

The Distribution of a Linear Combination of Two Correlated Chi-Square Variables

Distribución de una Combinación Lineal de Dos Variables Chi-Cuadrado Correlacionadas

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Abstract

The distribution of the linear combination of two chi-square variables is known if the variables are independent. In this paper, we derive the distribution of positive linear combination of two chi-square variables when they are correlated through a bivariate chi-square distribution. Some properties of the distribution, namely, the characteristic function, cumulative distribution function, raw moments, mean centered moments, coefficients of skewness and kurtosis are derived. Results match with the independent case when the variables are uncorrelated. The graph of the density function is presented.

Key words: Bivariate Chi-square Distribution, Correlated Chi-square Variables, Linear Combination, Characteristic Function, Cumulative Distribution, Moments.

Resumen

La distribución de una combinación lineal de dos variables chi cuadrado es conocida si las variables son independientes. En este artículo, se deriva la distribución de una combinación lineal positiva de dos variables chi cuadrado cuando éstas están correlacionadas a través de una distribución chi cuadrado bivariada. Algunas propiedades de esta distribución como la función característica, la función de distribución acumulada, sus momentos, momentos centrados alrededor de la media, los coeficientes de sesgo y curtosis son derivados. Los resultados coinciden con el caso independiente cuando las variables son no correlacionadas. La gráfica de la función de densidad es también presentada.

Palabras clave: combinación lineal, distribución acumulada, distribución chi cuadrado bivariada, función característica, momentos, variables chi cuadrado correlacionadas.

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1. Introduction

Let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ be ($N > 2$) two-dimensional independent normal random vectors from $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with mean vector $\bar{\mathbf{X}} = (\bar{X}_1, \bar{X}_2)'$, where $N\bar{X}_i = \sum_{j=1}^N X_{ij}$, ($i = 1, 2$) so that sums of squares and cross product matrix is given by $\sum_{j=1}^N (\mathbf{X}_j - \bar{\mathbf{X}})(\mathbf{X}_j - \bar{\mathbf{X}})' = \mathbf{A}$. Let the matrix \mathbf{A} be denoted by $\mathbf{A} = (a_{ik}), i = 1, 2; k = 1, 2$; where $a_{ii} = mS_i^2, (i = 1, 2), m = N - 1$ and $a_{12} = mRS_1S_2$. That is, S_1 and S_2 are the sample standard deviations based on the bivariate sample, and R is the related product moment correlation coefficient. Also let $\boldsymbol{\Sigma} = (\sigma_{ik}), i = 1, 2; k = 1, 2$ where $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2, \sigma_{12} = \rho\sigma_1\sigma_2$ with $\sigma_1 > 0, \sigma_2 > 0$. The quantity $\rho (-1 < \rho < 1)$ is the product moment correlation coefficient between X_{1j} and $X_{2j} (j = 1, 2, \dots, N)$.

The joint density function $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$, called the bivariate chi-square distribution, was derived by Joarder (2009) in the spirit of Krishnaiah, Haggis & Steinberg (1963) who studied the bivariate chi-distribution.

The distribution of linear function of random variables is useful in the theory of process capability indices and the study of two or more control variables. See, for example, Glynn & Inglehart (1989) and Chen & Hsu (1995). It also occurs in statistical hypothesis testing and high energy physics (See Bausch 2012).

The density function of positive linear combination of independent chi-square random variables was derived by Gunst & Webster (1973). Algorithms were written by Davies (1980) and Farebrother (1984) for the distribution of the linear combination of independent chi-square variables. The exact density function of a general linear combination of independent chi-square variables is a special case of a paper by Provost (1988) for a more general case of Gamma random variables. Interested readers may go through Johnson, Kotz & Balakrishnan (1994) for a detailed historical account.

By application of the inversion formula to the characteristic function of the sum of correlated chi-squares, Gordon & Ramig (1983) derived an integral form of the cumulative distribution function (CDF) of the sum and the used trapezoidal rule to evaluate it. Since this integral form of the CDF involves integration of complex variables, the percentage points depends on the type of numerical technique you employ. Recently Bausch (2012) has developed an efficient algorithm for numerically computing the linear combination of independent chi-square random variables. He has shown its application in string theory.

In Section 2, some mathematical preliminaries are provided. In Section 3, we derive the density function and the Cumulative Distribution Function of the positive linear combination of two correlated chi-square variables when they are governed through a bivariate chi-square density function given by (6). In Section 4, we derive the characteristic function of the distribution. In Section 5, we also derive some properties of the distribution, namely, raw moments, mean centered moments, coefficient of skewness and kurtosis. The results match with the independent case when the variables are uncorrelated. The results also match with the special case of the sum of two correlated chi-square variables considered by

Joarder & Omar (2013). The graph of the density function of the sum is presented at the end of the paper.

2. Mathematical Preliminaries

Let $f_{X,Y}(x, y)$ be the joint density function of X and Y . Then the following lemma is well known.

Lemma 1. *Let X and Y be two random variables with common probability density function $f_{X,Y}(x, y)$. Further let $Z = X + Y$. Then the density function of Z at z is given by*

$$h_Z(z) = \int_0^\infty f_{X,Y}(z - y, y) dy \tag{1}$$

The duplication of the Gamma function is given below:

$$\Gamma(2z)\sqrt{\pi} = 2^{2z-1}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) \tag{2}$$

The incomplete Gamma is defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt \tag{3}$$

where $Re(\alpha) > 0$ (Gradshteyn & Ryzhik 1994, Equation 8.350, p. 949).

The hypergeometric function ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z)$ is defined by

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{k=0}^\infty \frac{(a_1)_{\{k\}}(a_2)_{\{k\}} \dots (a_p)_{\{k\}} z^k}{(b_1)_{\{k\}}(b_2)_{\{k\}} \dots (b_q)_{\{k\}} k!} \tag{4}$$

where $a_{\{k\}} = a(a + 1) \dots (a + k - 1)$

The following integral will be used:

$$\int_0^\infty x^{a-1} e^{-bx} \gamma(c, dx) dx = \frac{d^c \Gamma(a + c)}{c(b + d)^{a+c}} {}_2F_1\left(1, a + c; c + 1; \frac{d}{b + d}\right) \tag{5}$$

with $Re(a + b) > 0, b > 0, (a + c) > 0$, (Gradshteyn & Ryzhik 1994).

The following theorem is due to Joarder (2009), although it can be followed from Krishnaiah et al. (1963).

Theorem 1. *The random variables U and V are said to have a correlated bivariate chi-square distribution each with $m(> 2)$ degrees of freedom, if its density function is given by*

$$f_{U,V}(u, v) = \frac{(uv)^{(m/2)-1}}{2^m \Gamma^2(m/2) (1 - \rho^2)^{m/2}} \exp\left(-\frac{u + v}{2 - 2\rho^2}\right) {}_0F_1\left(\frac{m}{2}; \frac{\rho^2 uv}{(2 - 2\rho^2)^2}\right) \tag{6}$$

where ${}_0F_1(; b; z)$ is defined in 4 and $-1 < \rho < 1$.

In case $\rho = 0$, the density function of the joint probability distribution in Theorem 1, would be $f_{U,V}(u, v) = f_U(u)f_V(v)$ where $U \sim X_m^2$ and $V \sim X_m^2$. The product moment correlation coefficient between U and V can be calculated to be ρ^2 . For the estimation of correlation coefficient ρ by modern techniques, we refer to Ahmed (1992).

3. The Density Function and the Cumulative Distribution Function

Let c_1 and c_2 be positive numbers so that $T_1 = c_1U + c_2V$. Equivalently, let $T_1 = c_1T$ where $T = U + cV$, $c = c_2/c_1$ defines a general linear combination of the variables U and V .

Theorem 2. *Let U and V be two chi-square variables each having m degrees of freedom with density function given in Theorem 1. Then for any positive constant c , the density function of $T = U + cV$ is given by*

$$f_T(t) = \frac{\Gamma((m+1)/2)t^{m-1}}{2^m\Gamma(m)[c(1-\rho^2)]^{m/2}} \exp\left(-\frac{t}{2-2\rho^2}\right) \times \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+(m+1)/2)} \frac{(t\rho)^{2k}}{(4-4\rho^2)^{2k}c^k k!} {}_1F_1\left(k+\frac{m}{2}; 2k+m; \frac{(c-1)t}{(2-2\rho^2)c}\right) \quad (7)$$

where $m > 2$, $-1 < \rho < 1$ and $0 \leq t < \infty$.

Proof. It follows from (6) that the joint density function of $X = U$ and $Y = cV$ is given by

$$f_{X,Y}(x, y) = \frac{(1-\rho^2)^{-m/2}}{2^m\Gamma^2(m/2)} \left(\frac{xy}{c}\right)^{(m/2)-1} \exp\left(-\frac{1}{2-2\rho^2}\left(x+\frac{y}{c}\right)\right) {}_0F_1\left(\frac{m}{2}; \frac{\rho^2}{(2-2\rho^2)^2} \frac{xy}{c}\right) \frac{1}{c}$$

so that, by Lemma 1, the density function of $T = X + Y$ is given by

$$f_T(t) = \frac{(c(1-\rho^2))^{-m/2}}{2^m\Gamma^2(m/2)} \exp\left(-\frac{t}{2-2\rho^2}\right) I(t; m, \rho, c) \quad (8)$$

where

$$I(t; m, \rho, c) = \Gamma(m/2) \sum_{k=0}^{\infty} \frac{1}{\Gamma[k+(m/2)]} \frac{\rho^{2k}}{(2-2\rho^2)^{2k}c^k k!} J(t; m, \rho, c) \quad (9)$$

with $J(t; m, \rho, c) = \int_0^t (t-y)^{k-1+(m/2)} y^{k-1+(m/2)} \exp\left(-\frac{(c-1)y}{c(2-2\rho^2)}\right) dy$ □

Substituting $y = st$ we have

$$J(t; m, \rho, c) = \Gamma(k + (m/2))t^{2k+m-1} \sum_{j=0}^{\infty} \frac{\Gamma[(k + j + (m/2))]}{\Gamma(2k + j + m)} \frac{(c - 1)^j t^j}{(2 - 2\rho^2)^j c^j j!} \quad (10)$$

which, by (4), can be expressed as

$$J(t; m, \rho, c) = t^{2k+m-1} \frac{\Gamma^2(k + (m/2))}{\Gamma(2k + m)} {}_1F_1\left(k + \frac{m}{2}; 2k + m; \frac{(c - 1)t}{(2 - 2\rho^2)c}\right)$$

Plugging this in (9) and simplifying, we have

$$I(t; m, \rho, c) = \frac{\Gamma(m/2)\sqrt{\pi}}{2^{m-1}} t^{m-1} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + (m + 1)/2)} \frac{(t\rho)^{2k}}{(4 - 4\rho^2)^{2k} c^k k!} \times {}_1F_1\left(k + \frac{m}{2}; 2k + m; \frac{(c - 1)t}{(2 - 2\rho^2)c}\right)$$

Substituting this in (8) and simplifying, we have (7).

Figure 1 provides a graphical display of this density function for $m = 5$ and various values of c and ρ .

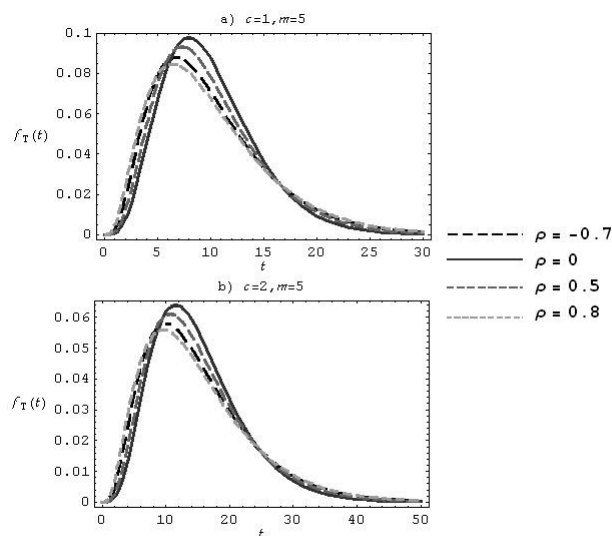


FIGURE 1: Linear combination of chi-square variables for $m = 5$ and various values of ρ .

Theorem 3. Let T have a density function given by (7). Then the Cumulative Distribution Function of T is given by

$$F_T(t) = \frac{\Gamma((m + 1)/2)}{2^m \Gamma(m) (c(1 - \rho^2))^{m/2}} \times \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + (m + 1)/2)} \frac{\rho^{2k}}{(4 - 4\rho^2)^{2k} c^k k!} I(k; m, \rho) \quad (11)$$

where $I(k; m, \rho) = \int_{y=0}^t y^{m+2k-1} \exp\left(-\frac{y}{(2-2\rho^2)}\right) {}_1F_1\left(k + \frac{m}{2}; 2k + m; \frac{(c-1)y}{(2-2\rho^2)c}\right) dy$, $0 \leq t < \infty$, $-1 < \rho < 1$, $m > 2$ and c is any positive constant.

Proof. It is immediate from Theorem 2 □

The CDF in (11) is not in closed form. However, if $\rho = 0$, a closed form expression is presented in Theorem 5.

Theorem 4. Let U and V be two independent chi-square variables each having $m(> 2)$ degrees of freedom. Then for any positive constant c , the density function of $T = U + cV$ is given by

$$f_T(t) = \frac{t^{m-1} e^{-t/2}}{2^m c^{m/2} \Gamma(m)} {}_1F_1\left(\frac{m}{2}; m; \frac{(c-1)t}{2c}\right), \quad 0 \leq t < \infty \quad (12)$$

Proof. Putting $\rho = 0$ in Theorem 2, we have (12). □

If $c = 1$, then (12) simplifies to the density function of X_{2m}^2 as expected. The equation (10) is a special case of Provost (1988)

Theorem 5. Let U and V be two independent chi-square variables each having $m(> 2)$ degrees of freedom. Then the Cumulative Density Function of $T = U + cV$ is given by

$$F(t) = \frac{1}{c^{m/2}} \sum_{k=0}^{\infty} \frac{(m/2)_{\{k\}}}{\Gamma(k+m)} \frac{(c-1)^k}{c^k k!} \gamma(k+m, t/2) \quad (13)$$

where $m > 2$ and $\gamma(\alpha, x)$ is defined in (3).

Proof. By substituting $\rho = 0$ in (12), we have

$$F(t) = \frac{1}{2^m \Gamma(m) c^{m/2}} \int_0^t y^{m-1} \exp(-y/2) {}_1F_1\left(\frac{m}{2}; \frac{(c-1)y}{2c}\right) dy$$

which simplifies to (13). □

By substituting $c = 1$ in (13), we have $F(t) = \gamma(m, t/2)/\Gamma(m)$ which is the Cumulative Distribution Function X_{2m}^2 . Bausch (2012) developed an efficient algorithm for computing linear combination of independent chi-square variables.

4. The Characteristic Function

The quantity i in this section is defined by the imaginary number $i = \sqrt{-1}$.

Theorem 6. Let U and V be two chi-square variables each having $m(> 2)$ degrees of freedom $-1 < \rho < 1$ with density function given in Theorem 1. Then the characteristic function $\phi_{U,V}(w_1, w_2) = E(e^{iw_1 U + iw_2 V})$ of U and V at w_1 and w_2 is given by

$$\phi_{U,V}(w_1, w_2) = [(1 - 2iw_1)(1 - 2iw_2) + 4w_1 w_2 \rho^2]^{-m/2} \quad (14)$$

where $m > 2$ and $-1 < \rho < 1$.

Proof. See Omar & Joarder (2010). □

The characteristic function of the linear combination of two correlated chi-square variables is derived below.

Theorem 7. *Let U and V be two chi-square variables each having m degrees of freedom. Then for any known constant c , the characteristic function of $T = U + cV$ at w is given by the following:*

$$\phi_T(w) = [(1 - 2iw)(1 - 2icw) + 4w^2c\rho^2]^{-m/2} \tag{15}$$

where $m > 2$ and $-1 < \rho < 1$.

Proof. By definition, the characteristic function of $T = U + cV$ is given by $\phi_T(w) = E(e^{iwT}) = E[e^{iw(U+cV)}] = E[e^{i(wU+cwV)}]$. □

By (14), $E[e^{i(wU+cwV)}] = \phi_{U,V}(w, cw)$ and can be written as $\phi_{U,V}(w, cw) = [(1 - 2iw)(1 - 2icw) + 4w^2c\rho^2]^{-m/2}$, which is (15).

The corollary below follows from Theorem 7.

Corollary 1. *Let U and V be two independent chi-square variables each having the same degrees of freedom m . Then for any positive constant c , the characteristic function of $T = U + cV$ at w is given by the following:*

$$\phi_T(w) = [(1 - 2iw)(1 - 2iwc)]^{-m/2}, \quad m > 2 \tag{16}$$

Since the above can be expressed as $\phi_T(w) = \phi_U(w)\phi_{cV}(w)$, clearly the random variable T is the linear combination of two independent random variables U and V . In case $c = 1$, the equation (16) will be specialized to the characteristic function of a chi-square variable with $2m$ degrees of freedom.

The following results are for any general bivariate distribution.

Theorem 8. *Let X and Y have a bivariate distribution with density function $f_{X,Y}(x, y)$ and characteristic function $\varphi_{X,Y}(w_1, w_2) = E(e^{iw_1X+iw_2Y})$. Then for any constant c , the characteristic function of $T = X + cY$ at w is given by the following:*

$$\phi_T(w) = \phi_{X,Y}(w, cw) \tag{17}$$

Proof. By definition, the characteristic function of $T = X + cY$ is given by $\phi_T(w) = E(e^{iwT}) = E[e^{iw(X+cY)}] = E[e^{i(wX+cwY)}] = \phi_{X,Y}(w, cw)$. □

Corollary 2. *Let X and Y have a bivariate distribution with density function $f_{X,Y}(x, y)$ and characteristic function $\varphi_{X,Y}(w_1, w_2) = E(e^{iw_1X+iw_2Y})$. Then, the characteristic function of $T = X + Y$ at w is given by the following:*

$$\phi_T(w) = \phi_{X,Y}(w, w) \tag{18}$$

5. Moments, Coefficient of Skewness and Kurtosis

The following theorem is due to Joarder, Laradji, & Omar (2012).

Theorem 9. Let U and V have the bivariate chi-square distribution with density function with common degrees of freedom m and density function in Theorem 1. Then for $a > -m/2, b > -m/2$ and $-1 < \rho < 1$, the (a, b) -th product moment of U and V , denoted by $\mu'_{a,b;\rho}(U, V) = E(U^a V^b)$, is given by

$$\mu'_{a,b;\rho}(U, V) = 2^{a+b}(1-\rho^2)^{a+b+(m/2)} \frac{\Gamma(a+(m/2))\Gamma(b+(m/2))}{\Gamma^2(m/2)} \times {}_2F_1\left(a+\frac{m}{2}, b+\frac{m}{2}; \frac{m}{2}; \rho^2\right) \quad (19)$$

where $m > 2, -1 < \rho < 1$ and ${}_2F_1(a_1, a_2; b; z)$ is defined in (4).

Theorem 10. Let T have a density function given by (7). Then the first four moments of T are respectively given by

$$E(T) = (c+1)m \quad (20)$$

$$E(T^2) = (c^2+1)m(m+2) + 2cm(m+2\rho^2) \quad (21)$$

$$E(T^3) = (c^3+1)m(m+2)(m+4) + 3c(c+1)(m(m+2)(m+4\rho^2)) \quad (22)$$

$$E(T^4) = (c^4+1)[m(m+2)(m+4)(m+6)] + 4c(c^2+1)[m(m+2)(m+4)(m+6\rho^2)] + 6c^2m(m+2)[m(m+2) + 8(m+2)\rho^2 + 8\rho^4] \quad (23)$$

where $c > 0, m > 2$ and $-1 < \rho < 1$.

Proof. The moment expressions between (20) and (23) inclusively follow from Theorem 9 by tedious algebraic simplification. \square

Let T have a density function given by (7). Then the a -th moment of T denoted by $E(T^a) = E(U+cV)^a$, where c is any non-negative constant, is given by

$$\mu'_a(T) = \sum_{j=0}^a \binom{a}{j} c^{a-j} \mu'_{j,a-j;\rho}(U, V) \quad (24)$$

where $\mu'_{j,a-j;\rho}(U, V) = E(U^j V^{a-j})$ is given by Theorem 9.

The centered moments of T of order a is given by $\mu_a = E(T - E(T))^a, a = 1, 2, \dots$. That is the second, third and fourth order mean corrected moments are respectively given by

$$\mu_2 = E(T^2) - \mu^2 \quad (25)$$

$$\mu_3 = E(T^3) - 3E(T^2)\mu + 2\mu^3 \quad (26)$$

$$\mu_4 = E(T^4) - 4E(T^3)\mu + 6E(T^2)\mu^2 - 3\mu^4 \quad (27)$$

Where $\mu = E(T)$. The explicit forms for the centered moments of the linear combination of bivariate chi-square random variables are given in the following theorem.

Theorem 11. *Let T have a density function given by (7). The second to fourth centered moments of T are given by the following:*

$$\mu_2 = 2m(1 + c^2 + 2c\rho^2) \tag{28}$$

$$\mu_3 = 8(c + 1)m(c^2 - c + 1 + 3c\rho^2) \tag{29}$$

$$\begin{aligned} \mu_4 = 12m[2c^2m + (c^4 + 1)(m + 4) \\ + 4c(4c^2 + 4c + 4 + c^2m + m)\rho^2 + 4c^2(m + 2)\rho^4] \end{aligned} \tag{30}$$

where $m > 2, c$ is any positive constant and $-1 < \rho < 1$.

Proof. The moments between (28) to (30) inclusively follow from (25),(26) and (27) with tedious algebraic simplifications. □

In case $\rho = 0$, the moments match with that of $T = U + cV$ where U and V have independent chi-square distributions each with degrees of freedom $m(> 2)$.

The skewness and kurtosis of a random variable T are given by the moment ratios $\alpha_i(T) = \mu_i\mu_2^{-i/2}, i = 3, 4$. The theorem below follows from Theorem 11.

Theorem 12. *Let T have a density function given by (7). The coefficient of skewness and kurtosis of T where c is any non-negative constant, are given by*

$$\alpha_3(T) = \frac{2\sqrt{2}(c + 1)(3c\rho^2 + c^2 - c + 1)}{\sqrt{m}(2c\rho^2 + c^2 + 1)^{3/2}} \tag{31}$$

and

$$\alpha_4(T) = 3 + \frac{12}{m(2\rho^2c + c^2 + 1)^2}(2c^2\rho^4 + 4c(c^2 + c + 1) + c^4 + 1) \tag{32}$$

respectively, where $m > 2, c$ is any positive constant and $-1 < \rho < 1$.

In case $\rho = 0$, the above coefficient of skewness and kurtosis simplifies to, as expected, that for $T = U + cV$ where U and V are independent chi-square with the same degrees of freedom $m(> 2)$. In case $c = 1, \rho$ decreases to 0 and the degrees of freedom m increases indefinitely, then the coefficient of skewness and that of kurtosis converges to 0 and 3 as expected.

6. Conclusion

We have developed the distributional characteristics of linear combination of correlated chi-square variables. Based on the results in the paper, efficient computational algorithms can be developed along the line of Bausch (2012) who developed an efficient algorithm for computing linear combination of independent chi-square variables.

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