Rend. Sem. Mat. Univ. Pol. Torino
Vol. 57, 3 (1999)

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THE ROLE OF THE LAGRANGE CONSTANT IN SOME NONLINEAR WAVES EQUATIONS

Sommario. Let $M(\alpha)$ be the Lagrange constant associated to an irrational number α. In this note we point out how this constant plays a role in the study of some partial differential equations, more precisely nonlinear waves equations.

1. Introduction and Motivation

In what follows we shall see how the study of solutions of some partial differential equations leads to problems in number theory.

Our motivation was the study of certain nonlinear wave equations. The technic used to solve such problems depend in an essential way on the space dimension (for example the parity) or/and on the rationality of the ratio between the period and the interval lenght (when one search for periodic solutions). Hence some results in number theory, especially in diophantine approximations are needed. It is an established fact, today, that the diophantine approximations play a fundamental role in dynamical systems. We begin by considering two problems:

1.1. Problem 1

Consider the existence of weak solutions for the following periodic-Dirichlet problem for a one-dimensional semilinear wave equation

$$\begin{array}{rcll} u_{tt} - u_{xx} + g(u) &=& f(t,x) & \text{on }]0, 2\pi/\alpha[\times]0,\pi[\\ u(t,0) = u(t,\pi) &=& 0 & \text{on } [0, 2\pi/\alpha] \\ u(0,x) - u(2\pi/\alpha, x) &=& u_t(0,x) - u_t(2\pi/\alpha, x) = 0 & \text{on } [0,\pi], \end{array}$$

where α is a positive irrational number which is not the square root of an integer, $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f \in H := L^2(]0, 2\pi/\alpha[\times]0,\pi[)$.

We shall denote by L the abstract realization in H of the wave operator with the periodic-Dirichlet conditions on $]0, 2\pi/\alpha[\times]0,\pi[$. Thus L is self-adjoint and its spectrum is the closure of the set of the eigenvalues:

$$\sigma(L) = \overline{\{n^2 - \alpha^2 m^2 : n \in \mathbb{N}_0, m \in \mathbb{N}\}}.$$

Then it is essential to know the structure of the spectrum $\sigma(L)$ and consequently the properties of the operator L. Indeed, we have for the linear associated problem, the following simple result:

THEOREM 2.1. *The linear periodic-Dirichlet problem*

$$\begin{array}{rcll} u_{tt}-u_{xx} & = & f(t,x) & \text{on }]0,2\pi/\alpha[\times]0,\pi[\\ u(t,0)=u(t,\pi) & = & 0 & \text{on } [0,2\pi/\alpha] \\ u(0,x)-u(2\pi/\alpha,x) & = & u_t(0,x)-u_t(2\pi/\alpha,x)=0 & \text{on } [0,\pi], \end{array}$$

has a weak solution for each $f \in H$ *if and only if*

(1) $$\inf_{(m,n)\in\mathbb{Z}\times\mathbb{Z}_0} |(\alpha m)^2-n^2|>0.$$

We clearly see that the condition (1.1) which is crucial is a problem of diophantine approximations. If (1.1) is satisfied, $0 \notin \sigma(L)$, hence L is invertible and we can solve the nonlinear problem above by (for example) fixed point theory. For more details in this direction we refer to [3].

1.2. Problem 2

We consider the Dirichlet problem for the semilinear equation of the vibrating string:

(2) $$\left\{\begin{array}{rcll} u_{xy}+f(u) & = & 0, & (x,y)\in\Omega, \\ u|_{\partial\Omega} & = & 0, & \end{array}\right.$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain, convex relative to the characteristic lines $x\pm y=const$. It is assumed that $\Gamma=\partial\Omega=\cup_{j=1}^4\Gamma_j$, where $\Gamma_j\in C^k$ for each j, for some $k\geq 2$, and the endpoints of the curve Γ_j are the so-called vertices of Γ with respect to the lines $x\pm y=const$. A point $(x_0,y_0)\in\Gamma$ is said to be a vertex of Γ with respect to the lines $x\pm y=const$ if one of the two lines $x\pm y=x_0\pm y_0$ has an empty intersection with Ω. The domain Ω can be regarded as a "curved rectangle". More precily:

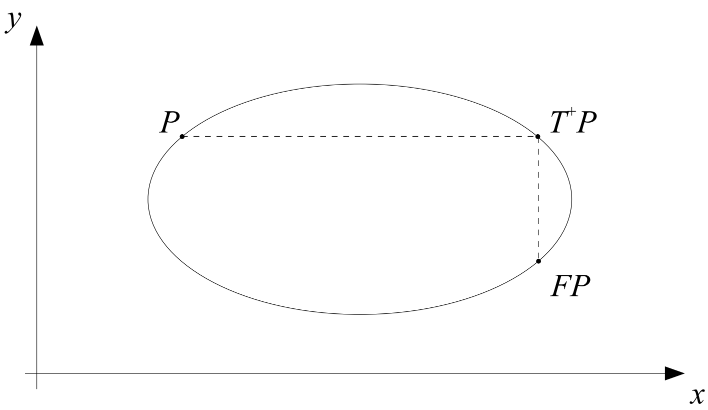
The domain $\Omega\subset\mathbb{R}^2$ is assumed to be bounded, with a boundary $\Gamma=\partial\Omega$ satisfying:

- A_1) $\Gamma=\partial\Omega=\cup_{j=1}^4\Gamma_j$, $\Gamma_j=\{(x,y_j(x))|x_j^0\leq x\leq x_j^1\}$, $y_j(x)\in C^k([x_j^0,x_j^1])$ for any $j=1,2,3,4$ and for some $k\geq 2$.
- A_2) $|y_j'(x)|>0, x\in[x_j^0,x_j^1]$, $j=1,2,3,4$.
- A_3) The endpoints $P_j=(x_j^0,y_j(x_j^0))$ of the curves $\Gamma_1,...,\Gamma_4$ are the vertices of Γ with respect to the lines $x=const.$, $y=const$. By this we mean that for any $j=1,...,4$ one of the two lines $x=x_j^0$, $y=y_j(x_j^0)$ has empty intersection with Ω and there are no other points on Γ with this property.

These conditions imply that the domain Ω is strictly convex relative to the lines $x=const.$, $y=const$. Therefore, following [8], we can define homeomorphisms T^+,T^- on the boundary Γ as follows:

T^+ assigns to a point on the boundary the other boundary point with the same y coordinate. T^- assigns to a point on the boundary the other boundary point with the same x coordinate.

Notice that each vertex P_j is fixed point of either T^+ or T^-. We define $F := T^+ \circ T^-$. It is easy to see that F preserves the orientation of the boundary. (See the following figure).



Let $\Gamma = \{(x(s), y(s)) | \ 0 \leq s < l\}$ be the parametrization of Γ by arc length, so that l is the total length of Γ. For each point $P \in \Gamma$, we denote its coordinate by $S(P) \in [0, l[$. Then the homeomorphism F can be lifted to a continuous map $f_1 : \mathbb{R} \to \mathbb{R}$, which is an increasing function onto $\mathbb{R}$ such that $0 \leq f_1(0) < l$ and

$$f_1(s + l) = f_1(s) + l, \quad s \in \mathbb{R}, \text{ and } \quad S(F(P)) = f_1(S(P)) \pmod{l}, \quad P \in \Gamma.$$

The function f_1 is called *the lift* of F [13]. If we inductively set $f_k(s) := f_1(f_{k-1}(s))$ for integer $k \geq 2$, then it is known that the limit

$$\lim_{k \to \infty} \frac{f_k(s)}{kl} =: \alpha(F) \in [0, 1]$$

exists and is independent of $s \in \mathbb{R}$. The number $\alpha(F)$ is called *the winding number* or *rotation number* of F. The following cases are possible:

(A) $\alpha(F) = \frac{m}{n}$ is a rational number, and $F^n = I$ where I is the identity mapping of Γ onto itself.

(B) $\alpha(F) = \frac{m}{n}$ is a rational number, F^n has a fixed point on Γ, but $F^n \neq I$.

(C) $\alpha(F)$ is an irrational number, and F^k has no fixed point on Γ for any $k \in \mathbb{N}$.

The solvability of problem (2) is quite different in the three cases (A), (B), (C) (see [6] and [3]). The cases (A) and (B) are classical. For the case (C) we have the following result due to Fokin [6].

Let L be the linear differential operator on $H := L^2(\Omega)$ associated to problem (2) and $\sigma(L)$ its spectrum.

THEOREM 2.2. [6] *Suppose that for the domain Ω condition (C) holds. Then L is selfadjoint and the linear problem*

$$\begin{cases} u_{xy} + h(x, y) = 0, & (x, y) \in \Omega, \\ u|_{\partial\Omega} = 0, & \end{cases}$$

has a unique solution u in H for any $h \in H$ if and only if for some $C(\alpha) > 0$ and any rational number m/n,

$$|\alpha - m/n| \geq C(\alpha)/n^2 \tag{3}$$

We see that this problem leads again to diophantine appoximations. Moreover it has been shown that the conditions 1 and 3 are equivalent. In the sequel we shall characterize the irrational numbers which satisfies these conditions and we shall give further results. For the solvability of the nonlinear problem and more details we refer to our recent papers [1], [2] and the works of Lyashenko [9],[10], and Lyashenko and Smiley [11].

2. Diophantine Approximations

As we have seen, these existence theorems require some results of number theory. Those results can essentially be found in [12] but we reproduce them here for the reader's convenience, because of the lack of availability of [12] and because our presentation is simpler [3].

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let Q_α be the quadratic form defined on $\mathbb{Z} \times \mathbb{Z}_0$ by

$$Q_\alpha(m, n) := (\alpha m)^2 - n^2.$$

We want to determine a class of α such that

$$|Q_\alpha(m, n)| \geq c_\alpha > 0,$$

for some $c_\alpha > 0$ and all

$$(m, n) \in \mathbb{Z} \times \mathbb{Z}_0,$$

such that $Q_\alpha(m, n) \neq 0$. Now, $|Q_\alpha(0, n)| = n^2 \geq 1$ for all $n \in \mathbb{Z}_0$, and hence we can restrict ourself to the $(m, n) \in \mathbb{Z}_0 \times \mathbb{Z}_0$ such that $Q_\alpha(m, n) \neq 0$, i.e. to all $(m, n) \in \mathbb{Z}_0 \times \mathbb{Z}_0$, because, α being irrationnal, $Q_\alpha(m, n) \neq 0$ for $(m, n) \in \mathbb{Z}_0 \times \mathbb{Z}_0$. As

$$Q_\alpha(m, n) = Q_{|\alpha|}(|m|, |n|),$$

we can further assume, without loss of generality, that $\alpha > 0$ and

$$(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0.$$

Define Δ_α and Δ'_α respectively by

$$\Delta_\alpha := \inf_{(m,n)\neq(0,0)} |Q_\alpha(m, n)|, \quad \Delta'_\alpha := \liminf_{|m|+|n|\to\infty} |Q_\alpha(m, n)|.$$

Clearly, $\Delta_\alpha \leq \Delta'_\alpha$ and $\Delta'_\alpha > 0$ if and only if $\Delta_\alpha > 0$. Indeed, if $\Delta'_\alpha > 0$, there exists $R > 0$ such that

$$\inf_{|m|+|n|\geq R} |Q_\alpha(m, n)| \geq \Delta'_\alpha/2 > 0,$$

and, α being irrationnal,

$$|Q_\alpha(m, n)| = |\alpha m + n||\alpha m - n| \neq 0,$$

for all $(m, n) \neq (0, 0)$, and hence has a positive lower bound on the finite set $\{(m, n) \neq (0, 0) : |m| + |n| < R\}$.

Let

$$\alpha := [a_0, a_1, \ldots]$$

be the continuous fraction decomposition of α. Recall that it is obtained as follows; put $a_0 := [\alpha]$, where [.] denotes the integer part. Then $\alpha = a_1 + \frac{1}{\alpha_1}$ with $\alpha_1 > 1$, and we set $a_1 := [\alpha_1]$. If $a_0, a_1, \ldots, a_{n-1}$ and $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are known, then $\alpha_{n-1} = a_{n-1} + \frac{1}{\alpha_n}$, with $\alpha_n > 1$ and we set $a_n := [\alpha_n]$. It can be shown that this process does not terminate if and only if α is irrational. The integers $a_0, a_1, \ldots$ are the partial quotients of α; the numbers $\alpha_1, \alpha_2, \ldots$ are the complete quotients of α and the rationals

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \ldots + \frac{1}{a_n}\ ,$$

with p_n, q_n relatively prime integers, are the convergents of α and are such that $p_n/q_n \to \alpha$ as $n \to \infty$. It is well known that the p_n, q_n are recursively defined by the relations

$$p_0 := a_0,\ \ q_0 := 1,\ \ p_1 := a_0 a_1 + 1,\ \ q_1 := a_1,$$

$$p_n := a_n p_{n-1} + p_{n-2},\ \ q_n := a_n q_{n-1} + q_{n-2}.$$

The following Lemma is useful to find Δ'_α.

LEMMA 2.1. *To each irrational number α corresponds a unique (extended) number $M(\alpha) \in [\sqrt{5}, \infty]$ (called the Lagrange constant) having the following properties*

(i) *For each positive number $\mu < M(\alpha)$ there exist infinitely many pairs (p_i, q_i) with $q_i \neq 0$, such that*

$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{\mu q_i^2}.$$

(ii) *If $M(\alpha)$ is finite, then, for each $\mu > M(\alpha)$, there are only finitely many pairs (p_i, q_i) satisfying the inequality*

$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{\mu q_i^2}.$$

Dimostrazione. Let

$$\mu_i := q_i^{-2}\left|\alpha - \frac{p_i}{q_i}\right|^{-1} = q_i^{-1}|\alpha q_i - p_i|^{-1}, \quad i \geq 1,$$

$$M(\alpha) := \limsup_{i\to\infty} \mu_i \in \mathbb{R} \cup \{+\infty\}.$$

It then follows from the elementary properties of the upper limit that $M(\alpha)$ satisfies the conditions of the lemma, with the exception of the estimate $M(\alpha) \geq \sqrt{5}$. But a well known theorem of Hurwitz [14] asserts that for infinitely many pairs (p_i, q_i) one has

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{\sqrt{5} q_i^2},$$

so that the proof is complete.

$\square$

If we set

$$\mathcal{M}(\alpha) := \left\{ M \in \mathbb{R}_0^+ : \text{ infinitely many } (p_i, q_i) \text{ satisfy } \left| \alpha - \frac{p_i}{q_i} \right| \le \frac{1}{Mq_i^2} \right\},$$

then the above Lemma clearly states that $M(\alpha) = \sup \mathcal{M}(\alpha)$.

PROPOSITION 2.1. *$M(\alpha)$ is finite if and only if the sequence $(a_i)_{i\in\mathbb{N}}$ of partial quotients of α is bounded.*

Dimostrazione. We have

$$\begin{aligned} \mu_i &= q_i^{-2}|\alpha - \frac{p_i}{q_i}|^{-1} = q_i^{-2}|(-1)^i q_i(\alpha_{i+1}q_i + q_{i-1})| \\ &= \left|\alpha_{i+1} + \frac{q_{i-1}}{q_i}\right| = \left|[a_{i+1}, a_{i+2}, \ldots] + \frac{1}{[a_i, a_{i-1}, \ldots, a_1]}\right| \\ &= |[a_{i+1}, a_{i+2}, \ldots] + [0, a_i, a_{i-1}, \ldots, a_1]| \\ &= |[a_{i+1}] + \theta_i + \eta_i|, \end{aligned}$$

with $0 < \theta_i, \eta_i < 1$ for all positive integers i. Thus, if $(a_i)_{i\in\mathbb{N}}$ is unbounded, one has

$$\limsup_{i\to\infty} \mu_i \ge \limsup_{i\to\infty}([a_{i+1}] - 2) = +\infty,$$

and $M(\alpha) = \infty$. If $(a_i)_{i\in\mathbb{N}}$ is bounded, say, by M, then

$$M(\alpha) = \limsup_{i\to\infty} \mu_i \le \limsup_{i\to\infty}([a_{i+1}] + 2) < \infty.$$

□

PROPOSITION 2.2. *If $\alpha \in \mathbb{R}^+ \setminus \mathbb{Q}$, then*

$$\Delta'_\alpha = 2\alpha/M(\alpha).$$

Dimostrazione. We have

$$|Q_\alpha(p_i, q_i)| = |\alpha q_i - p_i||\alpha q_i + p_i| = \mu_i^{-1}|\alpha + (p_i/q_i)|,$$

and hence

$$\liminf_{i\to\infty} |Q_\alpha(p_i, q_i)| = 2\alpha/M(\alpha).$$

Now let

$$\begin{aligned} \mathcal{N}(\alpha) := \{M \in \mathbb{R}_0^+ :\ & \text{infinitely many pairs of integers } (p, q) \\ & \text{with } q \ne 0 \text{ satisfy } |\alpha - (p/q)| \le 1/Mq^2\} \supset \mathcal{M}(\alpha). \end{aligned}$$

It is known [14] (see also the interesting paper [15]) that if $M > 2$ and $M \in \mathcal{N}(\alpha)$, then $M \in \mathcal{M}(\alpha)$, and that, for each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\sqrt{5} \in \mathcal{M}(\alpha)$. Thus,

$$M(\alpha) = \sup \mathcal{M}(\alpha) = \sup \mathcal{N}(\alpha),$$

and hence, for $\mu > M(\alpha)$, only finitely many pairs of integers (p, q) with $q \ne 0$ satisfy the inequalities

$$Q_\alpha(p, q) \le \mu^{-1}(\alpha + (p/q)) \le \mu^{-1}(2\alpha + (1/\mu q^2)),$$

which imply that

$$\Delta'_\alpha = \liminf_{|p|+|q|\to\infty} \left[Q_\alpha(p,q) - \frac{1}{\mu^2 q^2} \right] \geq 2\alpha/\mu.$$

Consequently, $\Delta'_\alpha \geq 2\alpha/M(\alpha)$, so that the equality holds.

□

Now, as $\Delta'_\alpha > 0$ if and only if $\Delta_\alpha > 0$, we also have the following characterizations.

COROLLARY 2.1. $\Delta_\alpha > 0$ *if and only if* $M(\alpha) < \infty$, *i.e. if and only if the sequence* $(a_j)_{j\in\mathbb{N}}$ *is bounded above.*

Below we give a straightforward approach to Corollary 2.1, but which need much material.

Dimostrazione. Noting that the minimum Δ_α is preserved under equivalence of forms, we construct an equivalent form which is more natural. Let $\alpha = [a_0; a_1, \cdots]$. The form of Q_α allows us to assume that $\alpha > 0$. Also, we have that $Q_\alpha(x,y) = -\alpha^2 Q_{\frac{1}{\alpha}}(y,x)$, implying that $\Delta_\alpha = \alpha^2 Q_{\frac{1}{\alpha}}$. When $\alpha < 1$ we have $\frac{1}{\alpha} = [a_1; a_2, \cdots]$. Therefore, we may assume that $\alpha > 1$. We consider the equivalent form

$$f(x,y) = Q_\alpha(y, x - a_0 y) = -(x - (\alpha + a_0)y)(x - (-\alpha + a_0)y).$$

We note that this is of the form $g(x,y) = \pm(x - ry)(x - sy)$ where $r = [c_0; c_1, \cdots] > 1$ and $s = -[0, c_{-1}; c_{-2}, \cdots]$. In 1879 A. Markoff (see also T.Cusick and M. Flahive 's book [4], Appendix 1) proved in his original paper that the minimum of such g equals

$$\frac{r - s}{\sup\{[c_i; c_{i+1}, \cdots] + [0; c_{i-1}, \cdots]\}}.$$

Noting that for all i,

$$c_i < [c_i; c_{i+1}, \cdots] + [0; c_{i-1}, \cdots] < c_i + 2,$$

we obtain that the minimum of any form equivalent to g is zero if and only if c_i is unbounded. In our case we have $r = \alpha + a_0 = [2a_0; a_1, \cdots]$ and $-s = \alpha - a_0 = [0; a_1, \cdots]$. □

For example, for the golden number $\alpha := \frac{1+\sqrt{5}}{2}$, we have $\alpha = [1, 1, 1\cdots]$ and then $\Delta_\alpha > 0$. Finaly let

$$\Sigma := \{\alpha : \alpha \in \mathbb{R} \setminus \mathbb{Q},\ M(\alpha) < \infty\},$$

then it can be shown that Σ is a dense, uncountable, and null subset of the real line.

3. Further Results

In this section we continue the study of Σ. Two reals numbers α, β are said to be equivalent, if there exist integers a, b, c, d, such that $|ad - bc| = 1$, and

$$\beta = \frac{a\alpha + b}{c\alpha + d}.$$

There is an old result which states that if α and β are two equivalent irrational numbers, then $M(\alpha) = M(\beta)$. This result was generalized by T.Cusick and M. Mendes France in 1979 [5] proving (among others results) that if $\beta = \frac{a\alpha+b}{c\alpha+d}$ with $ad - bc \neq 0$, and $a, b, c, d \in \mathbb{Z}$, then $M(\alpha) \leq M(\beta)|ad - bc|$. As consequence, $M(\alpha)$ is finite if and only if $M(\beta)$ is finite (which was already observed by O. Perron in the begining of this century).

THEOREM 2.3. *Let α and β two irrational numbers such that*

$$\beta = \frac{a\alpha + b}{c\alpha + d}$$

with $ad - bc \neq 0$, and $a, b, c, d \in \mathbb{Z}$. Then

$$\frac{M(\alpha)}{|ad - bc|} \leq M(\beta) \leq |ad - bc| M(\alpha)$$

Dimostrazione. Let $M \in \mathcal{M}(\alpha)$. Then there exist infinitely many pairs (p_i, q_i) with $q_i \neq 0$, such that

$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{Mq_i^2}.$$

Now

$$\left|\beta - \frac{ap_i + bq_i}{cp_i + dq_i}\right| = |ad - bc| \frac{|\alpha - (p_i/q_i)|}{|c(p_i/q_i) + d||c\alpha + d|}$$

Let $\epsilon > 0$. Then there exist i_ϵ such that

$$\frac{1}{|c\alpha + d|} \leq \frac{1 + \epsilon}{|c(p_i/q_i) + d|}, \text{ for all } i \geq i_\epsilon$$

and

$$\left|\beta - \frac{ap_i + bq_i}{cp_i + dq_i}\right| \leq \frac{(1 + \epsilon)|ad - bc|}{M(c(p_i/q_i) + d)^2 q_i^2} = \frac{(1 + \epsilon)|ad - bc|}{M} \frac{1}{(cp_i + dq_i)^2}$$

for all $i \geq i_\epsilon$. Therefore

$$\frac{M}{(1 + \epsilon)|ad - bc|} \in \mathcal{N}(\beta), \quad \text{for all } \epsilon > 0$$

Now if $\epsilon \to 0$, we get

$$\frac{M}{|ad - bc|} \leq M(\beta)$$

and then

$$\frac{M(\alpha)}{|ad - bc|} \leq M(\beta).$$

Rewrite $\alpha = \frac{-d\beta+b}{c\beta-a}$. Then the second inequality follows immediatly, so that the proof is complete.

□

As a first simple consequence of this theorem we have the following classical result

COROLLARY 2.2. *If α and β are equivalent then $M(\alpha) = M(\beta)$.*

Dimostrazione. The proof is immediate. By the theorem $M(\alpha) \le M(\beta)$. On the other hand, we have

$$\alpha = \frac{-d\beta + b}{c\beta - a},$$

where $(-d)(-a) - bc = ad - bc$. Then $M(\beta) \le M(\alpha)$, which finishes the proof.

□

Now using the above propositions we easily obtain the following result.

COROLLARY 2.3. *Under the hypothesis of Theorem 2.3, we have*

$$M(\alpha) < \infty \Longleftrightarrow M(\beta) < \infty$$

i.e.

$$\Delta_\alpha > 0 \Longleftrightarrow \Delta_\beta > 0$$

i.e. the sequence of partial quotients of α is bounded above if and only if the sequence of partial quotients of β is bounded above.

To illustrate the results of this section we return to the Problem 2 in the Introduction. For Ω we consider the following domain:

$$\Omega(a, b) := \{(x, y) \in \mathbb{R}^2 \mid 0 < x + y < a,\ 0 < x - y < b\}.$$

In this particular case, the winding number $\alpha(F)$ of the corresponding diffeomorphism F for $\Omega(a, b)$ is given by

PROPOSITION 2.3. $\alpha(F) = \frac{a}{a+b}$ *for all* $\Omega(a, b)$ *where* F *is the corresponding diffeomorphism.*

Dimostrazione. It is easy to see that $S(F(P)) - S(P) = \sqrt{2}\, a$, for all $P \in \Gamma$. Moreover the function

$$g(s) := f_1(s) - s - \sqrt{2}\, a$$

where f_1 is the lift of F, is such that $g(s + l) = g(s)$ and $g(0) = f_1(0) - \sqrt{2}\, a \in]-l, l[$ where $l = \sqrt{2}(a + b)$. Since $S(F(P)) = f_1(S(P))$ (mod l), we can write:

$$g(S(P)) = f_1(S(P)) - S(P) - \sqrt{2}\, a = S(F(P)) - S(P) - \sqrt{2}\, a + n_P$$

where $n_P \in \mathbb{Z}$, and from above $g(S(P)) = n_P l$. If $P = 0$, then $n_0 l = g(0) \in]-l, l[$ and $n_0 = 0$. Since g is continuous, n_P is constant and then $n_P = n_0 = 0$ and hence $g(s) = 0$ i.e. $f_1(s) = s + \sqrt{2}\, a$. Therefore

$$\alpha(F) = \lim_{k\to\infty} \frac{f_k(0)}{kl} = \frac{k\sqrt{2}\, a}{k\sqrt{2}(a + b)} = \frac{a}{a + b}$$

which finishes the proof.

□

Consequently we can write$\alpha(F) = \frac{a/b}{a/b+1}$ and if we set $\beta := a/b \notin \mathbb{Q}$ it is clear that $\alpha(F)$ and β are equivalent and from corollary 2.2,

$$\beta \in \Sigma \Longleftrightarrow \alpha(F) \in \Sigma.$$

More generally if $\alpha(F)$ can be written $\alpha(F) = \frac{a\beta+b}{c\beta+d}$, with $a, b, c, d \in \mathbb{Z},\ ad - bc \neq 0$ and $\beta \notin \mathbb{Q}$, then from corollary 2.3

$$\beta \in \Sigma \Longleftrightarrow \alpha(F) \in \Sigma.$$

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AMS Subject Classification: 35L05, 11D09.

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Lavoro pervenuto in redazione il 25.9.1999 e, in forma definitiva, il 15.12.1999.