[1-F(y)\right]^{r-j} dy;\; j = 1, 2, \ldots, n \end{aligned} \tag{2}$$

The first and second L-moments (λ_1, λ_2)are equal to the measure of location and dispersion respectively. The ratio of the third L-moment (λ_3) to the second L-moment and ratio of the fourth L-moments (λ_4)to the second L-moment are the measure of skewness $\tau_{cs}^L = \lambda_3/\lambda_2$ and kurtosis, $\tau_{ck}^L = \lambda_4/\lambda_2$ respectively. The sample L-moments (l_r)are $l_1 = d_0$, $l_2 = 2d_1 - d_0$, $l_3 = 6d_2 - 6d_1 + d_0$and$l_4 = 20d_3 - 30d_2 + 12d_1 - d_0$and the sample L-skewness and L-kurtosis are $t_{cs}^L = l_3/l_2,\ t_{ck}^L = l_4/l_2$respectively. These ratios are less biased than for the conventional moments in estimation. The above mentioned d_r ($r = 1, 2, 3, 4$) are given by

$$d_r = \frac{1}{n}\sum_{j=r+1}^{n}\frac{(j-1)(j-2)\cdots(j-r)}{(n-1)(n-2)\cdots(n-r)}y_{j:n} \tag{3}$$

where the size of data is n.

Elamir & Seheult (2003) introduced the TL-moments. TL-moments does not have the assumption of the existence of the mean. The TL-moments for the Cauchy distribution are derived by Shabri, Ahmad & Zakaria (2011), even though the mean of this distribution does not exist. According to Elamir & Seheult (2003), the r^{th} TL-moments and sample TL-moment are given by

$$\lambda_r^{(t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left(Y_{r+t-k:r+2t}\right), \qquad \begin{array}{l} r = 1,2,3,\ldots \\ t = 1,2,3,\ldots \end{array} \tag{4}$$

and

$$l_r^{(t)} = \frac{1}{r}\sum_{j=t+1}^{n-t}\left[\sum_{k=0}^{r-1}(-1)^k C\right] Y_{j:n} \tag{5}$$

where $C = \binom{r-1}{k}\binom{j-1}{r+t-1-k}\binom{n-j}{t+k}\Big/\binom{n}{r+2t}$ respectively. The sample TL-skewness and TL-kurtosis are defined as $t_{cs}^{(t)} = l_3^{(t)}\Big/l_2^{(t)}$ and $t_{ck}^{(t)} = l_4^{(t)}\Big/l_2^{(t)}$, respectively.

3. Dagum Distribution

The Dagum distribution is a special case of the Generalized Beta type-II distribution ($DD(a,b,p) = GB2(a,b,p,1)$)as mentioned by Kleiber (1996). It is often used to model wage, wealth and income data. It was introduced by Dagum (1977*b*). The pdf of the distribution is given by

$$f(y) = \frac{ap\,(y)^{ap-1}}{b^{ap}\left[1+(y/b)^a\right]^{p+1}}, \tag{6}$$

where $p > 0$ and $a > 0$ are the shape parameters and $b > 0$ is the scale parameter. The cdf and r^{th} moment about zero are given by $F(y) = \left[(y/b)^{-a}+1\right]^{-p}$ and $E(Y^r) = b^r\Gamma(p+r/a)\Gamma(1-r/a)/\Gamma p$ respectively. The three-parameter DD provides a flexible distribution (Dagum & Lemmi 1988), and has better performace than other commonly used models (Kleiber 1996).

To evaluate the best method of estimation among the considered methods, we used the criteria of bias and RMSE. Bias is the expected difference between estimated and true value of the parameter. According to Daud, Kassim, Desa & Nguyen (2002) the $RMSE = \left[\sum_{i=1}^{n}\frac{(y_i-\hat{y}_i)}{n-m}\right]^{1/2}$, where y_i is actual observations, $\hat{y}_i$ is the estimated value obtained from the fitted distribution, $n-m$ is the difference between the number of observations in the sample and the number of parameters being estimated.

4. L-Moments and TL-Moments for the DD

As mentioned earlier, to best of our knowledge, there is no derivation of the L & TL-moments for DD in the literature. In this section, we derive L & TL-moments of DD using a general rule. The derivation of the L & TL-moments is given in the following Subsections 4.1-4.3.

4.1. L-moments of DD

Let $Y_{1:n} \leq Y_{2:n} \leq Y_{3:n} \leq \cdots \leq Y_{n:n}$ denote the order statistics from DD. The expected value of the r^{th} order statistics $Y_{r:n}$ is

$$E(Y_{r:n}) = \frac{n!ap}{(r-1)!(n-r)} \int (y_{r:n}/b)^{ab} \left[(y_{r:n}/b)^{-a}+1\right]^{-pr-1} \times \left[1-\left((y_{r:n}/b)^{-a}+1\right)^{-p}\right]^{n-r} dy_{r:n} \tag{7}$$

where $y(F) = b\left(F^{-1/p}-1\right)^{-1/a}$ is the quantile function of the DD. Now using the general form of L-moments, we have the first four L-moments for DD as follows

$$\lambda_1 = E(Y) = b\Gamma(1-\alpha)G_1 \tag{8}$$

$$\lambda_2 = b\Gamma(1-\alpha)\left(-G_1+G_2\right) \tag{9}$$

$$\lambda_3 = b\Gamma(1-\alpha)\left(G_1-3G_2+2G_3\right) \tag{10}$$

$$\lambda_4 = b\Gamma(1-\alpha)\left(-G_1+6G_2-10G_3+5G_4\right) \tag{11}$$

where $G_i = \Gamma(ip+\alpha)/\Gamma ip$; $i = 1,2,3,4,5$.

Equating the population L-moments with sample L-moments and after simplification, we get the following results that could be used for the parameter estimation of DD

$$l_1 = bp \times Beta\left(1-\alpha, p+\alpha\right) \tag{12}$$

$$l_2 = -l_1 + 2bp \times Beta\left(1-\alpha, 2p+\alpha\right) \tag{13}$$

$$l_3 = -2l_1 - 3l_2 + 6bp \times Beta\left(1-\alpha, 3p+\alpha\right) \tag{14}$$

$$l_4 = -15l_1 - 24l_2 - 10l_3 + 20bp \times Beta\left(1-\alpha, 4p+\alpha\right) \tag{15}$$

where 'Beta' is the beta function $(Beta(\theta_1,\theta_2) = \Gamma\theta_1\Gamma\theta_2/\Gamma(\theta_1+\theta_2))$.

4.2. TL-moments of DD

L-moments are the foundation of TL-moments. TL-moments are more robust than L-moments (Elamir and Seheult, 2003) because they trim the extreme values on the data. The close from of the first four TL-moments are

$$\lambda_1^{(t)} = b\Gamma(1-\alpha)\left(3G_2-2G_3\right) \tag{16}$$

$$\lambda_2^{(t)} = b\Gamma(1-\alpha)\left(3G_2+6G_3-3G_4\right) \tag{17}$$

$$\lambda_3^{(t)} = (10b\Gamma(1-\alpha)/3)\left(G_2-4G_3+5G_4-2G_5\right) \tag{18}$$

$$\lambda_4^{(t)} = 15b\Gamma(1-\alpha)\left(-G_2/4+5G_3/3-15G_4/4+7G_5/2-7G_6/6\right) \tag{19}$$

4.3. L & TL Coefficient of Variation, Skewness and Kurtosis

The population coefficient of variation $\left(\tau_{cv}^L\right)$lies between 0 and 1, τ_{cs}^L also has the range 0 and 1, andτ_{ck}^Lmeasure the peakness of any distribution, lies within the range of $\left(5\left(\tau_{cs}^L\right)^2-1\right)/4 \leq \tau_{ck}^L < 1$ according to Hosking (1990) The $\tau_{cv}^L, \tau_{cs}^L$and $\tau_{ck}^L$of DD are expressed as follows:

$$\tau_{cv}^L = \frac{G_2}{G_1} - 1 \tag{20}$$

$$\tau_{cs}^L = \frac{G_1-3G_2+2G_3}{-G_1+G_2} \tag{21}$$

$$\tau_{ck}^L = \frac{-G_1+6G_2-10G_3+5G_4}{-G_1+G_2} \tag{22}$$

The population TL-moments CV, CS and CK are represented with the notation $\tau_{cv}^{(t)}, \tau_{cs}^{(t)}$and $\tau_{ck}^{(t)}$of DD and expressed as follows:

$$\tau_{cv}^{(t)} = \frac{3G_2+6G_3-3G_4}{3G_2-2G_3} \tag{23}$$

$$\tau_{cs}^{(t)} = \frac{10\left(G_2-4G_3+5G_4-2G_5\right)}{3\left(3G_2+6G_3-3G_4\right)} \tag{24}$$

$$\tau_{ck}^{(t)} = \frac{5\left(-G_2+5G_3-15G_4+7G_5-7G_6\right)}{\left(G_2+2G_3-G_4\right)} \tag{25}$$

5. Monte Carlo Simulation Study

In this section, we use Monte Carlo simulated experiments to compare the three methods of moment estimators, conventional, L & TL-moments estimators of DD. This comparison is based on a measure of biasedness, root mean square estimators (RMSEs), sample CV, sample CS and sample CK. We use MATLAB-7 software to conduct our experiment. We perform our experiments for various sample sizes (15, 30, 50, 100, 500 and 1,000) as well as for different values of parameters. We have repeated each of our experiment 10,000 times. We use same parametric values for DD as were used by Ye, Oluyede & Pararai (2012).

In each case, for the estimation of b (scale parameter), we equate the sample moments to the corresponding population moments, and finally get the biasness and RMSEs of the b assuming the MMEs, LMEs & TLMEs of DD. Graphical shapes of the distribution on the bases of these parameters are given in Figure 1.

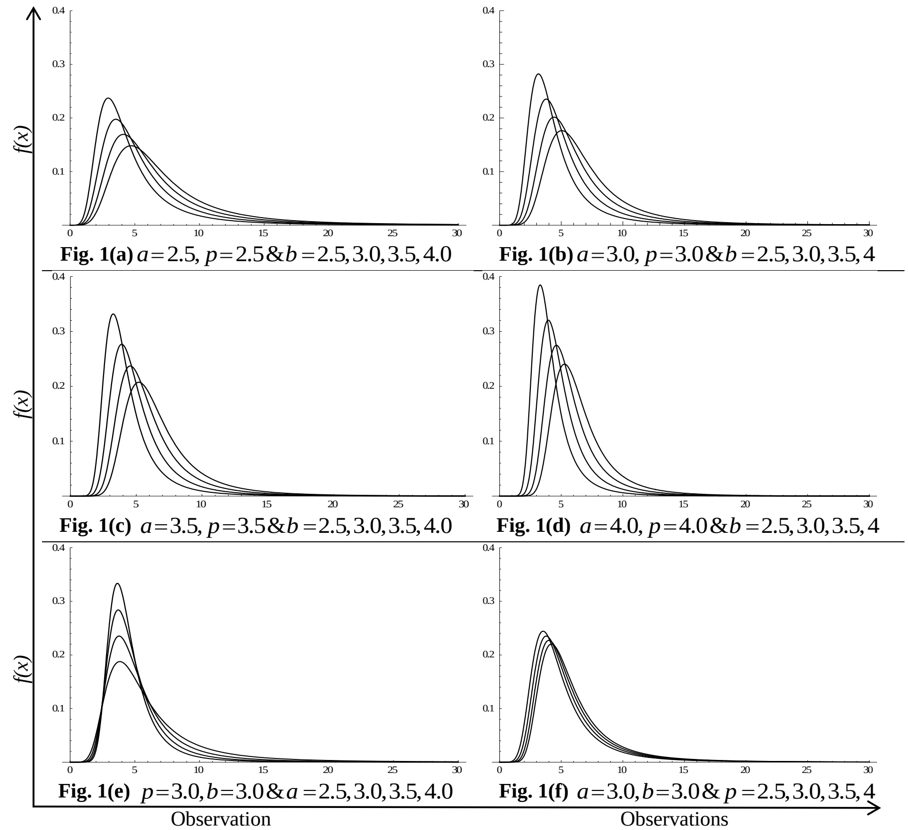

FIGURE 1: Dagum Distribution trend with different values of the parameters.

The results are presented in the Table 1 - 5. We find that the method of moments estimator (MME) gave biased results of the scale parameter with higher RMSEs. The L-moments estimator (LME) gave biased results with lower RMSEs than MME, and the TLMEs has a smaller bias with respect to the scale parameter and with lowest RMSEs. TLME results are very close to the true parametric values. So the TL-moments provide an unbiased estimator. According to the RMSEs, we can define the relation of these three moments estimators asTLME $<$ LME $<$ MME. These results hold for all the sample sizes we have considered. Therefore, the TL-moments provide precise and accurate estimates of the scale parameters of DD. If we do not want to trim the extreme values then L-moments provide better results.

The mean, L-moment standard deviation (LSD), $\tau_{\text{cv}}^{\text{L}}$, $\tau_{\text{cs}}^{\text{L}}$and $\tau_{\text{ck}}^{\text{L}}$ are computed using equations (21), (22) and (23) respectively. TL-moment standard deviation (TLSD),$\tau_{\text{cv}}^{\text{TL}}$, $\tau_{\text{cs}}^{\text{TL}}$and $\tau_{\text{ck}}^{\text{TL}}$ are also computed using equations (24), (25) and (26) respectively. These results are presented in Table A.6, assuming the same parametric values as those used for the scale parameter estimation. We observe that L & TL-moments coefficients are in the defined range and TL-moments coefficients have a relatively smaller value than the conventional and L-moments ones. We also observe that the shape parameters (a and p)make some effect on the coefficients value but for different values of scale parameters (b) coefficients remain constant. Finally, we sum up all the above description in the favour of TL-moments for DD.

6. Conclusion

We have derived L and TL-moments for DD, compared parameter estimates and descriptive statistics with the conventional methods of moment estimates, assuming different parametric values for small to large samples. For parameter estimation, we found TL-moments provide unbiased and efficient results compared to the remaining moments because it is more robust against outliers. L-moments also provide more or less unbiased results and is more efficient than conventional moments. In distribution fitting, according to the location, scale, RMSE, skewness and kurtosis, TL-moments are better for DD parameter estimation. We find that TL moments estimators are the best, and L-moments are better than conventional moments for untrimmed data. These results hold for all sample sizes and parametric values which we have considered in our study.

TABLE 1: Biases and RMSEs of the parameter estimations for different types of estimators assuming DD for b when $b = 2.5$

Parameters			**n = 50**			**n = 100**		
$\boldsymbol{a}$	$\boldsymbol{p}$		**MME**	**LME**	**TLME**	**MME**	**LME**	**TLME**
2.5	2.5	Bias	-0.0645	-0.0002	-0.0016	-0.0370	-0.0004	0.0003
		RMSEs	0.4567	0.3497	0.3442	0.3268	0.2487	0.2430
	3.5	Bias	-0.0494	0.0016	0.0032	-0.0318	-0.0043	-0.0045
		RMSEs	0.3571	0.3069	0.3264	0.2543	0.2141	0.2255
	5.0	Bias	-0.0034	-0.0222	0.0034	-0.0222	0.0004	0.001
		RMSEs	0.3182	0.2893	0.3191	0.2222	0.1020	0.2204
3.5	2.5	Bias	-0.0535	-0.0050	-0.0035	-0.0226	0.0021	0.0017
		RMSEs	0.3422	0.2991	0.3215	0.2460	0.2123	0.2123
	3.5	Bias	-0.0427	-0.0014	-0.0036	-0.0215	-0.0002	-0.0010
		RMSEs	0.2933	0.2741	0.3066	0.2086	0.1943	0.2164
	5.0	Bias	-0.0417	-0.0011	0.0011	-0.0196	0.0002	0.0012
		RMSEs	0.2685	0.2634	0.3041	0.1899	0.1846	0.2118
5.0	2.5	Bias	-0.0444	-0.0018	-0.0021	-0.0209	0.0002	-0.0002
		RMSEs	0.2971	0.2815	0.3109	0.2067	0.1944	0.2145
	3.5	Bias	-0.0422	-0.0033	-0.0045	-0.0208	-0.0010	-0.0006
		RMSEs	0.2680	0.2654	0.3074	0.1885	0.1860	0.2128
	5.0	Bias	-0.0325	0.0052	0.0033	-0.0213	-0.0022	-0.0020
		RMSEs	0.2566	0.2594	0.3029	0.1810	0.1816	0.2103
Parameters			**n = 500**			**n = 1,000**		
$\boldsymbol{a}$	$\boldsymbol{p}$		**MME**	**LME**	**TLME**	**MME**	**LME**	**TLME**
2.5	2.5	Bias	-0.0051	0.0009	0.00003	-0.0039	-0.0002	-0.0004
		RMSEs	0.1612	0.1107	0.1069	0.0754	0.0783	0.0754
	3.5	Bias	-0.0058	-0.0003	0.0001	-0.0040	-0.0011	-0.0010
		RMSEs	0.1202	0.0983	0.1025	0.0843	0.0687	0.0717
	5.0	Bias	-0.0045	-.00001	-.00005	-0.0027	-0.0003	-0.0002
		RMSEs	0.1023	0.0908	0.0978	0.0712	0.0631	0.0684
3.5	2.5	Bias	-0.0056	-0.0004	0.00001	-0.0027	-0.0002	0.0001
		RMSEs	0.1138	0.0953	0.1004	0.0815	0.0670	0.0699
	3.5	Bias	-0.0043	0.0002	0.0009	-0.0024	-0.0005	-0.0010
		RMSEs	0.0958	0.0876	0.0958	0.0668	0.0612	0.0673
	5.0	Bias	-0.0044	-0.0006	-0.0009	-0.0026	-0.0007	-0.0008
		RMSEs	0.0836	0.0813	0.0939	0.0602	0.0582	0.0665
5.0	2.5	Bias	-0.0050	-0.0006	-0.0001	-0.0038	-0.0017	-0.0018
		RMSEs	0.0945	0.088	0.0974	0.0665	0.0620	0.0679
	3.5	Bias	-0.0043	-0.0003	-0.0002	-0.0019	0.0001	0.0007
		RMSEs	0.0844	0.0830	0.0943	0.0599	0.0588	0.0667
	5.0	Bias	-0.0040	-0.0002	0.0002	-0.0029	-0.0011	-0.0014
		RMSEs	0.0801	0.0807	0.0934	0.0568	0.0573	0.0665

Table 2: Biases and RMSEs of the parameter estimations for different types of estimators assuming DD for b when $b = 3.5$

Parameters			**n = 50**			**n = 100**		
a	***p***		**MME**	**LME**	**TLME**	**MME**	**LME**	**TLME**
2.5	2.5	Bias	-0.0896	0.0022	0.0046	-0.0479	0.0012	0.0024
		RMSEs	0.6254	0.4854	0.4810	0.4653	0.3447	0.3347
	3.5	Bias	-0.0701	0.0016	0.0017	-0.0409	-0.0048	-0.0057
		RMSEs	0.5065	0.4349	0.4584	0.3680	0.3074	0.3200
	5.0	Bias	-0.0620	-0.0011	-0.0042	-0.0382	-0.0059	-0.0052
		RMSEs	0.4364	0.3964	0.4377	0.3103	0.2805	0.3099
3.5	2.5	Bias	-0.0686	-0.0010	-0.0019	-0.0319	0.0014	-0.0019
		RMSEs	0.4849	0.4228	0.4483	0.3482	0.2971	0.3125
	3.5	Bias	-0.0689	-0.0094	-0.0102	-0.0349	-0.0041	-0.0032
		RMSEs	0.4114	0.3858	0.4313	0.2870	0.2689	0.3014
	5.0	Bias	-0.0614	-0.0056	-0.0036	-0.0290	-0.0011	0.0003
		RMSEs	0.3760	0.3678	0.4258	0.2654	0.2586	0.2967
5.0	2.5	Bias	-0.0596	-0.0010	-0.0031	-0.0276	0.0016	0.0015
		RMSEs	0.4127	0.3923	0.4346	0.2915	0.2739	0.3046
	3.5	Bias	-0.0504	0.0027	-0.0022	-0.0265	0.0003	-0.0016
		RMSEs	0.3723	0.3686	0.4228	0.2635	0.2596	0.2963
	5.0	Bias	-0.0540	-0.0005	-0.0001	-0.0232	0.0031	0.0007
		RMSEs	0.3608	0.3641	0.4256	0.2497	0.2522	0.2928
Parameters			**n = 500**			**n = 1,000**		
a	***p***		**MME**	**LME**	**TLME**	**MME**	**LME**	**TLME**
2.5	2.5	Bias	-0.0085	-.00003	-0.0012	-0.0052	0.0003	0.0004
		RMSEs	0.2297	0.1563	0.1494	0.1649	0.1113	0.1064
	3.5	Bias	-0.0045	0.0023	0.0020	-0.0047	-0.0002	-0.0001
		RMSEs	0.1693	0.1379	0.1431	0.1186	0.0973	0.1019
	5.0	Bias	-0.0061	-.00004	-0.0002	-0.0035	-0.0004	-0.0001
		RMSEs	0.1403	0.1248	0.1362	0.0999	0.0885	0.0964
3.5	2.5	Bias	-0.0062	0.0004	-.00004	-0.0038	-0.0007	-0.0009
		RMSEs	0.1578	0.1316	0.1391	0.1126	0.0926	0.0971
	3.5	Bias	-0.0066	-0.0010	-0.0020	-0.0039	-0.0007	-0.0002
		RMSEs	0.1327	0.1220	0.1343	0.0937	0.0861	0.0948
	5.0	Bias	-0.0053	0.0001	0.0003	-0.0045	-0.0018	-0.0020
		RMSEs	0.1182	0.1147	0.1313	0.0843	0.0816	0.0927
5.0	2.5	Bias	-0.0069	-0.0005	.00006	-0.0027	0.0003	0.0006
		RMSEs	0.1316	0.1221	0.1333	0.0927	0.0862	0.0948
	3.5	Bias	-0.0034	0.0017	0.0015	-0.0022	0.0005	0.0005
		RMSEs	0.1152	0.1151	0.1312	0.0847	0.0830	0.0937
	5.0	Bias	-0.0060	-0.0006	-0.0003	-0.0031	-0.0004	-0.0001
		RMSEs	0.1116	0.1122	0.1303	0.0792	0.0797	0.0925

Table 3: Biases and RMSEs of the parameter estimations for different types of estimators assuming DD for b when $b = 5$

Parameters			n = 50			n = 100		
a	p		MME	LME	TLME	MME	LME	TLME
2.5	2.5	Bias	-0.1247	0.0074	0.0054	-0.0795	-0.0035	-0.0001
		RMSEs	0.9049	0.7027	0.6920	0.6469	0.4887	0.4805
	3.5	Bias	-0.0996	-0.0019	-0.0076	-0.0528	-0.0013	-0.0007
		RMSEs	0.7260	0.6181	0.6502	0.5265	0.4393	0.4588
	5.0	Bias	-0.0962	-0.0962	-0.0124	-0.0507	-0.0058	-0.0061
		RMSEs	0.6166	0.5616	0.6176	0.4481	0.4047	0.4439
3.5	2.5	Bias	-0.1018	-0.0061	-0.0101	-0.0451	0.0016	-0.0026
		RMSEs	0.6879	0.6015	0.6371	0.5018	0.4248	0.4466
	3.5	Bias	-0.0937	-0.0105	-0.0159	-0.0378	0.0045	0.0053
		RMSEs	0.5907	0.5554	0.6179	0.4201	0.3901	0.4309
	5.0	Bias	-0.0808	-0.0017	-0.0020	-0.0391	-0.0004	-0.0014
		RMSEs	0.5389	0.5281	0.6078	0.3799	0.3698	0.4258
5	2.5	Bias	-0.0875	-0.0045	-0.0051	-0.0464	-0.0037	-0.0036
		RMSEs	0.5914	0.5611	0.6207	0.4152	0.3917	0.4342
	3.5	Bias	-0.0752	0.0024	0.0012	-0.0432	-0.0036	-0.0027
		RMSEs	0.5409	0.5344	0.6110	0.3821	0.3771	0.4301
	5.0	Bias	-0.0750	0.0016	0.0047	-0.0386	-0.0003	0.00058
		RMSEs	0.5133	0.5199	0.6120	0.3583	0.3613	0.4231
Parameters			**n = 500**			**n = 1,000**		
a	p		**MME**	**LME**	**TLME**	**MME**	**LME**	**TLME**
2.5	2.5	Bias	-0.0162	0.0015	0.0045	-0.0078	-0.0017	-0.0026
		RMSEs	0.3183	0.2176	0.2121	0.2338	0.1558	0.1515
	3.5	Bias	-0.0111	-0.0013	-0.0020	-0.0060	-0.0003	0.0005
		RMSEs	0.2371	0.1926	0.2023	0.1694	0.1366	0.1423
	5.0	Bias	-0.0091	-.00003	-0.0001	-0.0049	-0.0006	-0.0009
		RMSEs	0.2046	0.1816	0.1957	0.1431	0.1269	0.1382
3.5	2.5	Bias	-0.0117	-0.0014	-0.0012	-0.0078	-0.0023	-0.0021
		RMSEs	0.2266	0.1883	0.1972	0.1598	0.1333	0.1404
	3.5	Bias	-0.0085	0.0003	0.0007	-0.0041	0.0002	0.0003
		RMSEs	0.1869	0.1730	0.1925	0.1354	0.1238	0.1360
	5.0	Bias	-0.0074	0.0005	0.0016	-0.0042	-0.0005	-0.0011
		RMSEs	0.1704	0.1654	0.1895	0.1201	0.1159	0.1319
5	2.5	Bias	-0.0122	-0.0035	-0.0042	-0.0048	-.00028	0.00003
		RMSEs	0.1869	0.1742	0.1901	0.1323	0.1231	0.1348
	3.5	Bias	-0.0076	0.0004	0.00155	-0.0041	-0.0004	-0.0010
		RMSEs	0.1705	0.1676	0.1909	0.1200	0.1174	0.1328
	5.0	Bias	-0.0092	-0.0014	-0.0005	-0.0036	0.0002	0.00034
		RMSEs	0.1623	0.1635	0.1896	0.1146	0.1150	0.1325

Table 4: Biases and RMSEs of the parameter estimations for different types of estimators assuming DD for b when $b = 10$

Parameters			**n = 50**			**n = 100**		
a	p		**MME**	**LME**	**TLME**	**MME**	**LME**	**TLME**
2.5	2.5	Bias	-0.2562	0.0065	0.0132	-0.1483	-0.0074	-0.0070
		RMSEs	1.7869	1.3870	1.3742	1.3197	0.9804	0.9584
	3.5	Bias	-0.1635	-0.0045	-0.0097	-0.1153	-0.0129	-0.0188
		RMSEs	1.4408	1.2335	1.2969	1.0372	0.8624	0.8994
	5.0	Bias	-0.1772	-0.0032	-0.0122	-0.0883	-0.0020	-0.0058
		RMSEs	1.2469	1.1327	1.2507	0.8985	0.8096	0.8840
3.5	2.5	Bias	-0.1964	-0.0037	-0.0100	-0.0901	0.0065	0.0053
		RMSEs	1.3818	1.2082	1.2771	0.9798	0.8342	0.8827
	3.5	Bias	-0.1818	-0.0133	-0.0187	-0.0862	-0.0038	-0.0091
		RMSEs	1.1858	1.1119	1.2317	0.8413	0.7762	0.8647
	5.0	Bias	-0.1586	-0.0010	0.0004	-0.0839	-0.0022	0.0060
		RMSEs	1.0750	1.0537	1.2175	0.7585	0.7381	0.8469
5	2.5	Bias	-0.1621	0.0052	-.00006	-0.0879	0.0010	0.0071
		RMSEs	1.1603	1.1058	1.2293	0.8236	0.7823	0.8742
	3.5	Bias	-0.1635	-0.0045	-0.0097	-0.0826	-0.0043	-0.0060
		RMSEs	1.0567	1.0512	1.2127	0.7544	0.7424	0.8457
	5.0	Bias	-0.1418	0.0104	0.01018	-0.0852	-0.0090	-0.0071
		RMSEs	10.214	10.333	12.134	0.7252	0.7291	0.8485
Parameters			**n = 500**			**n = 1,000**		
a	p		**MME**	**LME**	**TLME**	**MME**	**LME**	**TLME**
2.5	2.5	Bias	-0.0310	-0.0004	0.0015	-0.0154	-0.0011	-0.0019
		RMSEs	0.6482	0.4411	0.4260	0.4611	0.3135	0.3018
	3.5	Bias	-0.0247	-0.0026	-0.0004	-0.0068	0.0037	0.0056
		RMSEs	0.4792	0.3910	0.4080	0.3467	0.2768	0.2851
	5.0	Bias	-0.0152	0.0021	0.00211	-0.0138	-0.0043	-0.0041
		RMSEs	0.3983	0.3538	0.3866	0.2834	0.2529	0.2767
3.5	2.5	Bias	-0.0236	-0.0027	-0.0035	-0.0108	-0.0007	0.0003
		RMSEs	0.4556	0.3812	0.4035	0.3263	0.2682	0.2798
	3.5	Bias	-0.0163	0.0006	-0.0004	-0.0124	-0.0047	-0.0062
		RMSEs	0.3777	0.3469	0.3830	0.2678	0.2459	0.2709
	5.0	Bias	-0.0159	0.0001	0.0002	-0.0093	-0.0021	-0.0042
		RMSEs	0.3378	0.3290	0.3780	0.2397	0.2316	0.2638
5	2.5	Bias	-0.0083	-0.0011	-0.0017	-0.0087	-0.0004	-0.0018
		RMSEs	0.2654	0.2470	0.2733	0.2669	0.2483	0.2718
	3.5	Bias	-0.0165	-0.0005	-0.0008	-0.0094	-0.0021	-0.0037
		RMSEs	0.3389	0.3332	0.3789	0.2414	0.2364	0.2671
	5.0	Bias	-0.0060	0.0017	0.0027	-0.0070	.00005	-0.0008
		RMSEs	0.2279	0.2294	0.2668	0.2280	0.2283	0.2616

Table 5: Mean, S.D, CV, CS and CK different parametric values assuming MMEs, LMEs and TLMEs

Parameters			Mean	S.D	CV	CS	CK
a	***b***	***p***	**Method of Moment Estimates**				
2.5	2.5	2.5	1.74618	0.96423	0.55219	1.59264	9.96322
3.5			1.87901	0.73405	0.39066	0.87000	5.18644
5.0			2.01460	0.55391	0.27495	0.41096	3.83527
10			2.22241	0.31206	0.14041	-0.1101	3.47769
2.5	2.5	2.5	1.74618	0.96423	0.55219	1.59264	9.96322
	3.5		2.44465	1.34992	0.55219	1.59264	9.96322
	5.0		3.49236	1.92847	0.55219	1.59264	9.96322
	10		6.98473	3.85693	0.55219	1.59264	9.96322
2.5	2.5	2.5	1.74618	0.96423	0.55219	1.59264	9.96322
		3.5	1.46679	0.74442	0.50751	1.08950	5.67080
		5.0	1.23674	0.59315	0.47961	0.81551	4.25391
		10	0.90892	0.41060	0.45174	0.56301	3.35399
a	***b***	***p***	**L-Moment Estimates**				
2.5	2.5	2.5	1.74618	0.50944	0.29174	0.18854	0.16014
3.5			1.87901	0.40268	0.21430	0.11113	0.14526
5.0			2.01460	0.30854	0.15315	0.05080	0.14071
10			2.22241	0.17429	0.07842	-0.0220	0.14334
2.5	2.5	2.5	1.74618	0.50944	0.29174	0.18854	0.16014
	3.5		2.44465	0.71321	0.29174	0.18854	0.16014
	5.0		3.49236	1.01888	0.29174	0.18854	0.16014
	10		6.98473	2.03776	0.29174	0.18854	0.16014
2.5	2.5	2.5	1.74618	0.50944	0.29174	0.18854	0.16014
		3.5	1.46679	0.40484	0.27600	0.15103	0.14100
		5.0	1.23674	0.32781	0.26506	0.12369	0.12821
		10	0.90892	0.23011	0.25316	0.09272	0.11494
a	***b***	***p***	**TL-Moment Estimates**				
2.5	2.5	2.5	1.65012	0.25671	0.15557	0.10855	0.07708
3.5			1.83426	0.20651	0.11258	0.06144	0.07183
5.0			1.99892	0.15907	0.07958	0.02539	0.07064
10			2.22625	0.08958	0.04024	-0.01743	0.07239
2.5	2.5	2.5	1.65012	0.25671	0.15557	0.10855	0.07708
	3.5		2.31018	0.35939	0.15557	0.10855	0.07708
	5.0		3.30025	0.51342	0.15557	0.10855	0.07708
	10		6.60051	1.02685	0.15557	0.10855	0.07708
2.5	2.5	2.5	1.65012	0.25671	0.15557	0.10855	0.07708
		3.5	1.40564	0.20865	0.14844	0.08752	0.06943
		5.0	1.19619	0.17147	0.14334	0.07193	0.06422
		10	0.88759	0.12219	0.13767	0.05396	0.05870

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