Rend. Sem. Mat. Univ. Pol. Torino
Vol. 57, 4 (1999)

**S. Challal**

# ELASTIC BEHAVIOR OF A TWO-DIMENSIONAL LATTICE BEAM

**Abstract.**
We study the linearized elasticity system and the dependence of the displacement on a small parameter $\varepsilon$ characterizing the length and the size of the period of the constitutive elements (bars or layers) of the structure. We show that, when $\varepsilon \to 0$, the structure becomes equivalent to a beam governed by the Bernoulli law.

## 1. Introduction

In this paper, we study the asymptotic behavior of linear elasticity equations in a two-dimensional domain perforated with holes periodically distributed in one direction. The size of the period is of the order of a small parameter $\varepsilon$ which is also the order of the thickness of the domain. This models many structures used in engineering such as lattice beams. We let $\varepsilon$ go to zero and look for laws governing the structure. The limit structure is governed by the Bernoulli law (see [4]). The main difficulties are to construct an extension operator of the displacement and the fact that the thickness and the period are in the same line of order. We overcome the last difficulty by using techniques developed by D. Caillerie [1] for thin elastic and periodic plates.

In section 2, we define the problem. In section 3, we give an a priori estimate and built an extension operator. Section 4 is devoted to a formal asymptotic study. In the last section we prove the convergence of our initial problem to the homogenized problem obtained in section 4.

## 2. Statement of the problem

The structure considered here is a two-dimensional periodic truss. The repeated element is named the basic cell. It may be simple (a single pattern in the basic cell) or complex (several patterns constituting the basic cell) (see Figure 1). To describe such family of structures, we denote by $Y$ the representative cell.

$$Y = (0, L) \times (-K/2, K/2), \qquad L, K > 0.$$

The part of $Y$ occupied by the material is denoted by $Y^*$, the "hole" $T = Y \setminus \overline{Y^*}$ does not intersect the boundary $\partial Y$. We assume that $\partial Y^*$ is Lipschitz continuous. We consider structures for which the number $n_0$ of elementary cells is large and the inverse $\varepsilon$ of $n_0$ will be taken as a small parameter. Our structure is then composed of identical cells which are homothetic in the ratio $\varepsilon$ to the basic cell $Y$ (see Figure 2). We denote by $L$ the length of the structure and we set:

| | | |
|---|---|---|
| $\Omega_\varepsilon$ | $=$ | $(0, L) \times (-\varepsilon K/2, \varepsilon K/2), \qquad Y_\varepsilon = \varepsilon Y, \qquad Y^*_\varepsilon = \varepsilon Y^*$ |
| $\Omega^*_\varepsilon$ | : | the part of $\Omega_\varepsilon$ occupied by the material $= \cup_{i=0}^{n_0-1} \tau_{(x_i,0)} Y^*_\varepsilon$ |
| $\tau_{(x_i,0)}$ | : | translation of vector $(x_i, 0), \qquad x_i = i\varepsilon L \qquad 0 \le i \le n_0$ |
| $T_\varepsilon$ | : | the set of holes. |

We also use the notations:

$$\begin{array}{rcl}
\Gamma_\varepsilon^1 & = & [0,L]\times\{\varepsilon K/2\}: \text{ the upper boundary of the lattice beam } \Omega_\varepsilon^* \\
\Gamma_\varepsilon^2 & = & [0,L]\times\{-\varepsilon K/2\}: \text{ the lower boundary of the lattice beam } \Omega_\varepsilon^* \\
\Gamma_0^\varepsilon & = & \{0\}\times(-\varepsilon K/2, \varepsilon K/2) \text{ (resp. } \Gamma_L^\varepsilon = \{L\}\times(-\varepsilon K/2, \varepsilon K/2)) : \\
 & & \text{the left (resp. the right) boundary of the lattice beam } \Omega_\varepsilon^*.
\end{array}$$

The current point in $\Omega_\varepsilon$ is denoted by $x = (x_1, x_2)$.

We assume the material to be anisotropic and satisfying the equations of linearized elasticity:

$$(1) \qquad \begin{cases}
\partial_j \widetilde{\sigma}_{ij}^\varepsilon + \widetilde{f}_i = 0 & i = 1,2 \quad \text{in } \Omega_\varepsilon^* \\
\widetilde{\sigma}_{ij}^\varepsilon = \widetilde{a}_{ijkh}^\varepsilon \varepsilon_{kh}(\widetilde{u}_\varepsilon) & i,j = 1,2 \\
\widetilde{\sigma}_{ij}^\varepsilon n_j = F_{i\varepsilon}^k & i = 1,2 \quad \text{on } \Gamma_\varepsilon^k \quad k = 1,2 \\
\widetilde{\sigma}_{ij}^\varepsilon n_j = 0 & i = 1,2 \quad \text{on } \partial T_\varepsilon \\
\widetilde{u}_{\varepsilon_i} = 0 & i = 1,2 \quad \text{on } \Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon
\end{cases}$$

where $n$ is the unit normal directed towards the exterior of $\Omega_\varepsilon^*$, $\widetilde{u}_\varepsilon = (\widetilde{u}_{\varepsilon_1}, \widetilde{u}_{\varepsilon_2})$ is the displacement, $(\widetilde{\sigma}_{ij}^\varepsilon)$ is the stress tensor and $\big(\varepsilon_{ij}(\widetilde{u}_\varepsilon) = \frac{1}{2}\big(\frac{\partial \widetilde{u}_{\varepsilon_i}}{\partial x_j} + \frac{\partial \widetilde{u}_{\varepsilon_j}}{\partial x_i}\big)\big)$ is the linearized strain tensor. The elasticity coefficients $\widetilde{a}_{ijkh}^\varepsilon$ are defined by:

$$\widetilde{a}_{ijkh}^\varepsilon(x) = \frac{1}{\varepsilon^2} a_{ijkh}\Big(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\Big)$$

where $a_{ijkh}(y)(i,j,k,h=1,2)$ are bounded functions defined for $y \in \mathcal{O} = \cup_{n\in\mathbb{Z}}(\tau_{(nL,0)}(Y^*))$, $Y_1$-periodic (i.e. periodic in $y_1 \in Y_1 = (0,L)$) and satisfy:

(2)
$$\begin{cases} i) & a_{ijkh}(y) = a_{jikh}(y) = a_{khij}(y) & \text{for a.e. } y \in Y^* \\ ii) & \exists m > 0 \text{ such that } \forall \tau = (\tau_{ij})_{1\le i,j\le 2} \in \mathbb{R}^4 & \tau_{ij} = \tau_{ji}\ i,j = 1,2 \\ & m\tau_{ij}\tau_{ij} \le a_{jikh}(y)\tau_{ij}\tau_{kh} & \text{for a.e. } y \in Y^* \\ iii) & \exists M \text{ such that } M = \sup_{y\in Y^*}\{a_{jikh}(y),\ i,j,k,h = 1,2\}. \end{cases}$$

We also set:

$$\begin{cases} a^{\varepsilon}_{ijkh}(x) &=& a_{ijkh}(\frac{x_1}{\varepsilon}, x_2) \\ \underline{a}^{\varepsilon}_{ijkh}(x) &=& \frac{1}{\varepsilon^2} a^{\varepsilon}_{ijkh}(x). \end{cases}$$

The structure is submitted to body forces $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2)$ and the upper and lower boundaries to forces $F^1_\varepsilon = (\dfrac{F^1_1}{\varepsilon}, F^1_2)$ and $F^2_\varepsilon = (\dfrac{F^2_1}{\varepsilon}, F^2_2)$. There is no applied surface force on the boundary of the holes and the lattice beam is supposed to be clamped on $\Gamma^\varepsilon_0 \cup \Gamma^\varepsilon_L$. We assume

$$\widetilde{f}(x_1,x_2) = f(x_1, x_2/\varepsilon) \in [L^2(\Omega_\varepsilon)]^2, \quad F^1_\varepsilon, F^2_\varepsilon \in [L^2(0,L)]^2.$$

A weak formulation of problem (1) is

$$\text{(3)} \qquad \begin{cases} \text{Find } \widetilde{u}_\varepsilon = (\widetilde{u}_{\varepsilon 1}, \widetilde{u}_{\varepsilon 2}) \in V_\varepsilon \text{ such that:} \\ \int_{\Omega^*_\varepsilon} \widetilde{\underline{a}}^{\varepsilon}_{ijkh}\varepsilon_{ij}(\widetilde{u}_\varepsilon)\varepsilon_{kh}(v) = \int_{\Omega^*_\varepsilon}\widetilde{f} + \int_{\Gamma^1_\varepsilon} F^1_\varepsilon v + \int_{\Gamma^2_\varepsilon} F^2_\varepsilon v \quad \forall v \in V_\varepsilon \end{cases}$$

where $V_\varepsilon = \{\, v \in [H^1(\Omega^*_\varepsilon)]^2 \ /\ v = 0 \text{ on } \Gamma^\varepsilon_0 \cup \Gamma^\varepsilon_L \,\}$ is a Hilbert space provided with the usual norm of $[H^1(\Omega^*_\varepsilon)]^2$.
By Lax-Milgram's theorem and Korn's inequality, we have a unique solution to problem (3).

REMARK 1. The elastic modulus depend on $\varepsilon$. This shows that the structure must be more rigid since it is more thin. The difference between the longitudinal and the transverse forces comes from the fact that the structure is more rigid under traction than under flexion.

We are interested in the dependence of $\widetilde{u}_\varepsilon$ on $\varepsilon$ (the length of the thickness of the lattice beam is $\varepsilon K$ and the period in the $x_1$ direction is $\varepsilon L$). For this, we first dilate our domain in the $x_2$ direction, then we construct an extension operator to get an estimate for the displacement in a fixed domain.

## 3. A priori estimate

Let us introduce the following new variables:

$$y_1 = x_1, \quad y_2 = \frac{x_2}{\varepsilon}.$$

Under this change of variables the set $\Omega^*_\varepsilon$ is expanded to $\mathcal{O}^*_\varepsilon$ and $\Gamma^\varepsilon_0$ (resp. $\Gamma^\varepsilon_L$, $\Gamma^k_\varepsilon$, $k = 1,2$) becomes $\Gamma_0$ (resp. $\Gamma_L$, $\Gamma^k$, $k = 1,2$). We set for any function $\widetilde{\varphi}$ defined on $\Omega^*_\varepsilon$: $\varphi(y_1,y_2) = \widetilde{\varphi}(y_1, \varepsilon y_2)$.

Then (1) can be written:

$$(4)\qquad \begin{cases} \partial_1\sigma^{\varepsilon}_{i1} + \frac{1}{\varepsilon}\partial_2\sigma^{\varepsilon}_{i2} + f_i = 0 & \text{in } \mathcal{O}^*_\varepsilon \quad i = 1,2 \\ \sigma^{\varepsilon}_{ij} = \left(\underline{a}^{\varepsilon}_{ij1k}\frac{\partial u_{\varepsilon k}}{\partial y_1} + \frac{1}{\varepsilon}\underline{a}^{\varepsilon}_{ij2k}\frac{\partial u_{\varepsilon k}}{\partial y_2}\right)(y_1,y_2) & i,j = 1,2 \\ \sigma^{\varepsilon}_{1j}n_j = \frac{F^k_1}{\varepsilon} & \text{on } \Gamma^k \quad k = 1,2 \\ \sigma^{\varepsilon}_{2j}n_j = F^k_2 & \text{on } \Gamma^k \quad k = 1,2 \\ \sigma^{\varepsilon}_{ij}n_j = 0 & \text{on } \partial\mathcal{T}_\varepsilon \quad i = 1,2 \\ u_{\varepsilon_i} = 0 & \text{on } \Gamma_0 \cup \Gamma_L \quad i = 1,2 \end{cases}$$

where $\partial\mathcal{T}_\varepsilon = \partial\mathcal{O}^*_\varepsilon \setminus \Gamma^1 \cup \Gamma^2 \cup \Gamma_0 \cup \Gamma_L$.
Let us take $\widetilde{u}_\varepsilon$ as a test function in (3) and use (2) *ii)*, we obtain:

$$(5)\qquad \frac{m}{\varepsilon^2}\int_{\Omega^*_\varepsilon} \varepsilon_{ij}(\widetilde{u}_\varepsilon)\varepsilon_{ij}(\widetilde{u}_\varepsilon) \le \int_{\Omega^*_\varepsilon} \widetilde{f}\widetilde{u}_\varepsilon + \int_{\Gamma^1_\varepsilon} F^1_\varepsilon\widetilde{u}_\varepsilon + \int_{\Gamma^2_\varepsilon} F^2_\varepsilon\widetilde{u}_\varepsilon.$$

Let $\hat{u}_\varepsilon = (\hat{u}_{\varepsilon 1}, \hat{u}_{\varepsilon 2})$ be the field defined on $\mathcal{O}^*_\varepsilon$ by:

$$(6)\qquad \hat{u}_{\varepsilon 1}(y_1,y_2) = \frac{1}{\varepsilon}\widetilde{u}_{\varepsilon 1}(y_1,\varepsilon y_2), \quad \hat{u}_{\varepsilon 2}(y_1,y_2) = \widetilde{u}_{\varepsilon 2}(y_1,\varepsilon y_2).$$

Since $\widetilde{u}_\varepsilon \in V_\varepsilon$, then $\hat{u}_\varepsilon \in H_\varepsilon$ where $H_\varepsilon$ is the Hilbert space defined by:

$$H_\varepsilon = \{\, v \in [H^1(\mathcal{O}^*_\varepsilon)]^2 \ / \ v = 0 \text{ on } \Gamma_0 \cup \Gamma_L \,\}$$

It is easy to see that we have:

$$\varepsilon_{11}(\hat{u}_\varepsilon)(y_1,y_2) = \tfrac{1}{\varepsilon}\varepsilon_{11}(\widetilde{u}_\varepsilon)(x_1,x_2), \quad \varepsilon_{22}(\hat{u}_\varepsilon)(y_1,y_2) = \varepsilon.\varepsilon_{22}(\widetilde{u}_\varepsilon)(x_1,x_2),$$
$$\varepsilon_{12}(\hat{u}_\varepsilon)(y_1,y_2) = \varepsilon_{12}(\widetilde{u}_\varepsilon)(x_1,x_2) \quad \text{with } (x_1,x_2) = (y_1,\varepsilon y_2).$$

So we deduce from (5)

$$(7)\qquad \begin{aligned} &m\varepsilon \textstyle\int_{\mathcal{O}^*_\varepsilon} (\varepsilon_{11}(\hat{u}_\varepsilon))^2 + \frac{1}{\varepsilon^4}(\varepsilon_{22}(\hat{u}_\varepsilon))^2 + \frac{2}{\varepsilon^2}(\varepsilon_{12}(\hat{u}_\varepsilon))^2 \\ &\le \textstyle\int_{\mathcal{O}^*_\varepsilon} \varepsilon[\varepsilon f_1\hat{u}_{\varepsilon 1} + f_2\hat{u}_{\varepsilon 2}] + \int_{\Gamma^1} F^1_1\hat{u}_{\varepsilon 1} + F^1_2\hat{u}_{\varepsilon 2} + \int_{\Gamma^2} F^2_1\hat{u}_{\varepsilon 1} + F^2_2\hat{u}_{\varepsilon 2}. \end{aligned}$$

Let us now prove the following lemma:

LEMMA 1. *Let $H$ be the Hilbert space defined by: $H = \{\, v \in [H^1(Y)]^2 \ / \ v = 0$ on $\Gamma_0 \cup \Gamma_L \,\}$. There exists an extension operator $P_\varepsilon \in \mathcal{L}(H_\varepsilon, H)$ such that:*

$$(8)\qquad \int_Y \varepsilon_{ij}(P_\varepsilon v)\varepsilon_{ij}(P_\varepsilon v) \le c\int_{\mathcal{O}^*_\varepsilon} (\varepsilon_{11}(v))^2 + \frac{1}{\varepsilon^4}(\varepsilon_{22}(v))^2 + \frac{2}{\varepsilon^2}(\varepsilon_{12}(v))^2 \ \forall v \in H_\varepsilon$$

*where $c$ is a constant independent of $\varepsilon$.*

*Proof.* It is done in two steps:
$1^{st}$ *step:* it is a lemma due to Conca [3].

LEMMA 2. *There exists an extension operator $S \in \mathcal{L}([H^1(Y^*)]^2, [H^1(Y)]^2)$ and a constant $c$ such that:*

$$(9)\qquad \int_Y \varepsilon_{ij}(Sw)\varepsilon_{ij}(Sw) \le c\int_{Y^*} \varepsilon_{ij}(w)\varepsilon_{ij}(w) \qquad \forall w \in [H^1(Y^*)]^2.$$

$2^{nd}$ *step:* from definition of $\mathcal{O}_\varepsilon^*$ we have: $\mathcal{O}_\varepsilon^* = \cup_{i=0}^{n-1} \tau_{(x_i,0)}(\varphi(Y^*))$ where $\varphi$ is the change of variable defined by: $\varphi : (y_1, y_2) \longmapsto (\varepsilon y_1, y_2)$.

First let $v \in [H^1(\varphi(Y^*))]^2$. Then $vo\varphi \in [H^1(Y^*)]^2$ and the function $w$ defined by $w = (v_1 o\varphi, \dfrac{1}{\varepsilon} v_2 o\varphi) \in [H^1(Y^*)]^2$. From Lemma 2, $Sw \in [H^1(Y)]^2$ and satisfies (9). Set

$$\widetilde{S}v = ((Sw)_1 o\varphi^{-1}, \varepsilon (Sw)_2 o\varphi^{-1}).$$

We have $\widetilde{S}v \in [H^1(\varphi(Y))]^2$ and the following inequality:

$$(10) \qquad \int_{\varphi(Y)} \varepsilon_{ij}(\widetilde{S}v)\varepsilon_{ij}(\widetilde{S}v) \le c \int_{\varphi(Y^*)} (\varepsilon_{11}(v))^2 + \frac{1}{\varepsilon^4}(\varepsilon_{22}(v))^2 + \frac{2}{\varepsilon^2}(\varepsilon_{12}(v))^2$$

indeed we have

$$\begin{array}{rcl} \int_{\varphi(Y)} \varepsilon_{ij}(\widetilde{S}v)\varepsilon_{ij}(\widetilde{S}v) & = & \varepsilon \int_Y \frac{1}{\varepsilon^2}(\varepsilon_{11}(Sw))^2 + \varepsilon^2(\varepsilon_{22}(Sw))^2 + 2(\varepsilon_{12}(Sw))^2 \\ & \le & \frac{1}{\varepsilon} \int_Y \varepsilon_{ij}(Sw)\varepsilon_{ij}(Sw) \qquad \text{(since } \varepsilon < 1) \\ & \le & \frac{c}{\varepsilon} \int_{Y^*} \varepsilon_{ij}(w)\varepsilon_{ij}(w) \qquad \text{(by (9))} \end{array}$$

and

$$\int_{Y^*} \varepsilon_{ij}(w)\varepsilon_{ij}(w) = \frac{1}{\varepsilon} \int_{\varphi(Y^*)} \varepsilon^2(\varepsilon_{11}(v))^2 + \frac{1}{\varepsilon^2}(\varepsilon_{22}(v))^2 + 2(\varepsilon_{12}(v))^2$$

then (10) holds.

Next, if $v \in [H^1(\mathcal{O}_\varepsilon^*)]^2$, we define the extension operator $P_\varepsilon$ by:

$$\begin{array}{rcl} P_\varepsilon v_{|_{\varphi(Y)}} & = & \widetilde{S}(v_{|_{\varphi(Y^*)}}) \\ P_\varepsilon v_{|_{\tau(x_i,0)(\varphi(Y))}} & = & \widetilde{S}(v_{|_{\tau(x_i,0)(\varphi(Y^*))}} o\tau(x_i,0))o\tau(-x_i,0) \quad i = 1, \dots, n_0 - 1 \end{array}$$

and we verify the inequality (8).

□

We have also the following inequalities:

LEMMA 3. *There exists a constant c independent of* $\varepsilon$ *such that:* $\forall v \in H_\varepsilon$*, we have:*

$$\begin{array}{llcl} i) & \int_Y (\varepsilon_{11}(P_\varepsilon v))^2 & \le & c \int_{\mathcal{O}_\varepsilon^*} (\varepsilon_{11}(v))^2 + \frac{1}{\varepsilon^4}(\varepsilon_{22}(v))^2 + \frac{2}{\varepsilon^2}(\varepsilon_{12}(v))^2 \\ ii) & \int_Y (\varepsilon_{22}(P_\varepsilon v))^2 & \le & c\varepsilon^4 \int_{\mathcal{O}_\varepsilon^*} (\varepsilon_{11}(v))^2 + \frac{1}{\varepsilon^4}(\varepsilon_{22}(v))^2 + \frac{2}{\varepsilon^2}(\varepsilon_{12}(v))^2 \\ iii) & \int_Y (\varepsilon_{12}(P_\varepsilon v))^2 & \le & c\varepsilon^2 \int_{\mathcal{O}_\varepsilon^*} (\varepsilon_{11}(v))^2 + \frac{1}{\varepsilon^4}(\varepsilon_{22}(v))^2 + \frac{2}{\varepsilon^2}(\varepsilon_{12}(v))^2. \end{array}$$

*Proof.* Using the same notations as in the proof of Lemma 2, we have for $v \in H_\varepsilon$:

$$\begin{array}{rcl} \int_{\varphi(Y)} (\varepsilon_{11}(\widetilde{S}v))^2 & = & \varepsilon \int_Y \frac{1}{\varepsilon^2}(\varepsilon_{11}(Sw))^2 \\ & \le & \frac{1}{\varepsilon} \int_Y \varepsilon_{ij}(Sw)\varepsilon_{ij}(Sw) \\ & \le & \frac{c}{\varepsilon} \int_{Y^*} \varepsilon_{ij}(w)\varepsilon_{ij}(w) \\ & \le & c \int_{\varphi(Y^*)} (\varepsilon_{11}(v))^2 + \frac{1}{\varepsilon^4}(\varepsilon_{22}(v))^2 + \frac{2}{\varepsilon^2}(\varepsilon_{12}(v))^2 \end{array}$$

then we deduce *i)*. To prove *ii)* and *iii)* one can see that we have:

$$\begin{array}{rcl} \int_{\varphi(Y)} (\varepsilon_{22}(\widetilde{S}v))^2 & = & \varepsilon \int_Y \varepsilon^2(\varepsilon_{22}(Sw))^2 \\ & \le & \varepsilon^3 \int_Y \varepsilon_{ij}(Sw)\varepsilon_{ij}(Sw) \\ & \le & c\varepsilon^3 \int_{Y^*} \varepsilon_{ij}(w)\varepsilon_{ij}(w) \\ & \le & c\varepsilon^4 \int_{\varphi(Y^*)} (\varepsilon_{11}(v))^2 + \frac{1}{\varepsilon^4}(\varepsilon_{22}(v))^2 + \frac{2}{\varepsilon^2}(\varepsilon_{12}(v))^2 \end{array}$$

and

$$\begin{array}{rcl}\int_{\varphi(Y)}(\varepsilon_{12}(\widetilde{S}v))^2 & = & \varepsilon\int_Y(\varepsilon_{12}(Sw))^2 \\ & \leq & \varepsilon\int_Y \varepsilon_{ij}(Sw)\varepsilon_{ij}(Sw) \\ & \leq & c\varepsilon\int_{Y^*}\varepsilon_{ij}(w)\varepsilon_{ij}(w) \\ & \leq & c\varepsilon^2\int_{\varphi(Y^*)}(\varepsilon_{11}(v))^2+\frac{1}{\varepsilon^4}(\varepsilon_{22}(v))^2+\frac{2}{\varepsilon^2}(\varepsilon_{12}(v))^2.\end{array}$$

□

COROLLARY 1. *Let $\hat{u}_\varepsilon$ defined by* (6). *Then we have:*

$$\begin{array}{llllllll} i) & |P_\varepsilon\hat{u}_\varepsilon|_H & \leq & c/\varepsilon, & ii) & |\varepsilon_{11}(P_\varepsilon\hat{u}_\varepsilon)|_{L^2(Y)} & \leq & c/\varepsilon \\ iii) & |\varepsilon_{22}(P_\varepsilon\hat{u}_\varepsilon)|_{L^2(Y)} & \leq & c\varepsilon, & iv) & |\varepsilon_{12}(P_\varepsilon\hat{u}_\varepsilon)|_{L^2(Y)} & \leq & c \end{array}$$

*where $c$ is a constant independent of $\varepsilon$.*

*Proof.* From (7), Korn's inequality and Lemma 1, we get:

$$\begin{array}{rcl}|P_\varepsilon\hat{u}_\varepsilon|_H^2 & \leq & \frac{c}{\varepsilon}\big[|\overline{f}|_{L^2(Y)}|P_\varepsilon\hat{u}_\varepsilon|_{L^2(Y)} \\ & + & |F^1|_{L^2(\Gamma^1)}|\gamma(P_\varepsilon\hat{u}_\varepsilon)|_{L^2(\Gamma^1)} \\ & + & |F^2|_{L^2(\Gamma^2)}|\gamma(P_\varepsilon\hat{u}_\varepsilon)|_{L^2(\Gamma^2)}\big]\end{array}$$

where '$-$' denotes the extension by 0 in $Y\setminus\mathcal{O}_\varepsilon^*$ and $\gamma$ the trace operator.
Now by Poincaré's inequality and the continuity of $\gamma$, we get *i)*.
*ii)*, *iii)* and *iv)* are consequences of lemma 3 and the following inequality:

$$\int_{\mathcal{O}_\varepsilon^*}(\varepsilon_{11}(\hat{u}_\varepsilon))^2+\frac{1}{\varepsilon^4}(\varepsilon_{22}(\hat{u}_\varepsilon))^2+\frac{2}{\varepsilon^2}(\varepsilon_{12}(\hat{u}_\varepsilon))^2\leq\frac{c}{\varepsilon}|P_\varepsilon\hat{u}_\varepsilon|_H\leq\frac{c}{\varepsilon^2}. \tag{11}$$

□

Now, we deduce estimates on the stress tensor defined by:

$$\hat{\sigma}_{ij}^\varepsilon=\varepsilon\sigma_{ij}^\varepsilon=\varepsilon\big(\underline{a}_{ij1k}^\varepsilon\frac{\partial u_{\varepsilon k}}{\partial y_1}+\frac{1}{\varepsilon}\underline{a}_{ij2k}^\varepsilon\frac{\partial u_{\varepsilon k}}{\partial y_2}\big)(y_1,y_2)\qquad(y_1,y_2)\in\mathcal{O}_\varepsilon^*$$

which can be written

$$\hat{\sigma}_{ij}^\varepsilon=\varepsilon\big[\varepsilon\underline{a}_{ij11}^\varepsilon\varepsilon_{11}(\hat{u}_\varepsilon)+2\underline{a}_{ij12}^\varepsilon\varepsilon_{12}(\hat{u}_\varepsilon)+\frac{1}{\varepsilon}\underline{a}_{ij22}^\varepsilon\varepsilon_{22}(\hat{u}_\varepsilon)\big].$$

Then by (11) we have

$$|\hat{\sigma}_{ij}^\varepsilon|_{L^2(\mathcal{O}_\varepsilon^*)}\leq c\big[|\varepsilon_{11}(\hat{u}_\varepsilon)|_{L^2(\mathcal{O}_\varepsilon^*)}+\frac{1}{\varepsilon}|\varepsilon_{12}(\hat{u}_\varepsilon)|_{L^2(\mathcal{O}_\varepsilon^*)}+\frac{1}{\varepsilon^2}|\varepsilon_{22}(\hat{u}_\varepsilon)|_{L^2(\mathcal{O}_\varepsilon^*)}\big]\leq c/\varepsilon$$

and

$$|\overline{\hat{\sigma}_{ij}^\varepsilon}|_{L^2(Y)}\leq c/\varepsilon. \tag{12}$$

As a consequence of estimates of Corollary 1 *i)*, (12) and the fact that the spaces $H^1(Y)$ and $L^2(Y)$ are reflexif, we obtain:

THEOREM 1. *There exists* $\hat{u}^* \in [H^1(Y)]^2$ *and* $\hat{\sigma}^* \in [L^2(Y)]^4$ *such that we have, up to a subsequence,*

$$\begin{aligned} \varepsilon.(P_\varepsilon(\hat{u}_\varepsilon)) &\rightharpoonup \hat{u}^* \quad \text{in } [H^1(Y)]^2 \\ \varepsilon.(\overline{\hat{\sigma}^\varepsilon}) &\rightharpoonup \hat{\sigma}^* \quad \text{in } [L^2(Y)]^4. \end{aligned} \tag{13}$$

In order to identify $\hat{u}^*$ and $\hat{\sigma}^*$, we first do a formal study in the following section which allows us to get the homogenized problem. In Section 5, we justify this result.

## 4. A formal asymptotic study

In this section, we use a formal method to give the asymptotic behavior of the structure. For this, we look for $u_\varepsilon$ and $\sigma^\varepsilon$ in the form:

$$\begin{aligned} u_{\varepsilon i}(x_1, y_2) &= \tfrac{1}{\varepsilon} u_i^{-1}(x_1) + u_i^0(x_1, \tfrac{x_1}{\varepsilon}, y_2) + \varepsilon u_i^1(x_1, \tfrac{x_1}{\varepsilon}, y_2) + \\ &\quad \varepsilon^2 u_i^2(x_1, \tfrac{x_1}{\varepsilon}, y_2) + \ldots \\ \sigma_{ij}^\varepsilon(x_1, y_2) &= \tfrac{1}{\varepsilon^3}\sigma_{ij}^{-3}(x_1) + \tfrac{1}{\varepsilon^2}\sigma_{ij}^{-2}(x_1, \tfrac{x_1}{\varepsilon}, y_2) + \\ &\quad \tfrac{1}{\varepsilon}\sigma_{ij}^{-1}(x_1, \tfrac{x_1}{\varepsilon}, y_2) + \ldots, \quad i, j = 1, 2 \end{aligned}$$

where the functions $u_i^m(x_1, y_1, y_2)$, $\sigma_{ij}^m(x_1, y_1, y_2)$ are $Y_1$-periodic ($y_1 \in Y_1 = (0, L)$) ($y_1 = \frac{x_1}{\varepsilon}$).

We take back these expansions in the equations of (4) and we identify the terms of the same line of order of $\varepsilon$. We obtain:

$$\left\{ \begin{aligned} &\frac{\partial \sigma_{i1}^{-3}}{\partial y_1} + \frac{\partial \sigma_{i2}^{-3}}{\partial y_2} &&= 0 \\ &\frac{\partial \sigma_{i1}^{m}}{\partial x_1} + \frac{\partial \sigma_{i1}^{m}}{\partial y_1} + \frac{\partial \sigma_{i2}^{m+1}}{\partial y_2} &&= 0 \quad \text{for } m \neq 0 \\ &\frac{\partial \sigma_{i1}^{0}}{\partial x_1} + \frac{\partial \sigma_{i1}^{1}}{\partial y_1} + \frac{\partial \sigma_{i2}^{1}}{\partial y_2} + f_i &&= 0 \quad \text{for } m = 0 \end{aligned} \right. \tag{14}$$

$$\left\{ \begin{aligned} &\sigma_{1j}^m n_j = 0 \quad \text{for } m \neq -1, \quad \sigma_{1j}^{-1} n_j = F_1^k \\ &\sigma_{2j}^m n_j = 0 \quad \text{for } m \neq 0, \quad \sigma_{2j}^{-1} n_j = F_2^k \qquad \text{on } \Gamma^k \quad k = 1, 2 \\ &\sigma_{ij}^m n_j = 0 \quad \forall m \text{ on } \partial Y_{int}^*, \quad \forall x_1 \in (0, L) \quad i = 1, 2 \end{aligned} \right. \tag{15}$$

where $\partial Y_{int}^*$ denotes the interior boundary of $Y^*$ and

$$\sigma_{ij}^m = a_{ij1k} \frac{\partial u_k^{m+2}}{\partial x_1} + a_{ijhk} \frac{\partial u_k^{m+3}}{\partial y_h} \qquad \forall m. \tag{16}$$

Let us define:

$$\left\{ \begin{aligned} &N^m(x_1) = \tfrac{1}{|Y_1|} \textstyle\int_{Y^*} \sigma_{11}^m(x_1, y_1, y_2) dy_1 dy_2 = \underline{\sigma}_{11}^m && : \text{the normal force} \\ &T^m(x_1) = \tfrac{1}{|Y_1|} \textstyle\int_{Y^*} \sigma_{12}^m(x_1, y_1, y_2) dy_1 dy_2 = \underline{\sigma}_{12}^m && : \text{the transverse shearing force} \\ &M^m(x_1) = \tfrac{1}{|Y_1|} \textstyle\int_{Y^*} y_2 \sigma_{11}^m(x_1, y_1, y_2) dy_1 dy_2 && : \text{the bending couple.} \end{aligned} \right.$$

Integrating the equilibrium equations (14) on $Y^*$ and using the boundary conditions (15), we obtain

$$\begin{cases} \frac{\partial \sigma_{i1}^m}{\partial x_1} = 0 & \forall m \neq -2, -1, 0 \\ \frac{\partial \sigma_{11}^{-2}}{\partial x_1} + F_1^1 + F_1^2 = 0 \quad \text{and} \quad \frac{\partial \sigma_{11}^{-2}}{\partial x_1} = 0 \\ \frac{\partial \sigma_{11}^{-1}}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial \sigma_{11}^{-1}}{\partial x_1} + F_2^1 + F_2^2 = 0. \end{cases}$$

Integrating the equation $\dfrac{\partial \sigma_{11}^{-2}}{\partial x_1} + \dfrac{\partial \sigma_{1j}^{-1}}{\partial y_j} = 0$ on $Y^*$ after multiplying by $y_2$, we get

$$\frac{\partial}{\partial x_1}\Big(\int_{Y^*} y_2 \sigma_{11}^{-2}\Big) - \int_{Y^*} \sigma_{12}^{-1} + |Y_1| \frac{K}{2}(F_1^1 - F_1^2) = 0.$$

Then we deduce the equilibrium equation of the homogenized structure:

$$\begin{cases} \frac{dN^{-2}}{dx_1}(x_1) = -(F_1^1 + F_1^2) \\ \frac{dT^{-1}}{dx_1}(x_1) = -(F_2^1 + F_2^2) \\ \frac{dM^{-2}}{dx_1}(x_1) - T^{-1}(x_1) = -\frac{K}{2}(F_1^1 - F_1^2). \end{cases} \tag{17}$$

Now, we are looking for the constitutive law of the structure.
First let us consider the problem satisfied by $\sigma_{ij}^{-3}$. We have:

$$\begin{cases} \frac{\partial \sigma_{ij}^{-3}}{\partial y_j} = 0 \quad \text{in } Y^*, \quad \sigma_{ij}^{-3} n_j = 0 \quad \text{on } \partial Y_{int}^* \cup \Gamma^1 \cup \Gamma^2 \\ \sigma_{ij}^{-3} \quad Y_1 - \text{periodic}, \quad \sigma_{ij}^{-3} = a_{ij1k} \frac{\partial u_k^{-1}}{\partial x_1} + a_{ijhk} \frac{\partial u_k^0}{\partial y_h}. \end{cases} \tag{18}$$

In these equations the variable $x_1$ appears like a parameter. If we suppose the function $u^{-1}$ well-known, then we can write problem (18) as:

$$\begin{cases} \frac{\partial}{\partial y_j}\Big(a_{ijhk} \frac{\partial u_k^0}{\partial y_h}\Big) = -\frac{\partial}{\partial y_j}\Big(a_{ij1k} \frac{\partial u_k^{-1}}{\partial x_1}\Big) \quad \text{in } Y^* \\ a_{ijhk} \frac{\partial u_k^0}{\partial y_h} n_j = -a_{ij1k} \frac{\partial u_k^{-1}}{\partial x_1} n_j \quad \text{on } \partial Y_{int}^* \cup \Gamma^1 \cup \Gamma^2 \\ u_k^0 \quad Y_1\text{-periodic}. \end{cases} \tag{19}$$

A weak formulation associated to (19) is:

$$\begin{cases} \text{Find } u^0 \in \mathcal{W}(Y^*) \text{ such that :} \\ \int_{Y^*} a_{ijhk} \frac{\partial u_k^0}{\partial y_h} \frac{\partial \psi_i}{\partial y_j} = -\int_{Y^*} a_{ij1k} \frac{\partial u_k^{-1}}{\partial x_1} \frac{\partial \psi_i}{\partial y_j} \quad \forall \psi \in \mathcal{W}(Y^*) \end{cases} \tag{20}$$

where $\mathcal{W}(Y^*)$ is the Hilbert space defined by:
$\mathcal{W}(Y^*) = \{\psi \in H_{loc}^1(\mathcal{O}), Y_1\text{-periodic such that} \displaystyle\int_{Y^*} \psi = 0\}$ and equiped with the norm $\|\psi\|_{\mathcal{W}} = \big(\varepsilon_{ij}(\psi)\varepsilon_{ij}(\psi)\big)^{1/2}$.
Applying Lax-Milgram's theorem, we conclude the existence and uniqueness of a solution $u^0$ of (19).
The linearity of problem (20) allows us to introduce the following functions

$$\begin{cases} \text{Find } \chi^{\alpha 1} \in \mathcal{W}(Y^*) \text{ such that :} \\ \int_{Y^*} a_{ijhk} \frac{\partial \chi_k^{\alpha 1}}{\partial y_h} \frac{\partial \psi_i}{\partial y_j} = -\int_{Y^*} a_{ij1\alpha} \frac{\partial \psi_i}{\partial y_j} \quad \forall \psi \in \mathcal{W}(Y^*) \end{cases} \tag{21}$$

and since $u^{-1}$ depends only on $x_1$, we can write for a function $\check{u}^0(x_1)$: $u_k^0 = \chi_k^{\alpha 1}\dfrac{\partial u_\alpha^{-1}}{\partial x_1} + \check{u}_k^0(x_1)$.
From definition (21) of $\chi^{\alpha 1}$, we can verify that $\chi_k^{21} = (m(y_2) - y_2)\delta_{k1}$ with $m(y_2) = \dfrac{1}{|Y^*|}\int_{Y^*} y_2$. So $u^0$ can be written:
$u_k^0 = \chi_k^{11}\frac{\partial u_1^{-1}}{\partial x_1} + (m(y_2) - y_2)\delta_{k1}\frac{\partial u_2^{-1}}{\partial x_1} + \check{u}_k^0(x_1)$ and $\sigma_{ij}^{-3} = \big(a_{ij11} + a_{ijkh}\frac{\partial \chi_k^{11}}{\partial y_h}\big)\frac{\partial u_1^{-1}}{\partial x_1}$.
Then integrating on $Y^*$, we get:

$$\underline{\sigma}_{ij}^{-3} = c_{ij}\frac{\partial u_1^{-1}}{\partial x_1} \quad \text{with} \quad c_{ij} = \frac{1}{|Y^*|}\int_{Y^*}\big(a_{ij11} + a_{ijkh}\frac{\partial \chi_k^{11}}{\partial y_h}\big).$$

Taking $\psi = (y_2 - m(y_2), 0)$ (resp. $\psi = (0, y_2 - m(y_2))$) in (21) for $\alpha = 1$, we obtain $c_{12} = c_{21} = 0$ (resp. $c_{22} = 0$). So $u_1^{-1}$ is a solution of the problem:

$$\text{(22)} \qquad \begin{cases} \frac{d}{dx_1}\big(c_{11}\frac{du_1^{-1}}{dx_1}\big) = 0 & \text{in } (0, L) \\ u_1^{-1}(0) = u_1^{-1}(L) = 0. \end{cases}$$

LEMMA 4. *We have :* $c_{11} > 0$.

*Proof.* Let $\psi = (y_1, 0)$ then $a_{11kh}\dfrac{\partial \psi_k}{\partial y_h} = a_{1111}$ and
$c_{11} = \dfrac{1}{|Y_1|}\int_{Y^*} a_{11kh}\dfrac{\partial}{\partial y_h}(\psi_k + \chi_k^{11})$.

Now take $\chi^{11}$ as a test function in (21) for $\alpha = 1$, we get:

$$\int_{Y^*} a_{ijkh}\frac{\partial}{\partial y_h}(\psi_k + \chi_k^{11})\frac{\partial \chi_i^{11}}{\partial y_j} = 0.$$

Then

$$\begin{aligned} c_{11} &= \frac{1}{|Y_1|}\int_{Y^*} a_{11kh}\frac{\partial}{\partial y_h}(\psi_k + \chi_k^{11}) + \frac{1}{|Y_1|}\int_{Y^*} a_{ijkh}\frac{\partial}{\partial y_h}(\psi_k + \chi_k^{11})\frac{\partial \chi_i^{11}}{\partial y_j} \\ &= \frac{1}{|Y_1|}\int_{Y^*} a_{ijkh}\frac{\partial}{\partial y_h}(\psi_k + \chi_k^{11})\frac{\partial}{\partial y_j}(\psi_i + \chi_i^{11}). \end{aligned}$$

By (2) *ii)*, we have: $c_{11} \geq \dfrac{m}{|Y_1|}\|\psi + \chi^{11}\|_{\mathcal{W}}$.
If $\|\psi + \chi^{11}\|_{\mathcal{W}} = 0$ then $\psi + \chi^{11} = (a + by_1, a - by_2)$ with $a, b \in \mathbb{R}$. So $\chi^{11} - (a, a - by_2) = (b-1)(y_1, 0)$. If $b = 1$, $\chi^{11} = (a, a - y_2)$ and satisfy $\displaystyle\int_{Y^*} a_{ijkh}\frac{\partial \chi_k^{11}}{\partial y_h}\frac{\partial \varphi_i}{\partial y_j} = -\int_{Y^*} a_{ij22}\frac{\partial \varphi_i}{\partial y_j}$ $\forall \varphi \in \mathcal{W}(Y^*)$ which contradicts (21). So $b \neq 1$. But $\chi^{11} - (a, a - by_2)$ is $Y_1$-periodic and $(b-1)(y_1, 0)$ is not $Y_1$-periodic. The lemma follows.

□

From this lemma and (22), we deduce that $u_1^{-1} = 0$ and $\sigma^{-1} = 0$. So $u^0$ becomes equal to:

$$\text{(23)} \qquad u_1^0 = (m(y_2) - y_2)\frac{\partial u_2^{-1}}{\partial x_1} + \check{u}_1^0(x_1) \quad \text{and} \quad u_2^0 = \check{u}_2^0(x_1).$$

To get more information, we compute now functions $u^1$ and $\sigma^{-2}$.
Using (14), (15) and (16), let us consider the problem:

$$(24)\qquad \begin{cases} \frac{\partial \sigma_{ij}^{-2}}{\partial y_j} = 0 \quad \text{in } Y^*, \quad \sigma_{ij}^{-2} n_j = 0 \quad \text{on } \partial Y^*_{int} \cup \Gamma^1 \cup \Gamma^2 \\ \sigma_{ij}^{-2} \quad Y_1 - \text{periodic}, \quad \sigma_{ij}^{-2} = a_{ij1k} \frac{\partial u_k^0}{\partial x_1} + a_{ijhk} \frac{\partial u_k^1}{\partial y_h}. \end{cases}$$

A weak formulation of (24) is:

$$\begin{cases} \text{Find } u^1 \in \mathcal{W}(Y^*) \text{ such that :} \\ \int_{Y^*} a_{ijhk} \frac{\partial u_k^1}{\partial y_h} \frac{\partial \psi_i}{\partial y_j} = -\int_{Y^*} a_{ij1k} \frac{\partial u_k^0}{\partial x_1} \frac{\partial \psi_i}{\partial y_j} \qquad \forall \psi \in \mathcal{W}(Y^*). \end{cases}$$

Noting the linearity of this problem, let us consider $\chi^{12}$ be the unique solution of :

$$(25)\qquad \begin{cases} \text{Find } \chi^{12} \in \mathcal{W}(Y^*) \text{ such that :} \\ \int_{Y^*} a_{ijhk} \frac{\partial \chi_k^{12}}{\partial y_h} \frac{\partial \psi_i}{\partial y_j} = -\int_{Y^*} a_{ij11} (m(y_2) - y_2) \frac{\partial \psi_i}{\partial y_j} \qquad \forall \psi \in \mathcal{W}(Y^*). \end{cases}$$

Then we can write:

$$u_k^1 = \chi_k^{11} \frac{\partial \check{u}_1^0}{\partial x_1} + (m(y_2) - y_2)\delta_{k1} \frac{\partial \check{u}_2^0}{\partial x_1} + \chi_k^{12} \frac{\partial^2 u_2^{-1}}{\partial x_1^2} + \check{u}_k^1(x_1)$$

and

$$\sigma_{ij}^{-2} = \big[a_{ij11} + a_{ijkh} \frac{\partial \chi_k^{11}}{\partial y_h}\big] \frac{\partial \check{u}_1^0}{\partial x_1} + \big[a_{ij11}(m(y_2) - y_2) + a_{ijkh} \frac{\partial \chi_k^{12}}{\partial y_h}\big] \frac{\partial^2 u_2^{-1}}{\partial x_1^2}.$$

Integrating on $Y^*$, we obtain the constitutive law of the structure

$$(26)\qquad N^{-2}(x_1) = s^{11} \frac{\partial \check{u}_1^0}{\partial x_1} + s^{12} \frac{\partial^2 u_2^{-1}}{\partial x_1^2}, \qquad M^{-2}(x_1) = s^{21} \frac{\partial \check{u}_1^0}{\partial x_1} + s^{22} \frac{\partial^2 u_2^{-1}}{\partial x_1^2}$$

where $(s^{\alpha\nu})$ $(\alpha, \nu = 1, 2)$ are given by:

$$(27)\qquad \begin{cases} s^{11} = \frac{1}{|Y_1|} \int_{Y^*} \big[a_{11kh} \frac{\partial \chi_k^{11}}{\partial y_h} + a_{1111}\big], \\ s^{12} = \frac{1}{|Y_1|} \int_{Y^*} \big[a_{11kh} \frac{\partial \chi_k^{12}}{\partial y_h} + a_{1111}(m(y_2) - y_2)\big] \\ s^{21} = \frac{1}{|Y_1|} \int_{Y^*} y_2 \big[a_{11kh} \frac{\partial \chi_k^{11}}{\partial y_h} + a_{1111}\big], \\ s^{22} = \frac{1}{|Y_1|} \int_{Y^*} y_2 \big[a_{11kh} \frac{\partial \chi_k^{12}}{\partial y_h} + a_{1111}(m(y_2) - y_2)\big]. \end{cases}$$

Taking into account the boundary conditions of $u_\varepsilon$ on $\Gamma_0 \cup \Gamma_L$ and (23), the homogenized problem is finally given by (17), (26), (27) and the boundary conditions:

$$(28)\qquad u_2^{-1}(0) = u_2^{-1}(L) = 0, \quad \frac{\partial u_2^{-1}}{\partial x_1}(0) = \frac{\partial u_2^{-1}}{\partial x_1}(L) = 0, \quad \check{u}_1^0(0) = \check{u}_1^0(L) = 0.$$

The remainder of this section is devoted to prove the existence and uniqueness of a solution to the homogenized problem. It suffices for this to verify that the matrix $S = (s^{\mu\nu})_{\mu,\nu=1,2}$ is invertible. First, we introduce the following matrix:

$$S^* = \begin{pmatrix} s_*^{11} & s_*^{12} \\ s_*^{21} & s_*^{22} \end{pmatrix} = \begin{pmatrix} s^{11} & s^{12} \\ m(y_2)s^{11} - s^{21} & m(y_2)s^{12} - s^{22} \end{pmatrix}$$

We remark that $det S^* = -det S = -(s^{11}s^{22} - s^{21}s^{12})$.
The advantage to introduce this matrix is to get a unified form for the coefficients $(s_*^{\mu\nu})$. Indeed we have:

(29) $$s_*^{\mu\nu} = \frac{1}{|Y_1|}\int_{Y^*}(m(y_2)-y_2)^{\mu-1}a_{ijkh}\Big[\frac{\partial \chi_k^{1\nu}}{\partial y_h} + (m(y_2)-y_2)^{\nu-1}\delta_{k1}\delta_{h1}\Big]\delta_{i1}\delta_{j1},$$
$\mu, \nu = 1, 2.$

From (21) and (25), we have:

(30) $$\int_{Y^*} a_{ijkh}\Big[\frac{\partial \chi_k^{1\nu}}{\partial y_h} + (m(y_2)-y_2)^{\nu-1}\delta_{k1}\delta_{h1}\Big]\frac{\partial \chi_i^{1\mu}}{\partial y_j} = 0.$$

Adding (29) and (30), we obtain:

$$s_*^{\mu\nu} = \frac{1}{|Y_1|}\int_{Y^*} a_{ijkh}\Big[\frac{\partial \chi_k^{1\nu}}{\partial y_h} + (m(y_2)-y_2)^{\nu-1}\delta_{k1}\delta_{h1}\Big]\Big[\frac{\partial \chi_i^{1\mu}}{\partial y_j} + (m(y_2)-y_2)^{\mu-1}\delta_{i1}\delta_{j1}\Big].$$

From coerciveness of coefficients $a_{ijkh}$, we get $s_*^{\mu\nu} = s_*^{\nu\mu}$ for $\mu, \nu = 1, 2$ and we deduce:

COROLLARY 2. *We have:* $s^{12} = m(y_2)s^{11} - s^{21}$.

Moreover we have:

LEMMA 5. *The matrix $S^*$ satisfies for some $\alpha > 0$:* $s_*^{\nu\mu}\xi^\nu\xi^\mu \geq \alpha|\xi|^2, \ \forall \xi \in \mathbb{R}^2$.

*Proof.* Let $\xi = (\xi^1, \xi^2) \in \mathbb{R}^2$ and set $w_{ij} = \xi^\mu[\varepsilon_{ij}(\chi^{1\mu}) + (m(y_2)-y_2)^{\mu-1}\delta_{i1}\delta_{j1}]$. Then we have:

$$s_*^{\nu\mu}\xi^\nu\xi^\mu = \frac{1}{|Y_1|}\int_{Y^*} a_{ijkh}w_{ij}w_{kh} \geq \frac{m}{|Y_1|}|w|^2_{0,Y^*}.$$

Suppose that there exists $\xi \in \mathbb{R}^2 \setminus \{0\}$ such that $s_*^{\nu\mu}\xi^\nu\xi^\mu = 0$. Then $w_{ij} = 0 \ \forall i, j = 1, 2$ i.e.

$$\varepsilon_{ij}(\xi^\mu\chi^{1\mu}) = -\xi^\mu(m(y_2)-y_2)^{\mu-1}\delta_{i1}\delta_{j1} \qquad \forall i, j = 1, 2.$$

For $(i, j) = (1, 1)$, we get for a function $f(y_2)$: $\xi^\mu\chi_1^{1\mu} = -\xi^\mu(m(y_2)-y_2)^{\mu-1}y_1 + f(y_2)$. Then taking into account the periodicity of $\chi^{11}$ or $\chi^{12}$ we get a contradiction, so $s_*^{\nu\mu}\xi^\nu\xi^\mu > 0$ $\forall \xi \in \mathbb{R}^2 \setminus \{0\}$.

□

Using Lemma 5, we deduce the following theorem:

THEOREM 2. *There exists a unique solution of the homogenized problem given by* (17), (26), (27) *and* (28).

## 5. The results of convergence

In this section, we pass to the limit and obtain the homogenized problem.
Let $S^\varepsilon$ be the subset of $(-K/2, K/2)$ defined by:

$$S^\varepsilon(x_1) = \{\, y_2 \;/\; (x_1, y_2) \in \mathcal{O}_\varepsilon^* \,\}.$$

We have $|S^\varepsilon(x_1)| \leq K$, where $|S^\varepsilon(x_1)|$ denotes the Lebesgue's measure of the set $S^\varepsilon(x_1)$.
Now, we set:

$$\begin{cases} N^\varepsilon(x_1) &= \int_{S^\varepsilon(x_1)} \hat{\sigma}_{11}(x_1, y_2)dy_2 \\ T^\varepsilon(x_1) &= \frac{1}{\varepsilon}\int_{S^\varepsilon(x_1)} \hat{\sigma}_{12}(x_1, y_2)dy_2 \\ N^\varepsilon(x_1) &= \int_{S^\varepsilon(x_1)} y_2\hat{\sigma}_{11}(x_1, y_2)dy_2. \end{cases}$$

Using (12), we get the following estimates:

$$|N^\varepsilon(x_1)|_{L^2(0,1)} \leq c/\varepsilon, \qquad |M^\varepsilon(x_1)|_{L^2(0,1)} \leq c/\varepsilon\ . \tag{31}$$

Then we have:

THEOREM 3. *There exists a subsequence of $N^\varepsilon$ (resp. $M^\varepsilon$, $T^\varepsilon$) still denoted by $N^\varepsilon$ (resp. $M^\varepsilon$, $T^\varepsilon$) such that*

$$\begin{array}{llll} \varepsilon N^\varepsilon \rightharpoonup N^* & \text{in } L^2(0, L), & \varepsilon M^\varepsilon \rightharpoonup M^* & \text{in } L^2(0, L), \\ & \varepsilon T^\varepsilon \rightharpoonup T^* & \text{weakly} * \text{ in } H^{-1}(0, L) & \text{when } \varepsilon \to 0. \end{array} \tag{32}$$

*Moreover, $N^*$, $M^*$ and $T^*$ satisfy in $H^{-1}(0, L)$:*

$$\begin{cases} \frac{dN^*}{dx_1} + F_1^1 + F_1^2 &= 0 \\ \frac{dT^*}{dx_1} + F_2^1 + F_2^2 &= 0 \\ \frac{dM^*}{dx_1} - T^* &= -\frac{K}{2}(F_1^1 - F_1^2). \end{cases} \tag{33}$$

*The limits $\displaystyle\int_{-K/2}^{K/2} \hat{\sigma}_{i2}^* dy_2$, $i = 1, 2$ also vanish.*

*Proof.* From estimates (31), we deduce the first and second convergences of (32). Now, since $\hat{\sigma}_{ij}^\varepsilon = \varepsilon\sigma_{ij}^\varepsilon$, we have from equations of problem (4):

$$\begin{aligned} \int_{\mathcal{O}_\varepsilon^*} \hat{\sigma}_{i1}^\varepsilon \frac{\partial v_i}{\partial x_1} + \frac{1}{\varepsilon}\hat{\sigma}_{i2}^\varepsilon \frac{\partial v_i}{\partial y_2} = &\int_{\mathcal{O}_\varepsilon^*} \varepsilon f v + \Big( \int_{\Gamma^1} \frac{F_1^1}{\varepsilon} v_1 + F_2^1 v_2 \Big) + \\ &\Big( \int_{\Gamma^2} \frac{F_1^2}{\varepsilon} v_1 + F_2^2 v_2 \Big), \qquad \forall v \in H_\varepsilon. \end{aligned} \tag{34}$$

To prove the theorem, we choose a suitable test function $v$ in (34).
i) Let us take $v = (\varphi(x_1), 0)$ with $\varphi \in H_0^1(0, L)$. Then we have $v \in H_\varepsilon$ and

$$\int_{\mathcal{O}_\varepsilon^*} \hat{\sigma}_{i1}^\varepsilon \frac{\partial\varphi}{\partial x_1} = \int_{\mathcal{O}_\varepsilon^*} \varepsilon f_1\varphi + \int_{\Gamma^1} \frac{F_1^1}{\varepsilon}\varphi + \int_{\Gamma^2} \frac{F_1^2}{\varepsilon}\varphi$$

which can be written

$$\int_0^L \varepsilon N^\varepsilon(x_1)\frac{\partial\varphi}{\partial x_1} = \int_0^L (F_1^1 + F_1^2)\varphi + \int_Y \varepsilon^2 \overline{f}_1\varphi. \tag{35}$$

Letting $\varepsilon \to 0$ in (35), we get:

$$\int_0^L N^*(x_1)\frac{\partial\varphi}{\partial x_1} = \int_0^L (F_1^1 + F_1^2)\varphi \quad \forall\varphi \in H_0^1(0,L)$$

then

$$-\frac{dN^*}{dx_1} = F_1^1 + F_1^2 \qquad \text{in } H^{-1}(0,L).$$

ii) Similarly if we take $v = (0, \varphi(x_1))$ with $\varphi \in H_0^1(0,L)$, we get

$$\int_0^L \varepsilon T^\varepsilon(x_1)\frac{\partial\varphi}{\partial x_1} = \int_0^L (F_2^1 + F_2^2)\varphi + \varepsilon \int_Y \overline{f}_2\varphi. \tag{36}$$

We have by iii) below $\varepsilon T^\varepsilon \rightharpoonup T^*$ weakly$*$ in $H^{-1}(0,L)$. So we get by letting $\varepsilon \to 0$ in (36):

$$\int_0^L T^*(x_1)\frac{\partial\varphi}{\partial x_1} = \int_0^L (F_2^1 + F_2^2)\varphi \quad \forall\varphi \in H_0^1(0,L) \quad \text{then} \quad -\frac{dT^*}{dx_1} = F_2^1 + F_2^2 \quad \text{in } H^{-1}(0,L).$$

iii) Now take $v = (y_2\varphi(x_1), 0)$ with $\varphi \in H_0^1(0,L)$. Then $v \in H_\varepsilon$ and we obtain:

$$\int_0^L \varepsilon M^\varepsilon(x_1)\frac{\partial\varphi}{\partial x_1} + \int_0^L \varepsilon T^\varepsilon(x_1)\varphi = \frac{K}{2}\int_0^L (F_1^1 - F_1^2)\varphi + \varepsilon^2 \int_Y \overline{f}_1 y_2\varphi.$$

Since $\varepsilon M^\varepsilon \rightharpoonup M^*$ in $L^2(0,L)$, we have

$$\lim_{\varepsilon\to 0}\int_0^L \varepsilon T^\varepsilon(x_1)\varphi = \frac{K}{2}\int_0^L (F_1^1 - F_1^2)\varphi - \int_0^L M^*(x_1)\frac{\partial\varphi}{\partial x_1} \qquad \forall\varphi \in H_0^1(0,L)$$

which means that $\varepsilon T^\varepsilon \rightharpoonup T^*$ weakly$*$ in $H^{-1}(0,L)$, where $T^*$ is defined by
$< T^*, \varphi >= \lim_{\varepsilon\to 0}\int_0^L \varepsilon T^\varepsilon(x_1)\varphi \ \forall\varphi \in H_0^1(0,L)$. Thus $-\frac{dM^*}{dx_1} + T^* = \frac{K}{2}(F_1^1 - F_1^2)$.
iv) Let us take $v = (0, y_2\varphi(x_1))$ in (34) with $\varphi \in H_0^1(0,L)$. Then $v \in H_\varepsilon$ and we obtain

$$\int_Y y_2\varepsilon\overline{\hat{\sigma}_{21}^\varepsilon}\frac{\partial\varphi}{\partial x_1} + \int_Y \overline{\hat{\sigma}_{22}^\varepsilon}\varphi = \varepsilon^3\int_Y \overline{f}_2 y_2\varphi + \varepsilon^2\Big(\int_{\Gamma^1} F_2^1 y_2\varphi + \int_{\Gamma^2} F_2^2 y_2\varphi\Big).$$

Letting $\varepsilon \to 0$, we get $\int_Y \hat{\sigma}_{22}^*\varphi = 0 \ \forall\varphi \in H_0^1(0,L)$ and then $\int_{-K/2}^{K/2} \hat{\sigma}_{22}^* dy_2 = 0$.
If we take $v = (y_2\varphi(x_1), 0)$ in (34) with $\varphi \in H_0^1(0,L)$, we get as in iii)

$$\varepsilon\int_Y y_2\overline{\varepsilon\hat{\sigma}_{11}^\varepsilon}\frac{\partial\varphi}{\partial x_1} + \int_Y \overline{\varepsilon\hat{\sigma}_{12}^\varepsilon}\varphi = \varepsilon^3\int_Y \overline{f}_1 y_2\varphi + \varepsilon^2\Big(\int_{\Gamma^1} F_1^1 y_2\varphi + \int_{\Gamma^2} F_1^2 y_2\varphi\Big)$$

and letting $\varepsilon \to 0$, we obtain: $\int_Y \hat{\sigma}_{12}^*\varphi = 0 \ \forall\varphi \in H_0^1(0,L)$ which leads to
$\int_{-K/2}^{K/2} \hat{\sigma}_{12}^* dy_2 = 0$.

□

Now, when $\varepsilon \to 0$, we have from corollary 1 and (13)

$$\varepsilon_{22}(\hat{u}^*) = 0 \quad \text{and} \quad \varepsilon_{12}(\hat{u}^*) = 0.$$

Then arguing as in [2], we get:

LEMMA 6. *There exists a function $\check{u}_1^* \in H_0^1(0, L)$ such that:*

$$\hat{u}_1^*(x_1, y_2) = -y_2 \frac{d\hat{u}_2^*}{dx_1} + \check{u}_1^* \tag{37}$$

*where the limit $\hat{u}_2^*$ is identified to a function of $H_0^2(0, L)$.*

In the remainder of this section, we are going to find the relations between $N^*$, $M^*$ and $\check{u}_1^*$, $\hat{u}_2^*$ which are the constitutive laws. To do this, we use the energy method developped by L. Tartar [5]. It consists in introducing suitable test functions in the weak formulation of the problem.

Let $\chi^{11}$ and $\chi^{12}$ be the two functions defined by (21) and (25) respectively. Set

$$X_\varepsilon^{1\nu}(x_1, y_2) = \chi^{1\nu}(\frac{x_1}{\varepsilon}, y_2).$$

We have

$$\frac{\partial X_\varepsilon^{1\nu}}{\partial x_1} = \frac{1}{\varepsilon}\frac{\partial \chi_\varepsilon^{1\nu}}{\partial y_1}, \qquad \frac{\partial X_\varepsilon^{1\nu}}{\partial y_2} = \frac{\partial \chi_\varepsilon^{1\nu}}{\partial y_2}$$

and $X_\varepsilon^{1\nu}$ satisfy the equations:

$$\left\{\begin{array}{rl} \varepsilon\frac{\partial}{\partial x_1}\Big(a_{i1k1}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial x_1}.\varepsilon + a_{i1k2}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial y_2} + a_{i111}^\varepsilon (m(y_2) - y_2)^{\nu-1}\Big) + & \\ \frac{\partial}{\partial y_2}\Big(a_{i2k1}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial x_1}.\varepsilon + a_{i2k2}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial y_2} + a_{i211}^\varepsilon (m(y_2) - y_2)^{\nu-1}\Big) = & 0 \text{ in } \mathcal{O}_\varepsilon^* \\ \Big(a_{ijk1}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial x_1}.\varepsilon + a_{ijk2}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial y_2} + a_{ij11}^\varepsilon (m(y_2) - y_2)^{\nu-1}\Big)n_j = & 0 \\ \text{in } \Gamma^1 \cup \Gamma^2 \cup \mathcal{T}^\varepsilon. & \end{array}\right. \tag{38}$$

Let $\psi \in \mathcal{D}(0, L)$. We multiply the equations (38) by $\psi u_\varepsilon$ and we integrate by parts, we obtain:

$$\begin{array}{rl} \int_{\mathcal{O}_\varepsilon^*} \varepsilon\Big(a_{i1k1}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial x_1}.\varepsilon + a_{i1k2}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial y_2} + a_{i111}^\varepsilon (m(y_2) - y_2)^{\nu-1}\Big)\frac{\partial}{\partial x_1}(u_{\varepsilon i}\psi) & + \\ \int_{\mathcal{O}_\varepsilon^*} \Big(a_{i2k1}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial x_1}.\varepsilon + a_{i2k2}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial y_2} + a_{i211}^\varepsilon (m(y_2) - y_2)^{\nu-1}\Big)\frac{\partial u_{\varepsilon i}}{\partial y_2}\psi & = \quad 0. \end{array} \tag{39}$$

Using $X_\varepsilon^{1\nu}\psi$ as a test function in (34), we get:

$$\begin{array}{l} \int_{\mathcal{O}_\varepsilon^*} \Big(\hat{\sigma}_{i1}^\varepsilon \frac{\partial X_{\varepsilon i}^{1\nu}}{\partial x_1} + \frac{1}{\varepsilon}\hat{\sigma}_{i2}^\varepsilon \frac{\partial X_{\varepsilon i}^{1\nu}}{\partial y_2}\Big)\psi + \int_{\mathcal{O}_\varepsilon^*} \hat{\sigma}_{i1}^\varepsilon X_{\varepsilon i}^{1\nu}\frac{\partial \psi}{\partial x_1} = \\ \int_{\mathcal{O}_\varepsilon^*} \varepsilon f X_\varepsilon^{1\nu}\psi + \Big(\int_{\Gamma^1} \frac{F_1^1}{\varepsilon} X_{\varepsilon 1}^{1\nu}\psi + F_2^1 X_{\varepsilon 2}^{1\nu}\psi\Big) + \\ \Big(\int_{\Gamma^2} \frac{F_1^2}{\varepsilon} X_{\varepsilon 1}^{1\nu}\psi + F_2^2 X_{\varepsilon 2}^{1\nu}\psi\Big). \end{array} \tag{40}$$

Taking into account the expression of $\hat{\sigma}_{ij}^\varepsilon$ and dividing by $\varepsilon^3$, (39) becomes:

$$\begin{array}{l} \int_{\mathcal{O}_\varepsilon^*} \Big(\hat{\sigma}_{i1}^\varepsilon \frac{\partial X_{\varepsilon i}^{1\nu}}{\partial x_1} + \frac{1}{\varepsilon}\hat{\sigma}_{i2}^\varepsilon \frac{\partial X_{\varepsilon i}^{1\nu}}{\partial y_2}\Big)\psi + \frac{1}{\varepsilon}\int_{\mathcal{O}_\varepsilon^*} \hat{\sigma}_{11}^\varepsilon (m(y_2) - y_2)^{\nu-1}\psi + \\ \frac{1}{\varepsilon}\int_{\mathcal{O}_\varepsilon^*} \Big(a_{i1k1}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial x_1} + \frac{1}{\varepsilon}a_{i1k2}^\varepsilon \frac{\partial X_{\varepsilon k}^{1\nu}}{\partial y_2} + \frac{1}{\varepsilon}a_{i111}^\varepsilon (m(y_2) - y_2)^{\nu-1}\Big)u_{\varepsilon i}\frac{\partial \psi}{\partial x_1} = 0. \end{array} \tag{41}$$

Subtracting (40) from (41) and multiplying the identity obtained by $\varepsilon^2$ we get:

$$\begin{aligned}(42)\qquad &\textstyle\int_{\mathcal{O}^*_\varepsilon} \varepsilon\hat{\sigma}^\varepsilon_{11}(m(y_2)-y_2)^{\nu-1}\psi = -\varepsilon\int_{\mathcal{O}^*_\varepsilon}(\varepsilon\hat{\sigma}^\varepsilon_{i1})X^{1\nu}_{\varepsilon i}\frac{\partial\psi}{\partial x_1} - \varepsilon^3\int_{\mathcal{O}^*_\varepsilon} fX^{1\nu}_\varepsilon\psi \\ &\textstyle -\varepsilon^2\Big(\int_{\Gamma^1}\frac{F^1_1}{\varepsilon}X^{1\nu}_{\varepsilon 1}\psi + F^1_2X^{1\nu}_{\varepsilon 2}\psi\Big) - \varepsilon^2\Big(\int_{\Gamma^2}\frac{F^2_1}{\varepsilon}X^{1\nu}_{\varepsilon 1}\psi + F^2_2X^{1\nu}_{\varepsilon 2}\psi\Big) \\ &\textstyle -\varepsilon\int_{\mathcal{O}^*_\varepsilon}\Big(a^\varepsilon_{i1k1}\frac{\partial X^{1\nu}_{\varepsilon k}}{\partial x_1} + \frac{1}{\varepsilon}a^\varepsilon_{i1k2}\frac{\partial X^{1\nu}_{\varepsilon k}}{\partial y_2} + \frac{1}{\varepsilon}a^\varepsilon_{i111}(m(y_2)-y_2)^{\nu-1}\Big)u_{\varepsilon i}\frac{\partial\psi}{\partial x_1}.\end{aligned}$$

To get the expressions of $M^*$ and $N^*$, it suffices to let $\varepsilon\to 0$ in the above equality. First, since $X^{1\nu}_\varepsilon$ is $Y_1$-periodic, we have

$$(43)\qquad X^{1\nu}_\varepsilon \rightharpoonup \frac{1}{L}\int_0^L \chi^{1\nu}(y_1,y_2)dy_1 \qquad \text{in } L^2(Y).$$

Then using (43) and the fact that the sequence $(\varepsilon\hat{\sigma}^\varepsilon_{ij})$ is bounded, we have the following limits:

$$(44)\qquad \begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon\int_{\mathcal{O}^*_\varepsilon}(\varepsilon\hat{\sigma}^\varepsilon_{i1})X^{1\nu}_{\varepsilon i}\frac{\partial\psi}{\partial x_1} = 0,\\ &\lim_{\varepsilon\to 0}\varepsilon^3\int_{\mathcal{O}^*_\varepsilon} fX^{1\nu}_\varepsilon\psi = 0,\\ &\lim_{\varepsilon\to 0}\varepsilon^2\Big(\int_{\Gamma^1}\frac{F^1_1}{\varepsilon}X^{1\nu}_{\varepsilon 1}\psi + F^1_2X^{1\nu}_{\varepsilon 2}\psi\Big) = 0,\\ &\lim_{\varepsilon\to 0}\varepsilon^2\Big(\int_{\Gamma^2}\frac{F^2_1}{\varepsilon}X^{1\nu}_{\varepsilon 1}\psi + F^2_2X^{1\nu}_{\varepsilon 2}\psi\Big) = 0.\end{aligned}$$

Next, to compute the last limit, we proceede as in [1]. Set

$$r^\nu_{ij} = a_{ijkh}\frac{\partial\chi^{1\nu}_k}{\partial y_h} + a_{ij11}(m(y_2)-y_2)^{\nu-1}.$$

We have from definition of $\chi^{1\nu}$:

$$(45)\qquad \begin{cases}\dfrac{\partial r^\nu_{ij}}{\partial y_j} = 0 & \text{in } Y^*\\ r^\nu_{ij}n_j = 0 & \text{on } \Gamma^1\cup\Gamma^2\cup\partial Y^*_{int}\quad i=1,2.\end{cases}$$

If we define $R^{\nu\varepsilon}_{ij}$ by:

$$R^{\nu\varepsilon}_{ij}(x_1,y_2) = r^\nu_{ij}(\frac{x_1}{\varepsilon},y_2),$$

then (45) leads to

$$\begin{cases}\dfrac{\partial R^{\nu\varepsilon}_{i1}}{\partial x_1} + \dfrac{1}{\varepsilon}\dfrac{\partial R^{\nu\varepsilon}_{i2}}{\partial y_2} = 0 & \text{in } \mathcal{O}^*_\varepsilon\\ R^{\nu\varepsilon}_{ij}n_j = 0 & \text{on } \Gamma^1\cup\Gamma^2\cup\partial Y^*_{int}\quad i=1,2.\end{cases}$$

Note that we have

$$R^{\nu\varepsilon}_{ij}(x_1,y_2) = \varepsilon a^\varepsilon_{ijk1}\frac{\partial X^{1\nu}_{\varepsilon k}}{\partial x_1} + a^\varepsilon_{ijk2}\frac{\partial X^{1\nu}_{\varepsilon k}}{\partial y_2} + a^\varepsilon_{ij11}(m(y_2)-y_2)^{\nu-1}.$$

Then by letting $\varepsilon\to 0$ in (42) and using (44), we get:

$$(46)\qquad \lim_{\varepsilon\to 0}\int_{\mathcal{O}^*_\varepsilon}\varepsilon\hat{\sigma}^\varepsilon_{11}(m(y_2)-y_2)^{\nu-1}\psi = -\lim_{\varepsilon\to 0}\int_{\mathcal{O}^*_\varepsilon} R^{\nu\varepsilon}_{i1}u_{\varepsilon i}\frac{\partial\psi}{\partial x_1}.$$

Let us now introduce the following problem:

$$
(47)\qquad \begin{cases} \text{Find } \varphi^{\nu} \;\; Y_1-\text{periodic such that }: \\ \frac{\partial}{\partial y_j}\Big(a_{ijkh}\frac{\partial \varphi_k^{\nu}}{\partial y_h}+a_{ij1h}\chi_h^{1\nu}\Big)+r_{i1}^{\nu} &=& 0 & \text{in } Y^* \\ \Big(a_{ijkh}\frac{\partial \varphi_k^{\nu}}{\partial y_h}+a_{ij1k}\chi_h^{1\nu}\Big)n_j &=& b_i^{k\nu} & \text{on } \Gamma^k \; k=1,2 \\ \Big(a_{ijkh}\frac{\partial \varphi_k^{\nu}}{\partial y_h}+a_{ij1k}\chi_h^{1\nu}\Big)n_j &=& 0 & \text{on } \partial Y^*_{int}. \end{cases}
$$

A necessary condition for the existence of $\varphi^{\nu}$ is:

$$
\int_{\Gamma^1} b_i^{1\nu}+\int_{\Gamma^2} b_i^{2\nu}+\int_{Y^*} r_{i1}^{\nu}=0 \qquad i=1,2
$$

which is satisfied if we choose in what follows:

$$
b_i^{1\nu}=b_i^{2\nu}=-\frac{1}{2|Y_1|}\int_{Y^*} r_{i1}^{\nu} \qquad i=1,2.
$$

Note that, using notations of section 4, we have $b_1^{11}=b_1^{21}=-s^{11}/2$ and $b_1^{12}=b_1^{22}=-s^{12}/2$. Now we consider:

$$
\tau_{ij}^{\nu}=a_{ijkh}\frac{\partial \varphi_k^{\nu}}{\partial y_h}+a_{ij1h}\chi_h^{1\nu} \quad \text{and} \quad T_{ij}^{\nu\varepsilon}(x_1,y_2)=\tau_{ij}^{\nu}(\frac{x_1}{\varepsilon},y_2).
$$

Then we have by (47),

$$
\begin{cases} \frac{\partial \tau_{ij}^{\nu}}{\partial y_j}+r_{i1}^{\nu} &=& 0 & \text{in } Y^* \\ \tau_{ij}^{\nu}n_j &=& b_i^{k\nu} & \text{on } \Gamma^k \quad k=1,2, \quad \tau_{ij}^{\nu}n_j=0 \quad \text{on } \partial Y^*_{int} \end{cases}
$$

and

$$
\begin{cases} \frac{\partial T_{i1}^{\nu\varepsilon}}{\partial x_1}+\frac{1}{\varepsilon}\frac{\partial T_{i2}^{\nu\varepsilon}}{\partial y_2} &=& -\frac{1}{\varepsilon}R_{i1}^{\nu\varepsilon} & \text{in } \mathcal{O}_\varepsilon^* \\ T_{ij}^{\nu\varepsilon}n_j &=& b_i^{k\nu} & \text{on } \Gamma^k \quad k=1,2, \quad T_{ij}^{\nu\varepsilon}n_j=0 \quad \text{on } \partial\mathcal{T}^{\varepsilon}. \end{cases}
$$

Let us compute

$$
\mathcal{A}^{\varepsilon}=\int_{\mathcal{O}_\varepsilon^*} T_{i1}^{\nu\varepsilon}u_{\varepsilon i}\frac{\partial^2\psi}{\partial x_1^2}.
$$

Since $\psi$ does not depend on $y_2$, we have

$$
(48)\qquad \begin{aligned} \mathcal{A}^{\varepsilon} &= \int_{\mathcal{O}_\varepsilon^*} T_{i1}^{\nu\varepsilon}u_{\varepsilon i}\frac{\partial^2\psi}{\partial x_1^2}+\frac{1}{\varepsilon}T_{i2}^{\nu\varepsilon}u_{\varepsilon i}\frac{\partial^2\psi}{\partial x_1\partial y_2} \\ &= \int_{\mathcal{O}_\varepsilon^*}\frac{1}{\varepsilon}R_{i1}^{\nu\varepsilon}u_{\varepsilon i}\frac{\partial\psi}{\partial x_1}-\int_{\mathcal{O}_\varepsilon^*}\Big(T_{i1}^{\nu\varepsilon}\frac{\partial u_{\varepsilon i}}{\partial x_1}+\frac{1}{\varepsilon}T_{i2}^{\nu\varepsilon}\frac{\partial u_{\varepsilon i}}{\partial y_2}\Big)\frac{\partial\psi}{\partial x_1} \\ &+ \frac{b_i^{1\nu}}{\varepsilon}\int_{\Gamma^1\cup\Gamma^2} u_{\varepsilon i}\frac{\partial\psi}{\partial x_1}. \end{aligned}
$$

By definition of $T_{ij}^{\nu\varepsilon}$ and $\hat{\sigma}_{ij}^{\varepsilon}$, one can see that we have

$$
(49)\qquad T_{i1}^{\nu\varepsilon}\frac{\partial u_{\varepsilon i}}{\partial x_1}+\frac{1}{\varepsilon}T_{i2}^{\nu\varepsilon}\frac{\partial u_{\varepsilon i}}{\partial y_2}=\varepsilon^2\hat{\sigma}_{k1}^{\varepsilon}\frac{\partial \Phi_k^{\nu\varepsilon}}{\partial x_1}+\varepsilon\hat{\sigma}_{k2}^{\varepsilon}\frac{\partial \Phi_k^{\nu\varepsilon}}{\partial y_2}+\varepsilon\hat{\sigma}_{1h}^{\varepsilon}X_h^{1\nu}
$$

where $\Phi^{\nu\varepsilon}(x_1, y_2) = \varphi(\frac{x_1}{\varepsilon}, y_2)$.
From (48) and (49) we deduce that

$$(50) \qquad \begin{aligned} -\int_{\mathcal{O}_\varepsilon^*} R_{i1}^{\nu\varepsilon} u_{\varepsilon i} \frac{\partial\psi}{\partial x_1} &= -\varepsilon\mathcal{A}^\varepsilon - \varepsilon \int_{\mathcal{O}_\varepsilon^*} \Big(\varepsilon^2 \hat{\sigma}_{k1}^\varepsilon \frac{\partial \Phi_k^{\nu\varepsilon}}{\partial x_1} + \varepsilon \hat{\sigma}_{k2}^\varepsilon \frac{\partial \Phi_k^{\nu\varepsilon}}{\partial y_2} + \varepsilon \hat{\sigma}_{1h}^\varepsilon X_h^{1\nu}\Big) \\ &+ \quad b_i^{1\nu} \int_{\Gamma^1\cup\Gamma^2} u_{\varepsilon i} \frac{\partial\psi}{\partial x_1}. \end{aligned}$$

Now we are going to let $\varepsilon$ goes to 0 in (50). First we have easily:

$$(51) \qquad \lim_{\varepsilon\to 0} \varepsilon \int_{\mathcal{O}_\varepsilon^*} \Big(\varepsilon^2 \hat{\sigma}_{k1}^\varepsilon \frac{\partial \Phi_k^{\nu\varepsilon}}{\partial x_1} + \varepsilon \hat{\sigma}_{k2}^\varepsilon \frac{\partial \Phi_k^{\nu\varepsilon}}{\partial y_2} + \varepsilon \hat{\sigma}_{1h}^\varepsilon X_h^{1\nu}\Big) = 0.$$

Next, we have

$$\begin{aligned} \varepsilon\mathcal{A}^\varepsilon &= \int_{\mathcal{O}_\varepsilon^*} \varepsilon T_{11}^{\nu\varepsilon} u_{\varepsilon 1} \frac{\partial^2\psi}{\partial x_1^2} + \varepsilon T_{21}^{\nu\varepsilon} u_{\varepsilon 2} \frac{\partial^2\psi}{\partial x_1^2} \\ &= \int_Y \varepsilon \overline{T_{11}^{\nu\varepsilon}} \big(\varepsilon (P\hat{u}_\varepsilon)_1\big) \frac{\partial^2\psi}{\partial x_1^2} + \int_Y \overline{T_{21}^{\nu\varepsilon}} \big(\varepsilon (P\hat{u}_\varepsilon)_2)\big) \frac{\partial^2\psi}{\partial x_1^2} \end{aligned}$$

Since $T_{ij}^{\nu\varepsilon}$ is $Y_1$-periodic, we get

$$\overline{T_{ij}^{\nu\varepsilon}} \rightharpoonup \frac{1}{L}\int_0^L \overline{\tau_{ij}^{\nu}}(y_1, y_2) dy_1 \qquad \text{in } L^2(Y)$$

then

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon\mathcal{A}^\varepsilon &= \int_Y \Big(\frac{1}{L}\int_0^L \overline{\tau_{21}^{\nu}}(y_1, y_2) dy_1\Big) \hat{u}_2^* \frac{\partial^2\psi}{\partial x_1^2} \\ &= \int_0^L \frac{1}{L}\Big(\int_{-K/2}^{K/2} \int_0^L \overline{\tau_{21}^{\nu}}(y_1, y_2) dy_1 dy_2\Big) \hat{u}_2^*(x_1) \frac{\partial^2\psi}{\partial x_1^2} \\ &= \int_0^L \Big(\frac{1}{L}\int_{Y^*} \tau_{21}^{\nu}\Big) \hat{u}_2^*(x_1) \frac{\partial^2\psi}{\partial x_1^2}. \end{aligned}$$

Moreover, one has

$$b_i^{1\nu} \int_{\Gamma^1\cup\Gamma^2} u_{\varepsilon i} \frac{\partial\psi}{\partial x_1} = b_1^{1\nu} \int_{\Gamma^1\cup\Gamma^2} \big(\varepsilon (P\hat{u}_\varepsilon)_1\big) \frac{\partial\psi}{\partial x_1} + b_2^{1\nu} \int_{\Gamma^1\cup\Gamma^2} (P\hat{u}_\varepsilon)_2 \frac{\partial\psi}{\partial x_1}$$

where $b_2^{1\nu} = -\frac{1}{2L}\int_{Y^*} r_{21}^{\nu} = -\frac{1}{2L}\int_{Y^*} \big[a_{21kh} \frac{\partial \chi_k^{1\nu}}{\partial y_h} + a_{2111}(m(y_2) - y_2)^{\nu-1}\big]$.
Taking $(y_2 - m(y_2), 0)$ as a test function in (21) and (25), one can see that: $b_2^{1\nu} = 0$ for $\nu = 1, 2$.
Thus

$$\lim_{\varepsilon\to 0} b_i^{1\nu} \int_{\Gamma^1\cup\Gamma^2} u_{\varepsilon i} \frac{\partial\psi}{\partial x_1} = b_1^{1\nu} \int_{\Gamma^1\cup\Gamma^2} u_1^* \frac{\partial\psi}{\partial x_1}.$$

Using (37), we have

$$\int_{\Gamma^1\cup\Gamma^2} \hat{u}_1^* \frac{\partial\psi}{\partial x_1} = \int_{\Gamma^1\cup\Gamma^2} \big(-y_2 \frac{d\hat{u}_2^*}{dx_1} + \check{u}_1^*\big) \frac{\partial\psi}{\partial x_1} = 2\int_0^L \check{u}_1^* \frac{\partial\psi}{\partial x_1}.$$

So

$$(52) \qquad \lim_{\varepsilon\to 0} b_i^{1\nu} \int_{\Gamma^1\cup\Gamma^2} u_{\varepsilon i} \frac{\partial\psi}{\partial x_1} = 2b_1^{1\nu} \int_0^L \check{u}_1^* \frac{\partial\psi}{\partial x_1}.$$

Finally, we have by (46), (50)-(52):

$$\lim_{\varepsilon\to 0} \int_{\mathcal{O}_\varepsilon^*} \varepsilon \hat{\sigma}_{11}^\varepsilon (m(y_2) - y_2)^{\nu-1} = -\int_0^L \Big(\frac{1}{L}\int_{Y^*} \tau_{21}^{\nu}\Big) \hat{u}_2^*(x_1) \frac{\partial^2\psi}{\partial x_1^2} + 2b_1^{1\nu} \int_0^L \check{u}_1^* \frac{\partial\psi}{\partial x_1}.$$

Then

$$\begin{cases} N^* & = \left( -\frac{1}{L}\int_{Y^*}\tau_{21}^1 \right)\frac{d^2\hat{u}_2^*}{dx_1^2} + s^{11}\frac{d\check{u}_1^*}{dx_1} \\ m(y_2)N^* - M^* & = \left( -\frac{1}{L}\int_{Y^*}\tau_{21}^2 \right)\frac{d^2\hat{u}_2^*}{dx_1^2} + s^{12}\frac{d\check{u}_1^*}{dx_1}. \end{cases}$$

Now, taking $(y_2, 0)$ as a test function in the weak formulation associated to (47), we get: $\dfrac{1}{L}\displaystyle\int_{Y^*}\tau_{21}^{\nu} = s^{2\nu}$. Then we have

$$\begin{cases} N^* & = s^{11}\frac{d\check{u}_1^*}{dx_1} - s^{21}\frac{d^2\hat{u}_2^*}{dx_1^2} \\ M^* & = (m(y_2)s^{11} - s^{12})\frac{d\check{u}_1^*}{dx_1} + (s^{22} - m(y_2)s^{21})\frac{d^2\hat{u}_2^*}{dx_1^2} \end{cases}$$

which can be written by using Corollary 2 and setting: $\underline{u}_1^* = \check{u}_1^* - m(y_2)\dfrac{d\hat{u}_2^*}{dx_1}$,

$$\begin{cases} N^* & = s^{11}\frac{d\underline{u}_1^*}{dx_1} + s^{12}\frac{d^2\hat{u}_2^*}{dx_1^2} \\ M^* & = s^{21}\frac{d\underline{u}_1^*}{dx_1} + s^{22}\frac{d^2\hat{u}_2^*}{dx_1^2} \end{cases} \tag{53}$$

with $\underline{u}_1^* \in H_0^1(0, L)$ and $\hat{u}_2^* \in H_0^2(0, L)$.
So we get the same homogenized problem obtained formally in section 4. Since the matrix $S = (s^{\mu\nu})_{\mu,\nu=1,2}$ is invertible (see Lemma 5), problem (33), (53) admits a unique solution. Thus at the limit, the asymptotic behavior of the lattice beam is governed by Bernoulli's law.

## References


[1] CAILLERIE D., *Thin elastic and periodic plates*, Math. Meth. in the Appl. Sci. **6** (1984), 159–191.

[2] CIARLET P.G. AND DESTUYNDER P., *A justification of the two-dimensional linear plate model*, J. de Mécanique **18** (1979), 315–344.

[3] CONCA C., *On the application of the homogenization theory to a class of problems arising in fluid mechanics*, J. Math. Pure et Appl. **64** (1985), 31–75.

[4] SALENCON J., *Mécanique des milieux continus. II Elasticité - milieux curvilignes* éllipses, (1988).

[5] TARTAR L., *Quelques remarques sur l'homogénéisation*, Functional and Numerical Analysis, Japan France Seminar, Tokyo and Kyoto (Ed. H. Fujita), Japan Society for the promotion of Science (1978), 469–481.


**AMS Subject Classification: 35JXX, 73B27.**

Samia CHALLAL
Département de Mathématiques
Université de Metz
Ile du Saulcy
57045 Metz Cedex 01, FRANCE
e-mail: `challal.samia@caramail.com`

*Lavoro pervenuto in redazione il 02.02.1998 e, in forma definitiva, il 20.06.1999.*