Some new results on common fixed points in certain topological spaces

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Abstract. The main purpose of this paper is to give some common fixed point theorems in $F$-type topological spaces.

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1. Introduction

In [1], Caristi proved that a selfmapping $T$ of a complete metric space $(X,d)$ has a fixed point if there exists a lower semi-continuous function $\phi : X \to \mathbb{R}^+$ such that

$$d(x,Tx) \leq \phi(x) - \phi(Tx), \quad \forall x \in X.$$ 

This result was frequently used to prove existence theorems in fixed point theory. However, it is not hard to see that if the graph of $T$ is closed and $T$ satisfies the above inequality for arbitrary function $\phi$, then $T$ will have a fixed point $x^*$ such that $x^*$ is the limit of the sequence $(x_n)$ defined by

$$\begin{cases}
    x_0 \in X, \\
    x_{n+1} = Tx_n.
\end{cases}$$

To support this remark, we give the following example. Let $X = [0, +\infty[$. Define $T$ and $\phi$ by

$$T x = \frac{1}{2} x, \quad \phi(x) = \begin{cases}
    x & \text{if } x \in [0, 1], \\
    2x & \text{if } x \in [1, +\infty[.
\end{cases}$$
Then we have $|x - Tx| = \frac{1}{2} x$ and

$$\phi(x) - \phi(Tx) = \begin{cases} 
\frac{1}{2} x & \text{if } x \in [0, 1] \\
\frac{3}{2} x & \text{if } x \in [1, 2] \\
x & \text{if } x \in [2, +\infty[. 
\end{cases}$$

Therefore

$$|x - Tx| \leq \phi(x) - \phi(Tx) \text{ for all } x \in X.$$ 

It is easy to see that $T$ has a closed graph and the function $\phi$ is not lower semi-continuous at 1 but $T0 = 0$.

On the other hand, Fang [4] introduced the concept of $F$-type topological space and gave a characterization of the kind of spaces. The usual metric spaces, Hausdorff topological vector spaces, and Menger probabilistic metric spaces are all the special cases of $F$-type topological spaces. Furthermore, Fang established a fixed point theorem in $F$-type topological spaces which extends Caristi’s theorem in the following way:

**Theorem 1.1** (Fang). Let $(X, \theta)$ be a sequentially complete $F$-type topological space generated by the family $\{d_\lambda, \lambda \in D\}$. Let $k : D \rightarrow ]0, +\infty[$ be a nonincreasing function and $\phi : X \rightarrow \mathbb{R}^+$ be a lower semi-continuous function. Let $T$ be a selfmapping of $X$ such that

$$d_\lambda(x, Tx) \leq k(\lambda)[\phi(x) - \phi(Tx)], \ \forall \lambda \in D, \forall x \in X.$$ 

Then $T$ has a fixed point in $X$.

The aim of this paper is to give some common fixed point theorems in $F$-type topological spaces. To do this, we first recall the definition of this space as given in [4].

**Definition 1.1** (Fang). A topological space $(E, \theta)$ is said to be $F$-type topological space if it is Hausdorff and for each $x \in E$, there exists a neighborhood base $F_x = \{U_x(\lambda, t) / \lambda \in D, t > 0\}$, where $D = (D, \prec)$ denotes a directed set such that:

\begin{align*}
(\text{F}_1) & \text{ If } y \in U_x(\lambda, t), \text{ then } x \in U_y(\lambda, t); \\
(\text{F}_2) & \text{ } U_x(\lambda, t) \subset U_x(\mu, s) \text{ for } \mu \prec \lambda, t \leq s; \\
(\text{F}_3) & \text{ } \forall \lambda \in D, \exists \mu \in D \text{ such that } \lambda \prec \mu \text{ and } U_x(\mu, t_1) \cap U_y(\mu, t_2) \neq \emptyset \text{ implies } y \in U_x(\lambda, t_1 + t_2); \\
(\text{F}_4) & \text{ } E = \bigcup_{t > 0} U_x(\lambda, t), \forall \lambda \in D, \forall x \in E.
\end{align*}

On the other hand, it is proved in [4] that for each $F$-type topological space $(E, \theta)$, there exists a family $M = \{d_\lambda, \lambda \in D\}$ of quasi-metrics on $E$ satisfying:

\begin{enumerate}
\item $d_\lambda(x, y) = 0 \ \forall \lambda \in D \text{ iff } x = y$;
\item $d_\lambda(x, y) = d_\lambda(y, x) \ \forall \lambda \in D$;
\item $d_\lambda(x, y) \leq d_\mu(x, y) \text{ for } \lambda \prec \mu$;
\item $\forall \lambda \in D, \exists \mu \in D \text{ such that } \lambda \prec \mu \text{ and } d_\lambda(x, y) \leq d_\mu(x, z) + d_\mu(z, y) \text{ for all } x, y, z \in E \text{ such that } \theta_M = \theta$.
\end{enumerate}
For more details we refer to [4].

2. Main results

**Theorem 2.1.** Let \((X, \theta)\) be a sequentially complete \(F\)-type topological space generated by the family \(\{d_\lambda, \lambda \in D\}\). Let \(k : D \rightarrow [0, +\infty)\) be a nonincreasing function and \(\phi : X \rightarrow \mathbb{R}^+\) be a function. Let \(T\) and \(S\) be two selfmappings of \(X\) with sequentially complete graphs such that \(TX \subset SX\) and

\[
\max\{d_\lambda(Sx, Tx), d_\mu(Tx, STx), d_\beta(Sx, TSTx)\} \\
\leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx) - \phi(Tx)],
\]

(1)

for all \((\lambda, \mu, \beta) \in D^3\), for all \(x \in X\). Then \(T\) and \(S\) have a common fixed point in \(X\).

**Proof.** Let \(x_0 \in X\). Choose \(x_1 \in X\) such that \(Tx_0 = Sx_1\). Choose \(x_2 \in X\) such that \(Tx_1 = Sx_2\). In general, choose \(x_n \in X\) such that \(Tx_{n-1} = Sx_n\). Let \((\lambda, \mu, \beta) \in D^3\). From (1), it follows

\[
d_\lambda(Sx_n, Sx_{n+1}) = d_\lambda(Sx_n, Tx_n) \leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx_n) - \phi(Tx_n)] \\
\leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx_n) - \phi(Sx_{n+1})].
\]

For all \((\lambda, \mu, \beta) \in D^3\), we consider the nonnegative real sequence \((a_n)\) defined by

\[
a_n = \max\{k(\lambda), k(\mu), k(\beta)\}\phi(Sx_n), \quad n = 1, 2, \ldots.
\]

It is easy to see that \((a_n)\) is nonincreasing and bounded below by 0. Hence it is a convergent sequence. On the other hand, for all \(\lambda \in D\), there exists \(\lambda_1 \in D\) such that \(\lambda < \lambda_1\) and

\[
d_\lambda(Sx_n, Sx_{n+m}) \leq d_\lambda(Sx_n, Sx_{n+1}) + d_\lambda(Sx_{n+1}, Sx_{n+m}).
\]

For this \(\lambda_1\), there exists \(\lambda_2 \in D\) such that \(\lambda_1 < \lambda_2\) and

\[
d_\lambda(Sx_{n+1}, Sx_{n+m}) \leq d_\lambda(Sx_{n+1}, Sx_{n+2}) + d_\lambda(Sx_{n+2}, Sx_{n+m}).
\]

Continuing in this fashion, there exists \((\lambda_1, \lambda_2, \ldots, \lambda_{m+1}) \in D^{m+1}\) such that \(\lambda < \lambda_1 < \lambda_2 < \cdots < \lambda_{m+1}\) and

\[
d_\lambda(Sx_n, Sx_{n+m}) \leq d_\lambda(Sx_n, Sx_{n+1}) + d_\lambda(Sx_{n+1}, Sx_{n+2}) + \cdots + d_\lambda(Sx_{n+m-1}, Sx_{n+m}).
\]

Hence

\[
d_\lambda(Sx_n, Sx_{n+m}) \leq \max\{k(\lambda_1), k(\mu), k(\beta)\}[\phi(Sx_n) - \phi(Sx_{n+1})] + \\
\max\{k(\lambda_2), k(\mu), k(\beta)\}[\phi(Sx_{n+1}) - \phi(Sx_{n+2})] + \cdots + \\
\max\{k(\lambda_{m+1}), k(\mu), k(\beta)\}[\phi(Sx_{n+m-1}) - \phi(Sx_{n+m})].
\]

Therefore, since the function \(k\) is nonincreasing, we have

\[
d_\lambda(Sx_n, Sx_{n+m}) \leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx_n) - \phi(Sx_{n+m})]
\]
which implies that \((Sx_n)\) is a Cauchy sequence. Since \(X\) is sequentially complete, there exists \(u \in X\) such that 
\[
\lim_{n \to \infty} Sx_n = u.
\]
Hence 
\[
\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sx_n = u.
\]
We shall show that 
\[
\lim_{n \to \infty} STx_n = u.
\]
Let \(\mu \in D\). There exists \(\mu_1 \in D\) such that 
\[
\mu \prec \mu_1 \text{ and } d_{\mu}(STx_n,u) \leq d_{\mu_1}(STx_n,Tx_n) + d_{\mu_1}(Tx_n,u).
\]
In view of (1), for all \(\mu \in D\) we have 
\[
d_{\mu}(Tx_n,STx_n) \leq a_n - a_{n+1}
\]
which implies that 
\[
\lim_{n \to \infty} d_{\mu}(Tx_n,STx_n) = 0 \forall \mu \in D.
\]
Therefore 
\[
\lim_{n \to \infty} STx_n = u.
\]
We have 
\[
\lim_{n \to \infty} STx_n = u \text{ and } \lim_{n \to \infty} TTx_n = u.
\]
Therefore since the graph of \(S\) is sequentially closed, we conclude that \(Su = u\). On the other hand, we have 
\[
\lim_{n \to \infty} TTx_n = u \text{ and } \lim_{n \to \infty} Sx_n = u.
\]
Therefore since the graph of \(T\) is sequentially closed, we obtain \(Tu = u\). □

Setting \(\lambda = \mu = \beta\) and \(S = \text{Id}_X\), we have the following result which gives a generalization of our earlier remark.

**Corollary 2.1.** Let \((X,\theta)\) be a sequentially complete \(F\)-type topological space generated by the family \(\{d_\lambda, \lambda \in D\}\). Let \(k : D \to ]0, +\infty[\) be a nonincreasing function and \(\phi : X \to \mathbb{R}^+\) be a function. Let \(T\) be a selfmapping of \(X\) such that:

1. \(d_\lambda(x,Tx) \leq k(\lambda)[\phi(x) - \phi(Tx)], \forall \lambda \in D, \forall x \in X;\)
2. \(T\) has a sequentially closed graph.

Then \(T\) has a fixed point in \(X\).

Taking \(\lambda = \mu = \beta\) and \(T = \text{Id}_X\), we get the following result.

**Corollary 2.2.** Let \((X,\theta)\) be a sequentially complete \(F\)-type topological space generated by the family \(\{d_\lambda, \lambda \in D\}\). Let \(k : D \to ]0, +\infty[\) be a nonincreasing function and \(\phi : X \to \mathbb{R}^+\) be a function. Let \(S\) be a surjective selfmapping of \(X\) such that:

1. \(d_\lambda(x,Sx) \leq k(\lambda)[\phi(Sx) - \phi(x)], \forall \lambda \in D, \forall x \in X;\)
2. \(S\) has a sequentially closed graph.

Then \(S\) has a fixed point in \(X\).

In the setting of metric space, we have the following

**Corollary 2.3.** Let \(T\) and \(S\) be two selfmappings of a complete metric space \((X,d)\). Let \(\phi : X \to \mathbb{R}^+\) be a function such that:

1. \(\max\{d(Sx,Tx), d(Tx,STx), d(Sx,TSx)\} \leq \phi(Sx) - \phi(Tx), \forall x \in X;\)
2. \(TX \subset SX;\)
(3) $T$ and $S$ have a sequentially closed graphs. Then $T$ and $S$ have a common fixed point in $X$.

Proof. Take an arbitrary directed set $D$ and let

$$d_\lambda(x, y) = d(x, y) \quad \forall x, y \in X, \ \forall \lambda \in D.$$ Taking $k(\lambda) = 1$ for all $\lambda \in D$, it is easy to see that all conditions of Theorem 2.1 are satisfied and the conclusion follows from this theorem immediately. 

As an example let $X = [0, +\infty]$ and consider $S, T : X \to X$ defined as follows:

$$S_x = \begin{cases} \tan x & \text{if } x \in [0, \pi/2[, \\ x & \text{if } x \in [\pi/2, +\infty[ \end{cases}$$

and $T_x = \arctan x, \ \forall x \in X$.

It is easy to see that $T$ and $S$ have closed graphs and $TX \subset SX$. Furthermore

$$|S_x - T_x| = \begin{cases} \tan x - \arctan x & \text{if } x \in [0, \pi/2[, \\ x - \arctan x & \text{if } x \in [\pi/2, +\infty[ \end{cases};$$

$$|S_x - TS_x| = \begin{cases} \tan x - x & \text{if } x \in [0, \pi/2[, \\ x - \arctan x & \text{if } x \in [\pi/2, +\infty[ \end{cases}$$

and $|Tx - STx| = x - \arctan x \ \forall x \in X$.

Therefore

$$\max\{|S_x - T_x|, |T_x - STx|, |S_x - TSx|\} = \begin{cases} \tan x - \arctan x & \text{if } x \in [0, \pi/2[, \\ x - \arctan x & \text{if } x \in [\pi/2, +\infty[ \end{cases}.$$

Consider the function $\phi$ defined on $X$ by

$$\phi(x) = 2x.$$

We have

$$\phi(Sx) - \phi(Tx) = \begin{cases} 2(tan x - \arctan x) & \text{if } x \in [0, \pi/2[, \\ 2(x - \arctan x) & \text{if } x \in [\pi/2, +\infty[ \end{cases}.$$

Subsequently, we have

$$\max\{|S_x - T_x|, |T_x - STx|, |S_x - TSx|\} \leq \phi(Sx) - \phi(Tx), \ \forall x \in X.$$ Therefore all conditions of Theorem 2.1 are verified and $T0 = S0 = 0$.

**Corollary 2.4.** Let $(X, \theta)$ be a Hausdorff sequentially complete topological vectorial space and $\{U_\lambda, \lambda \in D\}$ be a balanced neighborhood base of 0 in $X$. Let $\phi : X \to \mathbb{R}^+$ be a function and $k : D \to [0, +\infty[$ be a nonincreasing function. Suppose further that two mappings $T, S : X \to X$ satisfy the following conditions:

1. $\psi(x) = \phi(Sx) - \phi(Tx) \geq 0, \ \forall x \in X$;
Proof. Then $T$ and $S$ have a common fixed point in $X$.

$(2)$ for all $x \in X$ and for all $(\lambda, \mu, \beta) \in D^3$
\[
\begin{align*}
Tx - Sx & \in \max\{k(\lambda), k(\mu), k(\beta)\} \psi(x)U_\lambda, \\
Sx - TSx & \in \max\{k(\lambda), k(\mu), k(\beta)\} \psi(x)U_\mu, \\
Tx - STx & \in \max\{k(\lambda), k(\mu), k(\beta)\} \psi(x)U_\beta.
\end{align*}
\]

$(3)$ $TX \subset SX$.

$(4)$ $T$ and $S$ have a sequentially closed graphs.

Then $T$ and $S$ have a common fixed point in $X$.

**Proof.** As in [4], we define a partial order on $D$ as follows:
\[
\lambda < \mu \iff U_\mu \subset U_\lambda.
\]

Then $X$ is an $F$-type topological space generated by the family $\{d_\lambda : \lambda \in D\}$
where
\[
d_\lambda(x, y) = \inf\{t > 0 | x - y \in tU_\lambda\}, \quad \forall x, y \in X, \forall \lambda \in D.
\]

Therefore $\forall (\lambda, \mu, \beta) \in D^3$ and $\forall x \in X$, we have the following:
\[
\max\{d_\lambda(Sx, Tx), d_\mu(Tx, STx), d_\beta(Sx, TSx)\} \
\leq \max\{k(\lambda), k(\mu), k(\beta)\}[\phi(Sx) - \phi(Tx)]
\]

The conclusion follows immediately from Theorem 2.1.

\[\Box\]

3. Applications

Let $(D_1, \prec_{D_1})$ and $(D_2, \prec_{D_2})$ be directed sets.

**Theorem 3.1.** Let $(E, \theta_1)$ (resp. $(F, \theta_2)$) be a sequentially complete $F$-type topological space generated by the family $\{d_\lambda : \lambda \in D_1\}$ (resp. $\{d_\mu : \mu \in D_2\}$).

Let $v : E \to F$ be a function with sequentially closed graph. Let $k_1 : D_1 \to \mathbb{R}^+$ and $k_2 : D_2 \to \mathbb{R}^+$ be two nonincreasing functions. Let $\phi : E \to \mathbb{R}^+$ and $\psi : F \to \mathbb{R}^+$ be two arbitrary functions. Let $T$ and $S$ be selfmappings of $E$ with sequentially closed graphs such that $TE \subset SE$ and
\[
\max\{d_{\lambda_1}(Sx, Tx) + d_{\mu_2}(v(Sx), v(Tx)), d_{\lambda_2}(Sx, TSx) \\
+ d_{\mu_2}(v(Sx), v(TSx)), d_{\lambda_3}(Tx, STx) + d_{\mu_2}(v(Tx), v(STx))\}
\leq \max\{k_1(\lambda_1), k_1(\lambda_2), k_1(\lambda_3)\}[\phi(Sx) - \phi(Tx)] \\
+ \max\{k_2(\mu_1), k_2(\mu_2), k_2(\mu_3)\}[\psi(v(Sx)) - \psi(v(Tx))],
\]

for all $x \in E$ and for all $(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3) \in D_1^3 \times D_2^3$. Then $T$ and $S$ have a common fixed point in $E$.

**Proof.** We define on $D = D_1 \times D_2$ a relation “$\prec_D$” as follows: $(\lambda_1, \mu_1) \prec_D (\lambda_2, \mu_2) \iff \lambda_1 \prec_{D_1} \lambda_2$ and $\mu_1 \prec_{D_2} \mu_2$. For all $(\lambda, \mu) \in D$, we consider the function $\psi_{\lambda, \mu} : E \times E \to \mathbb{R}^+$ defined by
\[
\psi_{\lambda, \mu}(x, y) = d_\lambda(x, y) + d_\mu(v(x), v(y)).
\]
Next we show that $\psi_{\lambda,\mu}$ is a quasi-metric on $E$:

1. $\psi_{\lambda,\mu}(x, y) = 0 \implies d_{\lambda}(x, y) = 0 \implies x = y$.

2. $\psi_{\lambda,\mu}(x, y) = \psi_{\lambda,\mu}(y, x)$, $\forall (\lambda, \mu) \in D$.

3. Let $(\lambda, \alpha, \mu, \beta) \in D_1^2 \times D_2^2$ such that $(\lambda, \mu) \prec_D (\alpha, \beta)$. Then, $\forall (x, y) \in E^2$, $d_{\lambda}(x, y) \leq d_{\alpha}(x, z) + d_{\mu}(z, y)$ and $d_{\mu}(v(x), v(y)) \leq d_{\beta}(v(x), v(z)) + d_{\beta}(v(z), v(y))$. Hence $\psi_{\lambda,\mu}(x, y) \leq \psi_{\alpha,\beta}(x, y)$.

4. Let $(\lambda, \mu) \in D_1 \times D_2$. Then, $\exists (\alpha, \beta) \in D_1 \times D_2$, such that $(\lambda, \mu) \prec_D (\alpha, \beta)$, $d_{\lambda}(x, y) \leq d_{\alpha}(x, z) + d_{\mu}(z, y)$ and $d_{\mu}(v(x), v(y)) \leq d_{\beta}(v(x), v(z)) + d_{\beta}(v(z), v(y))$. Therefore, $\forall (\lambda, \mu) \in D_1 \times D_2$, $\exists (\alpha, \beta) \in D_1 \times D_2$, such that $(\lambda, \mu) \prec_D (\alpha, \beta)$ and $\psi_{\lambda,\mu}(x, y) \leq \psi_{\alpha,\beta}(x, z) + \psi_{\alpha,\beta}(z, y), \forall (x, y, z) \in E^3$.

Now we show that $E$, generated by the family of $(\psi_{\lambda,\mu} : (\lambda, \mu) \in D)$ and which we denote by $E'$, is sequentially complete. Let $(x_n)$ be a cauchy sequence of $E'$. Then $(x_n)$ (resp. $v(x_n)$) is a cauchy sequence in $(E, \theta_1)$ (resp. in $(F, \theta_2)$), which implies that there exists $(x, y) \in E \times F$ such that $\lim_{n \to \infty} x_n = x \in E$ and $\lim_{n \to \infty} v(x_n) = y$. As the function $v$ has a closed graph, we have $v(x) = y$. So, $(x_n)$ converges in $E'$. Therefore $E'$ is sequentially complete.

Next, it is clear that

$$\max\{k_1(\lambda_1), k_1(\lambda_2), k_1(\lambda_3), k_2(\mu_1), k_2(\mu_2), k_2(\mu_3)\}$$

$$= \max\{\max\{k_1(\lambda_1), k_1(\lambda_2), k_1(\lambda_3)\}, \max\{k_2(\mu_1), k_2(\mu_2), k_2(\mu_3)\}\}$$

$$= \max\{\max\{k_1(\lambda_1), k_2(\mu_1)\}, \max\{k_1(\lambda_2), k_2(\mu_2)\}, \max\{k_1(\lambda_3), k_2(\mu_3)\}\}.$$
nonincreasing functions. Let $\phi : E \rightarrow \mathbb{R}^+$ and $\psi : F \rightarrow \mathbb{R}^+$ be two arbitrary functions. Let $T$ be a selfmapping of $E$ such that:

$$d_\lambda(x, Tx) + d_\mu(v(x), v(Tx)) \leq k_1(\lambda)[\phi(x) - \phi(Tx)] + k_2(\mu)[\psi(v(x)) - \psi(v(Tx))],$$

$\forall x \in E$, $\forall (\lambda, \mu) \in D_1 \times D_2$;

(2) $T$ and $v$ have sequentially closed graphs.

Then $T$ has a fixed point.

**Corollary 3.2.** Let $(E, \theta_1)$ (resp. $(F, \theta_2)$) be a sequentially complete $F$-type topological space generated by the family $\{d_\lambda, \lambda \in D_1\}$ (resp. $\{d_\mu, \mu \in D_2\}$). Let $v : E \rightarrow F$ be a function. Let $k_1 : D_1 \rightarrow \mathbb{R}^+$ and $k_2 : D_2 \rightarrow \mathbb{R}^+$ be two nonincreasing functions. Let $\phi : E \rightarrow \mathbb{R}^+$ and $\psi : F \rightarrow \mathbb{R}^+$ be two arbitrary functions. Let $S$ be a surjective selfmapping of $E$ such that:

$$d_\lambda(x, Sx) + d_\mu(v(x), v(Sx)) \leq k_1(\lambda)[\phi(Sx) - \phi(x)] + k_2(\mu)[\psi(v(Sx)) - \psi(v(x))],$$

$\forall x \in E$, $\forall (\lambda, \mu) \in D_1 \times D_2$;

(2) $S$ and $v$ have a sequentially closed graphs.

Then $S$ has a fixed point.

**References**


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