Rend. Sem. Mat. Univ. Pol. Torino
Vol. 58, 4 (2000)

**M. Bramanti - L. Brandolini**

# $L^p$ ESTIMATES FOR UNIFORMLY HYPOELLIPTIC OPERATORS WITH DISCONTINUOUS COEFFICIENTS ON HOMOGENEOUS GROUPS

**Abstract.** Let $G$ be a homogeneous group and let $X_0, X_1, \ldots, X_q$ be left invariant real vector fields on $G$, satisfying Hörmander's condition. Assume that $X_1, \ldots, X_q$ be homogeneous of degree one and $X_0$ be homogeneous of degree two. We study operators of the kind:

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + a_0(x) X_0$$

where $a_{ij}(x)$ and $a_0(x)$ are real valued, bounded measurable functions belonging to the space "Vanishing Mean Oscillation", defined with respect to the quasidistance naturally induced by the structure of homogeneous group. Moreover, the matrix $\{a_{ij}(x)\}$ is uniformly elliptic and $a_0(x)$ is bounded away from zero. Under these assumptions we prove local estimates in the Sobolev space $S^{2,p}$ $(1 < p < \infty)$ defined by the vector fields $X_i$, for solutions to the equation $\mathcal{L}u = f$ with $f \in \mathcal{L}^p$. From this fact we also deduce the local Hölder continuity for solutions to $\mathcal{L}u = f$, when $f \in \mathcal{L}^p$ with $p$ large enough. Further (local) regularity results, in terms of Sobolev or Hölder spaces, are proved to hold when the coefficients and data are more regular. Finally, lower order terms (in the sense of the degree of homogeneity) can be added to the operator mantaining the same results.

## 1. Introduction

A classical result of Agmon-Douglis-Nirenberg [1] states that, for a given uniformly elliptic operator in nondivergence form with continuous coefficients,

$$Lu = \sum_{i,j} a_{ij}\ (x)\, u_{x_i x_j}$$

one has the following $L^p$-estimates for every $p \in (1, \infty)$, on a bounded smooth domain $\Omega$ of $\mathbb{R}^n$:

$$\left\| u_{x_i x_j} \right\|_{L^p(\Omega)} \leq c \left\{ \|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right\}.$$

While the above estimate is false in general if the coefficients are merely $L^\infty$, a remarkable extension of the above result, due to Chiarenza-Frasca-Longo [6],[7], replaces the continuity assumption with the weaker condition $a_{ij} \in VMO$, where $VMO$ is the Sarason's space of vanishing mean oscillation functions, a sort of uniform continuity in integral sense.

Roughly speaking, this extension relies on the classical theory of Calderón-Zygmu-nd operators, a theorem of Coifman-Rochberg-Weiss [8] (which we will recall later in detail) about the commutator of an operator of this type with a $BMO$ function, and the knowledge of the fundamental solution for constant coefficients elliptic operators on $\mathbb{R}^n$. All these ideas admit broad generalizations: the Calderón-Zygmund theory and the commutator theorem can be settled in the general framework of spaces of homogeneous type, in the sense of Coifman-Weiss (see [9], [20] and [4]); however the knowledge of the fundamental solution is a more subtle problem. Apart from the elliptic case, an explicit fundamental solution is also known for constant coefficients parabolic operators. This kernel is homogeneous with respect to the "parabolic dilations", so that the abstract Calderón-Zygmund theory can be applied to this situation to get $L^p$-estimates of the above kind for parabolic operators with $VMO$ coefficients (see Bramanti-Cerutti [3]).

In recent years it has been noticed by Lanconelli-Polidoro [23] that an interesting class of ultraparabolic operators of Kolmogorov-Fokker-Planck type, despite of its strong degeneracy, admits an explicit fundamental solution which turns out to be homogeneous with respect to suitable nonisotropic dilations, and invariant with respect to a group of (noncommutative) translations. These operators can be written as:

(1) $$Lu = \sum_{i,j=1}^{q} a_{ij} u_{x_i x_j} + \sum_{i,j=1}^{n} x_i b_{ij} u_{x_j} - u_t$$

where $(x,t) \in \mathbb{R}^{n+1}$, $\{b_{ij}\}$ is a constant real matrix with a suitable upper triangular structure, while $\{a_{ij}\}$ is a $q \times q$ uniformly elliptic matrix, with $q < n$. The structure of space of homogeneous type underlying the operator and the knowledge of a fundamental solution well shaped on this structure, suggest that an analog $L^p$ theory could be settled for operators of kind (1) with $a_{ij}$ in $VMO$. This has been actually done by Bramanti-Cerutti-Manfredini [5] . (In this case, only local estimates are proved).

The class of operators (1) contains prototypes of Fokker-Planck operators describing brownian motions of a particle in a fluid, as well as Kolmogorov operators describing systems with $2n$ degrees of freedom (see [23] ), and is still extensively studied (see for instance [22], [24], [25], [27] and references therein).

When $a_{ij} = \delta_{ij}$, (1) exhibits an interesting example of "Hörmander's operator", of the kind

$$Lu = \sum_{i=1}^{q} X_i^2 u + X_0 u$$

where $X_0 = \sum_{i,j=1}^{n} x_i b_{ij} \partial_{x_j} - \partial_t$, and $X_i = \partial_{x_i}$ for $i = 1, 2, , \dots q$. This introduces us to the point of view of hypoelliptic operators. Recall that a differential operator $P$ with $C^\infty$ coefficients is said to be hypoelliptic in some open set $U \subseteq \mathbb{R}^N$ if, whenever the equation $Pu = f$ is satisfied in $U$ by two distributions $u, f$, then the following condition holds: if $V$ is an open subset of $U$ such that $f_{|V} \in C^\infty(V)$, then $u_{|V} \in C^\infty(V)$. We recall the well-known

THEOREM 1 (HÖRMANDER, [16]). *Let $X_0, X_1, \dots, X_q$ be real vector fields with coefficients $C^\infty(\mathbb{R}^N)$. The operator*

(2) $$P = \sum_{i=1}^{q} X_i^2 + X_0$$

*is hypoelliptic in $\mathbb{R}^N$ if the Lie algebra generated at every point by the fields $X_0, X_1, \dots, X_q$ is $\mathbb{R}^N$. We will call this property "Hörmander's condition".*

The operator (1) with constant $a_{ij}$'s satisfies Hörmander's condition, by the structure assumption on the matrix $\{b_{ij}\}$, and is therefore hypoelliptic.

In '75, Folland [11] proved that any Hörmander's operator like (2) which is left invariant with respect to a group of translations, and homogeneous of degree 2 with respect to a family of (nonisotropic) dilations, which are group automorphisms, has a homogeneous left invariant fundamental solution. This allows to apply the abstract theory of singular integrals in spaces of homogeneous type, to get local $L^p$ estimates of the kind

$$\left\| X_i X_j u \right\|_{\mathcal{L}^p(\Omega')} \leq c \left\{ \| \mathcal{L}u \|_{\mathcal{L}^p(\Omega)} + \| u \|_{\mathcal{L}^p(\Omega)} \right\} \quad (i, j = 1, \dots, q) \tag{3}$$

for any $p \in (1, \infty)$, $\Omega' \subset\subset \Omega$.

Motivated by the results obtained by [3], [5], the aim of this paper is to extend the above techniques and results to the homogeneous setting considered by Folland, where good properties of the fundamental solution allow to obtain in a natural way the $L^p$ estimates, using the available real variable machinery.

More precisely, we study operators of the kind:

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + a_0(x) X_0$$

where $X_0, X_1, \dots, X_q$ form a system of $C^\infty$ real vector fields defined in $\mathbb{R}^N$ ($N \geq q+1$), satisfying Hörmander's condition. We also assume that $X_0, X_1, \dots, X_q$ are left invariant with respect to a "translation" which makes $\mathbb{R}^N$ a Lie group, and homogeneous with respect to a family of "dilations" which are group automorphisms. More precisely, $X_1, \dots, X_q$ are homogeneous of degree one and $X_0$ is homogeneous of degree two. The coefficients $a_{ij}(x)$, $a_0(x)$ are real valued bounded measurable functions, satisfying very weak regularity conditions (they belong to the class $VMO$, "Vanishing Mean Oscillation", defined with respect to the homogeneous distance; in particular, they can be discontinuous); moreover, the matrix $\{a_{ij}(x)\}$ is uniformly elliptic and not necessarily symmetric; the function $a_0(x)$ is bounded away from zero.

Under these assumptions (see §2 for precise statements) we prove that the local $\mathcal{L}^p$ estimates (3) hold for $p \in (1, \infty)$, every bounded domain $\Omega$, any $\Omega' \subset\subset \Omega$, and any $u$ for which the right hand side of (3) makes sense (see Theorem 3 for a precise statement). From this fact we also deduce the local Hölder continuity for solutions to the equation $\mathcal{L}u = f$, when $f \in \mathcal{L}^p(\Omega)$ with $p$ large enough (see Theorem 4).

To get (3) we will first prove the following estimate:

$$\left\| X_i X_j u \right\|_p \leq c \| \mathcal{L}u \|_p \quad (i, j = 1, \dots, q, 1 < p < \infty), \tag{4}$$

for every test function $u$ supported in a ball with sufficiently small radius (see Theorem 2). It is in this estimate that the $VMO$ regularity of the coefficients plays a crucial role.

Further (local) regularity results for solutions to the equation $\mathcal{L}u = f$, in terms of Sobolev or Hölder spaces, are proved to hold when the coefficients and data are more regular (see Theorems 5, 6). Finally, lower order terms (in a suitable sense) can be added to the operator maintaining the same results (see Theorem 7).

Since the operator $\mathcal{L}$ has, in general, nonsmooth coefficients, the above definition of hypoellipticity makes no sense for $\mathcal{L}$. However we will show (Theorem 8) that if the coefficients $a_{ij}(x)$ are smooth, then $\mathcal{L}$ is actually hypoelliptic. Moreover, for every fixed $x_0 \in \mathbb{R}^N$, the frozen operator

$$\mathcal{L}_0 = \sum_{i,j=1}^{q} a_{ij}(x_0) X_i X_j + a_0(x_0) X_0 \tag{5}$$

is always hypoelliptic and, by the results of Folland [11] (see Theorem 9 below), has a homogeneous fundamental solution, which we will prove to satisfy some uniform bounds, with respect to $x_0$ (Theorem 12). This perhaps justifies the (improper) name of "uniformly hypoelliptic operators" for $\mathcal{L}$, which appears in the title.

We point out that the results in this paper contain as particular cases the local estimates proved in [6], [3] and [5]. On the other side, global $L^p$ estimates on a domain are not available for hypoelliptic operators, even in simple model cases.

A natural issue is to discuss the necessity of our homogeneity assumptions. In a famous paper, Rothschild-Stein [28] introduced a powerful technique of "lifting and approximation", which allows to study a general Hörmander's operator by means of operators of the kind studied by Folland. As a consequence, they obtained estimates like (3) in this more general setting.

In a forthcoming paper [2], we shall use their techniques, combined with our results, to attack the general case where the homogeneous structure underlying the Hörmander's vector fields is lacking.

**Outline of the paper.** §§2.1, 2.2, 2.5 contain basic definitions and known results. In §2.3 we state our main results (Theorems 2 to 7 ). In §2.4 we illustrate the relations between our class of operators and the operators of Hörmander type, comparing our results with those of Rothschild-Stein [28].

In §3 we prove Theorem 2 (that is (4)). The basic tool is the fundamental solution of the frozen operator (5), whose existence is assured by [11] (see §3.1. The line of the proof consists of three steps:

$(i)$ we write a representation formula for the second order derivatives of a test function in terms of singular integrals and commutators of singular integrals involving derivatives of the fundamental solution (see §3.2);

$(ii)$ we expand the singular kernel in series of spherical harmonics, to get singular integrals of convolution-type, with respect to our group structure (see §3.3); this step is necessary due to the presence of the variable coefficients $a_{ij}(x)$ in the differential operator;

$(iii)$ we get $\mathcal{L}^p$-bounds for the singular integrals of convolution-type and their commutators, applying general results for singular integrals on spaces of homogeneous type (see §3.4).

This line is the same followed in [5], which in turn was inspired by [6], [7]. While the commutator estimate needed in [6], [7] to achieve point $(iii)$ is that proved by Coifman-Rochberg-Weiss in [8], the suitable extension of this theorem to spaces of homogeneous type has been proved by Bramanti-Cerutti in [4].

The basic difficulty to overcome in the present situation, due to the class of differential operators we are considering, is that an explicit form for the fundamental solution of the frozen operator $\mathcal{L}_0$ in (5) is in general unknown. Therefore we have to prove in an indirect way uniform bounds with respect to $x_0$ for the derivatives of the fundamental solutions corresponding to $\mathcal{L}_0$ (Theorem 12). This will be a key point, in order to reduce the proof of (3) to that of $\mathcal{L}^p$ boundedness for singular integrals of convolution type. We wish to stress that, although several deep results have been proved about sharp bounds for the fundamental solution of a hypoelliptic operator (see [26], [29], [19]), these bounds are proved for a fixed operator, and the dependence of the constants on the vector fields is not apparent: therefore, these results cannot be applied in order to get uniform bounds for families of operators. On the other side, a useful point of view on this problem has been developed by Rothschild-Stein [28], and we will adapt this approach to our situation. To make more readable the exposition, the proof of this uniform bound (Theorem 12) is postponed to §4.

To prove local estimates for solutions to the equation $\mathcal{L}u = f$, starting from our basic estimate (4), we need some properties of the Sobolev spaces generated by the vector fields $X_i$, which we investigate in §5: interpolation inequalities, approximation results, embedding theorems. Some of these results appear to be new and can be of independent interest, because they regard spaces of functions not necessarily vanishing at the boundary, whereas in [11] or [28], for instance, only Sobolev spaces of functions defined on the whole space are considered.

In §6 we apply all the previous theory to local estimates for solutions to $\mathcal{L}u = f$. First we prove (3) and the local Hölder continuity of solutions (see Theorems 4, 5). Then we prove some regularity results, in the sense of Sobolev or Hölder spaces (see Theorems 5, 6), when the coefficients are more regular, as well as the generalization of all the previous estimates to the operator with lower order terms (Theorem 7). Observe that, since the vector fields do not commute, estimates on higher order derivatives are not a straightforward consequence of the basic estimate (3). Instead, we shall prove suitable representation formulas for higher order derivatives and then apply again the machinery of §3.

## 2. Definitions, assumptions and main results

### 2.1. Homogeneous groups and Lie algebras

Following Stein (see [31], pp. 618-622) we call homogeneous group the space $\mathbb{R}^N$ equipped with a Lie group structure, together with a family of dilations that are group automorphisms. Explicitly, assume that we are given a pair of mappings:

$$[(x, y) \mapsto x \circ y] : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{and} \quad \left[x \mapsto x^{-1}\right] : \mathbb{R}^N \to \mathbb{R}^N$$

that are smooth and so that $\mathbb{R}^N$, together with these mappings, forms a group, for which the identity is the origin. Next, suppose that we are given an $N$-tuple of strictly positive exponents $\omega_1 \le \omega_2 \le \ldots \le \omega_N$, so that the dilations

$$D(\lambda) : (x_1, \ldots, x_N) \mapsto \left(\lambda^{\omega_1} x_1, \ldots, \lambda^{\omega_N} x_N\right) \tag{6}$$

are group automorphisms, for all $\lambda > 0$. We will denote by $G$ the space $\mathbb{R}^N$ with this structure of homogeneous group, and we will write $c(G)$ for a constant depending on the numbers $N$, $\omega_1, \ldots, \omega_N$ and the group law $\circ$.

We can define in $\mathbb{R}^N$ a homogeneous norm $\|\cdot\|$ as follows. For any $x \in \mathbb{R}^N$, $x \neq 0$, set

$$\|x\| = \rho \quad \Leftrightarrow \quad \left|D(\frac{1}{\rho})x\right| = 1,$$

where $|\cdot|$ denotes the Euclidean norm; also, let $\|0\| = 0$. Then:

(i) $\|D(\lambda)x\| = \lambda \|x\|$ for every $x \in \mathbb{R}^N$, $\lambda > 0$;

(ii) the set $\{x \in \mathbb{R}^N : \|x\| = 1\}$ coincides with the Euclidean unit sphere $\sum_N$;

(iii) the function $x \mapsto \|x\|$ is smooth outside the origin;

(iv) there exists $c(G) \ge 1$ such that for every $x, y \in \mathbb{R}^N$

$$\|x \circ y\| \le c(\|x\| + \|y\|) \quad \text{and} \quad \left\|x^{-1}\right\| \le c\,\|x\|\,; \tag{7}$$

(8) $$\frac{1}{c}\,|y| \le \|y\| \le c\,|y|^{1/\omega} \quad \text{if} \quad \|y\| \le 1, \quad \text{with } \omega = \max(\omega_1, \ldots, \omega_N)\,.$$

The above definition of norm is taken from [12]. This norm is equivalent to that defined in [31], but in addition satisfies (ii), a property we shall use in §3.3. The properties (i),(ii) and (iii) are immediate while (7) is proved in [31], p. 620 and (8) is Lemma 1.3 of [11].

In view of the above properties, it is natural to define the "quasidistance" $d$:

$$d(x, y) = \left\| y^{-1} \circ x \right\|.$$

For $d$ the following hold:

(9) $$d(x, y) \ge 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y;$$

(10) $$\frac{1}{c}\, d(y,x) \le d(x, y) \le c\, d(y,x);$$

(11) $$d(x, y) \le c\left(d(x, z) + d(z, y)\right)$$

for every $x, y, z \in \mathbb{R}^N$ and some positive constant $c(G) \ge 1$. We also define the balls with respect to d as

$$B(x, r) \equiv B_r(x) \equiv \left\{ y \in \mathbb{R}^N : d(x, y) < r \right\}.$$

Note that $B(0, r) = D(r)B(0, 1)$. It can be proved (see [31], p. 619) that the Lebesgue measure in $\mathbb{R}^N$ is the Haar measure of $G$. Therefore

(12) $$|B(x, r)| = |B(0, 1)|\, r^Q,$$

for every $x \in \mathbb{R}^N$ and $r > 0$, where $Q = \omega_1 + \ldots + \omega_N$ , with $\omega_i$ as in (6). We will call $Q$ the homogeneous dimension of $\mathbb{R}^N$. By (12) the Lebesgue measure $dx$ is a doubling measure with respect to $d$, that is

$$|B(x, 2r)| \le c \cdot |B(x, r)| \quad \text{for every } x \in \mathbb{R}^N \text{ and } r > 0$$

and therefore $(\mathbb{R}^N, dx, d)$ is a space of homogenous type in the sense of Coifman-Weiss (see [9]). To be more precise, the definition of space of homogenous type in [9] requires $d$ to be symmetric, and not only to satisfy (10). However, the results about spaces of homogeneous type that we will use still hold under these more general assumptions. (See Theorem 16).

We say that a differential operator $Y$ on $\mathbb{R}^N$ is homogeneous of degree $\beta > 0$ if

$$Y\left(f\left((D(\lambda)x\right)\right) = \lambda^\beta\,(Yf)(D(\lambda)x)$$

for every test function $f$, $\lambda > 0$, $x \in \mathbb{R}^N$. Also, we say that a function $f$ is homogeneous of degree $\alpha \in \mathbb{R}$ if

$$f\left((D(\lambda)x)\right) = \lambda^\alpha\, f(x) \quad \text{for every } \lambda > 0,\, x \in \mathbb{R}^N.$$

Clearly, if $Y$ is a differential operator homogeneous of degree $\beta$ and $f$ is a homogeneous function of degree $\alpha$, then $Yf$ is homogeneous of degree $\alpha - \beta$.

Let us consider now the Lie algebra $\ell$ associated to the group $G$ (that is, the Lie algebra of left-invariant vector fields). We can fix a basis $X_1, \ldots, X_N$ in $\ell$ choosing $X_i$ as the left invariant vector field which agrees with $\frac{\partial}{\partial x_i}$ at the origin. It turns out that $X_i$ is homogeneous of degree $\omega_i$ (see [11], p. 164). Then, we can extend the dilations $D(\lambda)$ to $\ell$ setting

$$D(\lambda)\, X_i = \lambda^{\omega_i}\, X_i .$$

$D(\lambda)$ turns out to be a Lie algebra automorphism, i.e.,

$$D(\lambda)\,[X, Y] = [D(\lambda)X, D(\lambda)Y] .$$

In this sense, $\ell$ is said to be a homogeneous Lie algebra; as a consequence, $\ell$ is nilpotent (see [31], p. 621-2).

Recall that a Lie algebra $\ell$ is said to be graded if it admits a vector space decomposition as

$$\ell = \bigoplus_{i=1}^{r} V_i \quad \text{with } \left[V_i, V_j\right] \subseteq V_{i+j} \ \text{ for } i + j \le r,\ \left[V_i, V_j\right] = \{0\} \text{ otherwise.}$$

In this paper, $\ell$ will always be graded and it will be possible to choose $V_i$ as the set of vector fields homogeneous of degree $i$.

Also, a homogeneous Lie algebra is called stratified if there exist $s$ vector spaces $\widetilde{V}_1, \dots, \widetilde{V}_s$ such that

$$\ell = \bigoplus_{i=1}^{s} \widetilde{V}_i \quad \text{with } \left[\widetilde{V}_1, \widetilde{V}_i\right] = \widetilde{V}_{i+1} \ \text{ for } 1 \le i < s \text{ and } \left[\widetilde{V}_1, \widetilde{V}_s\right] = \{0\}.$$

This implies that the Lie algebra generated by $\widetilde{V}_1$ is the whole $\ell$. Clearly, if $\ell$ is stratified then $\ell$ is also graded.

Throughout this paper, we will deal with two different situations:

Case A. There exist $q$ vector fields ($q \le N$) $X_1, \dots, X_q$, homogeneous of degree 1 such that the Lie algebra generated by them is the whole $\ell$. Therefore $\ell$ is stratified and $\widetilde{V}_1$ is spanned by $X_1, \dots, X_q$. In this case the "natural" operator to be considered is

$$\mathcal{L} = \sum_{i=1}^{q} X_i^2, \tag{13}$$

which is hypoelliptic, left invariant and homogeneous of degree two.

EXAMPLE 1. The simplest (nonabelian) example of Case A is the Kohn-Laplacian on the Heisenberg group $G = \left(\mathbb{R}^3, \circ, D(\lambda)\right)$ where:

$$(x_1, y_1, t_1) \circ (x_2, y_2, t_2) =$$

$$= (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_2 y_1 - x_1 y_2))$$

and

$$D(\lambda)\ (x, y, t) = \left(\lambda x, \lambda y, \lambda^2 t\right).$$

$$X = \frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial t}; \quad Y = \frac{\partial}{\partial y} - 2x\,\frac{\partial}{\partial t}; \quad [X, Y] = -4\frac{\partial}{\partial t};$$

$$\ell = V_1 \oplus V_2 \quad \text{with} \quad V_1 = \langle X, Y\rangle.$$

The fields $X, Y$ are homogeneous of degree 1, and the operator

$$\mathcal{L} = X^2 + Y^2$$

is hypoelliptic and homogeneous of degree two. Here the homogeneous dimension of $G$ is $Q = 4$.

Case B. There exist $q + 1$ vector fields $(q + 1 \leq N)$ $X_0$, $X_1, \ldots, X_q$, such that the Lie algebra generated by them is the whole $\ell$, $X_1, \ldots, X_q$ are homogeneous of degree 1 and $X_0$ is homogeneous of degree 2. In this case the "natural" operator to be considered is

$$\mathcal{L} = \sum_{i=1}^{q} X_i^2 \; + \; X_0. \tag{14}$$

Under these assumptions $\ell$ may or may not be stratified (see examples below).

EXAMPLE 2. (Kolmogorov-type operators, studied in [23]).
Consider $G = \left(\mathbb{R}^3, \circ, D(\lambda)\right)$ with:

$$(x_1, y_1, t_1) \circ (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2 - x_1 t_2, t_1 + t_2)$$

and

$$D(\lambda)\ (x, y, t) = \left(\lambda x, \lambda^3 y, \lambda^2 t\right).$$

$$X_1 = \frac{\partial}{\partial x};\ X_0 = \frac{\partial}{\partial t} - x\,\frac{\partial}{\partial y};\ \left[X_0, X_1\right] = \frac{\partial}{\partial y};$$

$$\ell = \widetilde{V}_1 \oplus \widetilde{V}_2 \quad \text{with} \quad \widetilde{V}_1 = \langle X_1, X_0 \rangle, \widetilde{V}_2 = \langle \frac{\partial}{\partial y} \rangle \tag{15}$$

therefore $\ell$ is stratified; the fields $X_1, X_0$ are homogeneous of degree 1 and 2, respectively, and the operator

$$\mathcal{L} = X_1^2 + X_0$$

is hypoelliptic and homogeneous of degree two. Note that in this case the stratification (15) of $\ell$ is different from the natural decomposition of $\ell$ as a graded algebra:

$$\ell = V_1 \oplus V_2 \oplus V_3 \quad \text{with} \quad V_1 = \langle X_1 \rangle, V_2 = \langle X_0 \rangle, V_3 = \langle \frac{\partial}{\partial y} \rangle.$$

This is the simplest (nonabelian) example of Case B; note that $Q = 6$. If, keeping the same group law $\circ$, we changed the definition of $D(\lambda)$ setting

$$D(\lambda)\,(x, y, t) = \left(\lambda x, \lambda^2 y, \lambda t\right),$$

then the fields $X_0, X_1$ would be homogeneous of degree one, and we should consider the operator

$$\mathcal{L} = X_1^2 + X_0^2,$$

as in Case A.

EXAMPLE 3. This is an example of the non-stratified case.
Consider $G = \left(\mathbb{R}^5, \circ, D(\lambda)\right)$ with:

$$(x_1, y_1, z_1, w_1, t_1) \circ (x_2, y_2, z_2, w_2, t_2) =$$

$$= (x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2 + x_1 y_2,$$

$$t_1 + t_2 - x_1 x_2 y_1 - x_1 x_2 y_2 - \frac{1}{2} x_2^2\, y_1 + x_1 w_2 + x_1 z_2)$$

and

$$D(\lambda)\,(x, y, z, w, t) = \left(\lambda x, \lambda y, \lambda^2 z, \lambda^2 w, \lambda^3 t\right).$$

The natural base for $\ell$ consists of:

$$X = \frac{\partial}{\partial x} - xy\frac{\partial}{\partial t};\; Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial w};\; Z = \frac{\partial}{\partial z} + x\frac{\partial}{\partial t};$$

$$W = \frac{\partial}{\partial w} + x\frac{\partial}{\partial t};\; T = \frac{\partial}{\partial t}.$$

We can see that $\ell$ is graded setting

$$\ell = V_1 \oplus V_2 \oplus V_3 \quad \text{with} \quad V_1 = \langle X, Y\rangle, V_2 = \langle Z, W\rangle, V_3 = \langle T\rangle.$$

The nontrivial commutation relations are:

$$[X, Y] = W; \quad [X, Z] = T; \quad [X, W] = T.$$

Therefore, if we set $\widetilde{V}_1 = \langle X, Y, Z\rangle$, we see that the Lie algebra generated by $\widetilde{V}_1$ is $\ell$; moreover $\widetilde{V}_2 = \left[\widetilde{V}_1, \widetilde{V}_1\right] = \langle W, T\rangle$ and $\widetilde{V}_3 = \left[\widetilde{V}_1, \widetilde{V}_2\right] = \langle T\rangle$, so that $\ell$ is not stratified. Noting that $X, Y, Z$ are homogeneous of degrees $1, 1, 2$ respectively, we have that the operator

$$\mathcal{L} = X^2 + Y^2 + Z$$

is hypoelliptic and homogeneous of degree two.

### 2.2. Function spaces

Before going on, we need to introduce some notation and function spaces. First of all, if $X_0$, $X_1, \ldots, X_q$ are the vector fields appearing in (13)-(14), define, for $p \in [1, \infty]$

$$\|Du\|_p \equiv \sum_{i=1}^{q} \|X_i u\|_p\;;$$

$$\left\|D^2 u\right\|_p \equiv \sum_{i,j=1}^{q} \left\|X_i X_j u\right\|_p + \|X_0 u\|_p\,.$$

More in general, set

$$\left\|D^k u\right\|_p \equiv \sum \left\|X_{j_1} \ldots X_{j_l} u\right\|_p$$

where the sum is taken over all monomials $X_{j_1} \ldots X_{j_l}$ homogeneous of degree $k$. (Note that $X_0$ has weight two while the remaining fields have weight one. Obviously, in Case A the field $X_0$

does not appear in the definition of the above norms). Let $\Omega$ be a domain in $\mathbb{R}^N$, $p \in [1,\infty]$ and $k$ be a nonnegative integer. The space $S^{k,p}(\Omega)$ consists of all $\mathcal{L}^p(\Omega)$ functions such that

$$\|u\|_{S^{k,p}(\Omega)} = \sum_{h=0}^{k} \left\| D^h u \right\|_{\mathcal{L}^p(\Omega)}$$

is finite. We shall also denote by $S_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $S^{k,p}(\Omega)$.

Since we will often consider the case $k = 2$, we will briefly write $S^p(\Omega)$ for $S^{2,p}(\Omega)$ and $S_0^p(\Omega)$ for $S_0^{2,p}(\Omega)$.

Note that the fields $X_i$, and therefore the definition of the above norms, are completely determined by the structure of $G$.

We define the Hölder spaces $\Lambda^{k,\alpha}(\Omega)$, for $\alpha \in (0,1)$, $k$ nonnegative integer, setting

$$|u|_{\Lambda^\alpha(\Omega)} = \sup_{\substack{x \neq y \\ x,y \in \Omega}} \frac{|u(x)-u(y)|}{d(x,y)^\alpha}$$

and

$$\|u\|_{\Lambda^{k,\alpha}(\Omega)} = \left| D^k u \right|_{\Lambda^\alpha(\Omega)} + \sum_{j=0}^{k-1} \left\| D^j u \right\|_{\mathcal{L}^\infty(\Omega)}.$$

In §4, we will also use the fractional (but isotropic) Sobolev spaces $H^{t,2}\left(\mathbb{R}^N\right)$, defined in the usual way, setting, for $t \in \mathbb{R}$,

$$\|u\|_{H^{t,2}}^2 = \int_{\mathbb{R}^N} |\widehat{u}(\xi)|^2 \left(1+|\xi|^2\right)^t d\xi,$$

where $\widehat{u}(\xi)$ denotes the Fourier transform of $u$.

The structure of space of homogenous type allows us to define the space of Bounded Mean Oscillation functions ($BMO$, see [18]) and the space of Vanishing Mean Oscillation functions ($VMO$, see [30]). If $f$ is a locally integrable function, set

(16) $$\eta_f(r) = \sup_{\rho<r} \frac{1}{|B_\rho|} \int_{B_\rho} \left| f(x) - f_{B_\rho} \right| dx \quad \text{for every } r > 0,$$

where $B_\rho$ is any ball of radius $\rho$ and $f_{B_\rho}$ is the average of $f$ over $B_\rho$.

We say that $f \in BMO$ if $\|f\|_* \equiv \sup_r \eta_f(r) < \infty$.

We say that $f \in VMO$ if $f \in BMO$ and $\eta_f(r) \to 0$ for $r \to 0$.

We can also define the spaces $BMO(\Omega)$ and $VMO(\Omega)$ for a domain $\Omega \subset \mathbb{R}^N$, just replacing $B_\rho$ with $B_\rho \cap \Omega$ in (16).

### 2.3. Assumptions and main results

We now state precisely our assumptions, keeping all the notation of §§2.1, 2.2.

Let $G$ be a homogeneous group of homogeneous dimension $Q \geq 3$ and $\ell$ its Lie algebra; let $\{X_i\}$ $(i = 1, 2, \ldots, N)$ be the basis of $\ell$ constructed as in §2.1, and assume that the conditions of

Case A or Case B hold. Accordingly, we will study the following classes of operators, modeled on the translation invariant prototypes (13), (14):

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j$$

or

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + a_0(x) X_0 \tag{17}$$

where $a_{ij}$ and $a_0$ are real valued bounded measurable functions and the matrix $\left\{a_{ij}(x)\right\}$ satisfies a uniform ellipticity condition:

$$\mu\, |\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(x)\, \xi_i \xi_j \le \mu^{-1} |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^q, \text{ a.e. } x, \tag{18}$$

for some positive constant $\mu$. Analogously,

$$\mu \le a_0(x) \le \mu^{-1}. \tag{19}$$

Moreover, we will assume

$$a_0, a_{ij} \in VMO.$$

Then:

THEOREM 2. *Under the above assumptions, for every $p \in (1, \infty)$ there exist $c = c(p, \mu, G)$ and $r = r(p, \mu, \eta, G)$ such that if $u \in C_0^\infty\left(\mathbb{R}^N\right)$ and $\operatorname{sprt} u \subseteq B_r$ ($B_r$ any ball of radius $r$) then*

$$\left\| D^2 u \right\|_p \le c\, \|\mathcal{L}u\|_p$$

*where $\eta$ denotes dependence on the "$VMO$ moduli" of the coefficients $a_0, a_{ij}$.*

THEOREM 3 (LOCAL ESTIMATES FOR SOLUTIONS TO THE EQUATION $\mathcal{L}u = f$ IN A DOMAIN). *Under the above assumptions, let $\Omega$ be a bounded domain of $\mathbb{R}^N$ and $\Omega' \subset\subset \Omega$. If $u \in S^p(\Omega)$, then*

$$\|u\|_{S^p(\Omega')} \le c \left\{ \|\mathcal{L}u\|_{\mathcal{L}^p(\Omega)} + \|u\|_{\mathcal{L}^p(\Omega)} \right\}$$

*where $c = c(p, G, \mu, \eta, \Omega, \Omega')$.*

THEOREM 4 (LOCAL HÖLDER CONTINUITY FOR SOLUTIONS TO THE EQUATION $\mathcal{L}u = f$ IN A DOMAIN). *Under the assumptions of Theorem 3, if $u \in S^p(\Omega)$ for some $p \in (1, \infty)$ and $\mathcal{L}u \in \mathcal{L}^s(\Omega)$ for some $s > Q/2$, then*

$$\|u\|_{\Lambda^\alpha(\Omega')} \le c \left\{ \|\mathcal{L}u\|_{\mathcal{L}^r(\Omega)} + \|u\|_{\mathcal{L}^p(\Omega)} \right\}$$

*for $r = \max(p, s)$, $\alpha = \alpha(Q, p, s) \in (0, 1)$, $c = c(G, \mu, p, s, \Omega, \Omega')$.*

THEOREM 5 (REGULARITY OF THE SOLUTION IN TERMS OF SOBOLEV SPACES). *Under the assumptions of Theorem 3, if $a_0$, $a_{ij} \in S^{k,\infty}(\Omega)$, $u \in S^p(\Omega)$ and $\mathcal{L}u \in S^{k,p}(\Omega)$ for some positive integer k (k even, in Case B), $1 < p < \infty$, then*

$$\|u\|_{S^{k+2,p}(\Omega')} \leq c_1 \left\{ \|\mathcal{L}u\|_{S^{k,p}(\Omega)} + c_2 \|u\|_{\mathcal{L}^p(\Omega)} \right\}$$

*where $c_1 = c_1(p, G, \mu, \eta, \Omega, \Omega')$ and $c_2$ depends on the $S^{k,\infty}(\Omega)$ norms of the coefficients.*

THEOREM 6 (REGULARITY OF THE SOLUTION IN TERMS OF HÖLDER SPACES). *Under the assumption of Theorem 3, if $a_0$, $a_{ij} \in S^{k,\infty}(\Omega)$, $u \in S^p(\Omega)$ and $\mathcal{L}u \in S^{k,s}(\Omega)$ for some positive integer k (k even, in Case B), $1 < p < \infty$, $s > Q/2$, then*

$$\|u\|_{\Lambda^{k,\alpha}(\Omega')} \leq c_1 \left\{ \|\mathcal{L}u\|_{S^{k,r}(\Omega)} + c_2 \|u\|_{\mathcal{L}^p(\Omega)} \right\}$$

*where $r = max(p, s)$, $\alpha = \alpha(Q, p, s) \in (0, 1)$, $c_1 = c_1(p, s, k, G, \mu, \eta, \Omega, \Omega')$ and $c_2$ depends on the $S^{k,\infty}(\Omega)$ norms of the coefficients.*

THEOREM 7 (OPERATORS WITH LOWER ORDER TERMS). *Consider an operator with "lower order terms" (in the sense of the degree of homogeneity), of the following kind:*

$$\mathcal{L} \equiv \left( \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + a_0(x) X_0 \right) + \left( \sum_{i=1}^{q} c_i(x) X_i + c_0(x) \right) \equiv$$
$$\equiv \mathcal{L}_2 + \mathcal{L}_1.$$

*i) If $c_i \in \mathcal{L}^\infty(\Omega)$ for $i = 0, 1, \ldots, q$, then:*
*if the assumptions of Theorem 3 hold for $\mathcal{L}_2$, then the conclusions of Theorem 3 hold for $\mathcal{L}$;*
*if the assumptions of Theorem 4 hold for $\mathcal{L}_2$, then the conclusions of Theorem 4 hold for $\mathcal{L}$.*
*ii) If $c_i \in S^{k,\infty}(\Omega)$ for some positive integer k, $i = 0, 1, \ldots, q$, then:*
*if the assumptions of Theorem 5 hold for $\mathcal{L}_2$, then the conclusions of Theorem 5 hold for $\mathcal{L}$;*
*if the assumptions of Theorem 6 hold for $\mathcal{L}_2$, then the conclusions of Theorem 6 hold for $\mathcal{L}$.*

REMARK 1. Since all our results are local, it is unnatural to assume that the coefficients $a_0$, $a_{ij}$ be defined on the whole $\mathbb{R}^N$. Actually, it can be proved that any function $f \in VMO(\Omega)$, with $\Omega$ bounded Lipschitz domain, can be extended to a function $\widetilde{f}$ defined in $\mathbb{R}^N$ with $VMO$ modulus controlled by that of $f$. (For more details see [3]). Therefore, all the results of Theorems 2, 7 still hold if the coefficients belong to $VMO(\Omega)$, but it is enough to prove them for $a_0$, $a_{ij} \in VMO$.

### 2.4. Relations with operators of Hörmander type

Here we want to point out the relationship between our class of operators and operators of Hörmander type (2).

THEOREM 8. *Under the assumptions of §2.3:*

(*i*) *if the coefficients $a_{ij}(x)$ are Lipschitz continuous (in the usual sense), then the operator $\mathcal{L}$ can be rewritten in the form*

$$\mathcal{L} = \sum_{i=1}^{q} Y_i^2 + Y_0$$

*where the vector fields $Y_i$ ($i = 1, \dots, q$) have Lipschitz coefficients and $Y_0$ has bounded measurable coefficients;*

($ii$) *if the coefficients $a_{ij}(x)$ are smooth ($C^\infty$), then $\mathcal{L}$ is hypoelliptic;*

($iii$) *if the coefficients $a_{ij}$ are constant, then $\mathcal{L}$ is left invariant and homogeneous of degree two; moreover, the transpose $\mathcal{L}^T$ of $\mathcal{L}$ is hypoelliptic, too.*

*Proof.* Let us split the matrix $a_{ij}(x)$ in its symmetric and skew-symmetric parts:

$$a_{ij}(x) = \frac{1}{2}\left(a_{ij}(x) + a_{ji}(x)\right) + \frac{1}{2}\left(a_{ij}(x) - a_{ji}(x)\right) \equiv b_{ij}(x) + \widetilde{b}_{ij}(x).$$

If the matrix $A = \{a_{ij}(x)\}$ satisfies condition (18), the same holds for $B = \{b_{ij}(x)\}$. Therefore we can write $B = MM^T$ where $M = \{m_{ij}(x)\}$ is an invertible, triangular matrix, whose entries are $C^\infty$ functions of the entries of $B$.

To see this, we can use the "method of completion of squares" (see $e.g.$ [17], p. 180), writing

$$\sum_{i,j=1}^{q} b_{ij}\,\xi_i\xi_j = \eta_1^2 + \sum_{i,j=2}^{q} b_{ij}^*\eta_i\eta_j$$

with

$$\eta_1 = \left(\sqrt{b_{11}}\,\xi_1 + \sum_{j=2}^{q}\frac{b_{1j}}{\sqrt{b_{11}}}\,\xi_j\right)\ ;\quad \eta_i = \xi_i \ \text{ for } i \geq 2;$$

$$b_{ii}^* = b_{ii} - \frac{b_{1i}^2}{b_{11}}\ ;\quad b_{ij}^* = b_{ij} \quad \text{for } i,j = 2,\dots,q,\ i \neq j.$$

Since $(\eta_1,\dots,\eta_q)$ are a linear invertible function of $(\xi_1,\dots,\xi_q)$, and the quadratic form $\sum_{i,j=1}^{q} b_{ij}\,\xi_i\xi_j$ is positive (on $\mathbb{R}^q$), also the quadratic form $\sum_{i,j=2}^{q} b_{ij}^*\eta_i\eta_j$ is positive (on $\mathbb{R}^{q-1}$), and we can iterate the same procedure. Note that $\eta_1 = \sum_{k=1}^{q} m_{1k}\xi_k$ with $m_{1k}$ smooth functions of the $b_{ij}$'s; moreover, $b_{ij}^*$ are smooth functions of the $b_{ij}$'s. Therefore iteration of this procedure allows us to write

$$\sum_{i,j=1}^{q} b_{ij}\,\xi_i\xi_j = \sum_{k=1}^{q}\lambda_k^2 \quad \text{with:}$$

$$\lambda_k = \sum_{h=k}^{q} m_{kh}\xi_h \ \text{ and }\ m_{kh} \ \text{ are smooth functions of the } b_{ij}\text{'s.}$$

This means that $b_{ij} = \sum_{k\geq i,j} m_{ki}\,m_{kj}$ with $m_{kh}$ smooth functions of the $b_{ij}$'s.

Therefore we can write:

$$\mathcal{L} = \sum_{i,j=1}^{q}\sum_{k=1}^{q} m_{ik}(x)\,m_{jk}(x)X_iX_j + \sum_{i<j}\widetilde{b}_{ij}(x)\left[X_i,X_j\right] + a_0(x)X_0$$

where the functions $m_{ik}(x)$ have the same regularity of the $a_{ij}(x)$'s. (To simplify the notation, from now on we forget the fact that $m_{ik} = 0$ if $k < i$). If the $a_{ij}(x)$'s are Lipschitz continuous, the above equation can be rewritten as

(20) $$\mathcal{L} = \sum_{k=1}^{q} Y_k^2 \ + \ Y_0$$

with

$$Y_k = \sum_{i=1}^{q} m_{ik}(x) X_i \quad \text{and}$$

$$Y_0 = \sum_{i<j} \widetilde{b}_{ij}(x) \left[X_i, X_j\right] + a_0(x) X_0 - \sum_{i,j=1}^{q} \sum_{k=1}^{q} m_{ik}(x) \cdot \left(X_i\, m_{jk}(x)\right) X_j,$$

which proves (i). If the coefficients $a_{ij}(x)$ are $C^\infty$, the $Y_i$'s are $C^\infty$ vector fields and satisfy Hörmander's condition, because every linear combination of the $X_i$ ($i = 0, 1, \ldots, q$) can be rewritten as a linear combination of the $Y_i$ and their commutators of length 2. Therefore, by Theorem 1, $\mathcal{L}$ is hypoelliptic, that is (ii). Finally, if the coefficients $a_{ij}$ are constant, then (20) holds with

$$Y_k = \sum_{i=1}^{q} m_{ik}\, X_i \;\text{ and }\; Y_0 = \sum_{i<j} \widetilde{b}_{ij}\, \left[X_i, X_j\right] + a_0\, X_0,$$

which means that $\mathcal{L}$ is left invariant and homogeneous of degree two. Moreover, since the fields $X_i$ are translation invariant, the transpose $X_i^T$ of $X_i$ equals $-X_i$ and as a consequence $\mathcal{L}^T$ is hypoelliptic as well. This proves (iii).

□

REMARK 2. By the above Theorem, if $a_{ij} \in C^\infty$, our class of operators is contained in that studied by Rothschild-Stein [28], so in this case our results follow from [28], without assuming the existence of a structure of homogeneous group. If the coefficients are less regular, but at least Lipschitz continuous, our operators can be written as “operators of Hörmander type”; however, in this case we cannot check Hörmander's condition for the fields $Y_i$ and therefore our estimates do not follow from known results about hypoelliptic operators. Finally, if the coefficients are merely $VMO$, we cannot even write $\mathcal{L}$ in the form (20).

### 2.5. More properties of homogeneous groups

We recall some known results which will be useful later. First of all, we define the convolution of two functions in $G$ as

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x \circ y^{-1})\, g(y)\, dy = \int_{\mathbb{R}^N} g(y^{-1} \circ x)\; f(y)\, dy,$$

for every couple of functions for which the above integrals make sense. From this definition we read that if $P$ is any left invariant differential operator,

$$P(f * g) = f * Pg$$

(provided the integrals converge). Note that, if $G$ is not abelian, we cannot write $f * Pg = Pf * g$. Instead, if $X$ and $X^R$ are, respectively, a left invariant and right invariant vector field which agree at the origin, the following hold (see [31], p. 607)

$$(Xf) * g = f * \left(X^R g\right); \quad X^R(f * g) = \left(X^R f\right) * g. \tag{21}$$

In view of the above identities, we will sometimes use the right invariant vector fields $X_i^R$ which agree with $\partial/\partial x_i$ (and therefore with $X_i$) at the origin ($i = 1, \ldots, N$), and we need some properties linking $X_i$ to $X_i^R$. It can be proved that

$$X_i = \frac{\partial}{\partial x_i} + \sum_{k=i+1}^{N} q_i^k(x) \frac{\partial}{\partial x_k}$$

$$X_i^R = \frac{\partial}{\partial x_i} + \sum_{k=i+1}^{N} \widetilde{q}_i^k(x) \frac{\partial}{\partial x_k}$$

where $q_i^k(x), \widetilde{q}_i^k(x)$ are polynomials, homogeneous of degree $\omega_k - \omega_i$ (the $\omega_i$'s are the exponents appearing in (6)). From the above equations we find that

$$X_i = \sum_{k=i}^{N} c_i^k(x)\, X_k^R$$

where $c_i^k(x)$ are polynomials, homogeneous of degree $\omega_k - \omega_i$. In particular, since $\omega_k - \omega_i < \omega_k$, $c_i^k(x)$ does not depend on $x_h$ for $h \geq k$ and therefore commutes with $X_k^R$, that is

$$X_i u = \sum_{k=i}^{N} X_k^R \left( c_i^k(x)\, u \right) \qquad (i = 1, \ldots, N) \tag{22}$$

for every test function $u$. This representation of $X_i$ in terms of $X_i^R$ will be useful in §6.

THEOREM 9. *(See Theorem 2.1 and Corollary 2.8 in* [11]*). Let $\mathcal{L}$ be a left invariant differential operator homogeneous of degree two on $G$, such that $\mathcal{L}$ and $\mathcal{L}^T$ are both hypoelliptic. Moreover, assume $Q \geq 3$. Then there is a unique fundamental solution $\Gamma$ such that:*

(*a*) $\Gamma \in C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$;

(*b*) *$\Gamma$ is homogeneous of degree $(2 - Q)$;*

(*c*) *for every distribution $\tau$,*

$$\mathcal{L}(\tau * \Gamma) = (\mathcal{L}\tau) * \Gamma = \tau.$$

THEOREM 10. *(See Proposition 8.5 in* [13]*), Proposition 1.8 in* [11]*)). Let $K_h$ be a kernel which is $C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$ and homogeneous of degree $(h - Q)$, for some integer $h$ with $0 < h < Q$; let $T_h$ be the operator*

$$T_h f = f * K_h$$

*and let $P^h$ be a left invariant differential operator homogeneous of degree $h$.*

*Then:*

$$P^h T_h f = \text{P.V.}\left( f * P^h K_h \right) + \alpha f$$

*for some constant $\alpha$ depending on $P^h$ and $K_h$;*

*the function $P^h K_h$ is $C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$, homogeneous of degree $-Q$ and satisfies the vanishing property:*

$$\int_{r<\|x\|<R} P^h K_h\,(x)\,dx = 0 \quad \text{for } 0 < r < R < \infty;$$

*the singular integral operator*

$$f \mapsto P.V.\Big(f * P^h K_h\Big)$$

*is continuous on* $\mathcal{L}^p$ *for* $1 < p < \infty$.

To handle the convolution of several kernels, we will need also the following

LEMMA 1. *Let* $K_1(\cdot,\cdot)$, $K_2(\cdot,\cdot)$ *be two kernels satisfying the following:*

($i$) *for every* $x \in \mathbb{R}^N$ $K_i(x,\cdot) \in C^\infty(\mathbb{R}^N \setminus \{0\})$ $(i = 1, 2)$*;*

($ii$) *for every* $x \in \mathbb{R}^N$ $K_i(x,\cdot)$ *is homogeneous of degree* $\alpha_i$, *with* $-Q < \alpha_i < 0, \alpha_1 + \alpha_2 < -Q$*;*

($iii$) *for every multiindex* $\beta$,

$$\sup_{x \in \mathbb{R}^N} \sup_{\|y\|=1} \left| \left(\frac{\partial}{\partial y}\right)^\beta K_i(x,y) \right| \le c_\beta .$$

*Then, for every test function* $f$ *and any* $x_0, y_0 \in \mathbb{R}^N$,

$$(f * K_1(x_0,\cdot)) * K_2(y_0,\cdot) = f * (K_1(x_0,\cdot) * K_2(y_0,\cdot)) .$$

*Moreover, setting* $K(x_0, y_0,\cdot) = K_1(x_0,\cdot) * K_2(y_0,\cdot)$, *we have the following:*

($iv$) *for every* $(x_0, y_0) \in \mathbb{R}^{2N}$, $K(x_0, y_0,\cdot) \in C^\infty(\mathbb{R}^N \setminus \{0\})$*;*

($v$) *for every* $(x_0, y_0) \in \mathbb{R}^{2N}$, $K(x_0, y_0,\cdot)$ *is homogeneous of degree* $\alpha_1 + \alpha_2 + Q$*;*

($vi$) *for every multiindex* $\beta$,

$$\sup_{(x,y) \in \mathbb{R}^{2N}} \sup_{\|z\|=1} \left| \left(\frac{\partial}{\partial z}\right)^\beta K(x,y,z) \right| \le c_\beta . \tag{23}$$

The above Lemma has been essentially proved by Folland (see Proposition 1.13 in [11]), apart from the uniform bound on $K$, which follows reading carefully the proof.

## 3. Proof of Theorem 2

All the proofs in this section will be written for the Case B. The results in Case A (which is easier) simply follow dropping the term $X_0$.

### 3.1. Fundamental solutions

For any $x_0 \in \mathbb{R}^N$, let us “freeze” at $x_0$ the coefficients $a_{ij}(x)$, $a_0(x)$ of the operator (17), and consider

$$\mathcal{L}_0 = \sum_{i,j=1}^{q} a_{ij}(x_0) X_i X_j + a_0(x_0) X_0 . \tag{24}$$

By Theorem 8, the operator $\mathcal{L}_0$ satisfies the assumptions of Theorem 9; therefore, it has a fundamental solution with pole at the origin which is homogeneous of degree $(2 - Q)$. Let us denote it by $\Gamma(x_0; \cdot)$, to indicate its dependence on the frozen coefficients $a_{ij}(x_0)$, $a_0(x_0)$. Also, set for $i, j = 1, \ldots, q$,

$$\Gamma_{ij}(x_0; y) = X_i X_j \left[\Gamma(x_0; \cdot)\right](y).$$

Next theorem summarizes the properties of $\Gamma(x_0; \cdot)$ and $\Gamma_{ij}(x_0; \cdot)$ that we will need in the following. All of them follow from Theorem 9 and Lemma 1.

THEOREM 11. *For every $x_0 \in \mathbb{R}^N$:*
$(a)$ $\Gamma(x_0, \cdot) \in C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$*;*
$(b)$ $\Gamma(x_0, \cdot)$ *is homogeneous of degree* $(2-Q)$*;*
$(c)$ *for every test function $u$ and every $x \in \mathbb{R}^N$,*

$$u(x) = (\mathcal{L}_0 u * \Gamma(x_0; \cdot))(x) = \int_{\mathbb{R}^N} \Gamma\left(x_0; y^{-1} \circ x\right) \mathcal{L}_0 u(y) dy;$$

*moreover, for every $i, j = 1, \dots, q$, there exist constants $\alpha_{ij}(x_0)$ such that*

$$X_i X_j u(x) = P.V. \int_{\mathbb{R}^N} \Gamma_{ij}\left(x_0; y^{-1} \circ x\right) \mathcal{L}_0 u(y) dy + \alpha_{ij}(x_0) \cdot \mathcal{L}_0 u(x); \tag{25}$$

$(d)$ $\Gamma_{ij}(x_0; \cdot) \in C^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$*;*
$(e)$ $\Gamma_{ij}(x_0; \cdot)$ *is homogeneous of degree* $-Q$*;*
$(f)$ *for every* $R > r > 0$,

$$\int_{r < \|y\| < R} \Gamma_{ij}(x_0; y)\, dy = \int_{\|y\|=1} \Gamma_{ij}(x_0; y)\, d\sigma(y) = 0.$$

The above properties hold for any fixed $x_0$. We also need some uniform bound for $\Gamma$, with respect to $x_0$. Next theorem contains this kind of result.

THEOREM 12. *For every multi-index $\beta$, there exists a constant $c_1 = c_1(\beta, G, \mu)$ such that*

$$\sup_{\substack{x \in \mathbb{R}^N \\ \|y\|=1}} \left| \left(\frac{\partial}{\partial y}\right)^\beta \Gamma_{ij}(x; y) \right| \leq c_1, \tag{26}$$

*for any $i, j = 1, \dots, q$. Moreover, for the $\alpha_{ij}$'s appearing in* (25)*, a uniform bound holds:*

$$\sup_{x \in \mathbb{R}^N} \left|\alpha_{ij}(x)\right| \leq c_2, \tag{27}$$

*for some constant $c_2 = c_2(G, \mu)$.*

We postpone the proof of the above Theorem to §4. The proof of Theorem 2 from Theorems 11, 12 proceeds in three steps, which are explained in §§3.2, 3.3, 3.4.

### 3.2. Representation formula and singular integrals

Let us consider (25). Writing $\mathcal{L}_0 = \mathcal{L} + (\mathcal{L}_0 - \mathcal{L})$ and then letting $x$ be equal to $x_0$, we get the following representation formula:

THEOREM 13. *Let $u \in C_0^\infty\left(\mathbb{R}^N\right)$. Then, for $i, j = 1, \dots, q$ and every $x \in \mathbb{R}^N$*

$$X_i X_j u(x) = P.V. \int \Gamma_{ij}(x; y^{-1} \circ x)\Bigg( \sum_{h,k=1}^{q} \left[a_{hk}(x) - a_{hk}(y)\right] X_h X_k u(y) +$$

(28)
$$+\left[a_0(x) - a_0(y)\right] X_0 u(y) + \mathcal{L}u(y)\Bigg) dy + \alpha_{ij}(x) \cdot \mathcal{L}u(x).$$

In order to rewrite the above formula in a more compact form, let us introduce the following singular integral operators:

(29)
$$K_{ij} f(x) = P.V. \int \Gamma_{ij}(x; y^{-1} \circ x)\, f(y)\, dy.$$

Moreover, for an operator $K$ and a function $a \in \mathcal{L}^\infty\left(\mathbb{R}^N\right)$, define the commutator

$$C[K, a](f) = K(af) - a \cdot K(f).$$

Then (28) becomes

$$X_i X_j u = K_{ij}(\mathcal{L}u) - \sum_{h,k=1}^{q} C\left[K_{ij}, a_{hk}\right](X_h X_k u) +$$

(30)
$$+C\left[K_{ij}, a_0\right](X_0 u) + \alpha_{ij} \cdot \mathcal{L}u$$

for $i, j = 1, \dots, q$.

Now the desired $\mathcal{L}^p$-estimate on $X_i X_j u$ depends on suitable singular integral estimates. Namely, we will prove the following:

THEOREM 14. *For every $p \in (1, \infty)$ there exists a positive constant $c = c(p, \mu, G)$ such that for every $a \in BMO$, $f \in \mathcal{L}^p\left(\mathbb{R}^N\right)$, $i, j = 1, \dots, q$:*

(31)
$$\left\| K_{ij}(f) \right\|_{\mathcal{L}^p(\mathbb{R}^N)} \le c \, \| f \|_{\mathcal{L}^p(\mathbb{R}^N)}$$

(32)
$$\left\| C\left[K_{ij}, a\right](f) \right\|_{\mathcal{L}^p(\mathbb{R}^N)} \le c \, \|a\|_* \, \| f \|_{\mathcal{L}^p(\mathbb{R}^N)}.$$

The estimate (32) can be localized in the following way (see [6] for the technique of the proof):

THEOREM 15. *If the function $a$ belongs to $VMO$, then for every $\varepsilon > 0$ there exists $r > 0$, depending on $\varepsilon$ and the $VMO$ modulus of $a$, such that for every $f \in \mathcal{L}^p$, with sprt $f \subseteq B_r$*

(33)
$$\left\| C\left[K_{ij}, a\right](f) \right\|_{\mathcal{L}^p(B_r)} \le c(p, \mu, G) \cdot \varepsilon \, \| f \|_{\mathcal{L}^p(B_r)}.$$

Finally, using the bounds (27), (31), (33) in the representation formula (30), we get Theorem 2. Note that the term $X_0 u$ can be estimated either by the same method used for $X_i X_j u$ for $i, j = 1, \dots, q$, or by difference.

So the proof of Theorem 2 relies on Theorem 14 (which will follow from §§3.3, 3.4), and Theorem 12 (which will follow from §4).

### 3.3. Expansion in series of spherical harmonics and reduction of singular integrals "with variable kernel" to singular integrals of convolution type

To prove Theorem 14, we have to handle singular integrals of kind (29), which are not of convolution type because of the presence of the first variable $x$ in the kernel, which comes from the variable coefficients $a_{ij}(x)$ of the differential operator $\mathcal{L}$. To bypass this difficulty, we can apply the standard technique of expanding the kernel in series of spherical harmonics. This idea dates back to Calderón-Zygmund [10], in the case of "standard" singular integrals, and has been adapted to kernels with mixed homogeneities by Fabes-Rivière [12]. We briefly describe this technique. (See [10] for details). Let

$$\{Y_{km}\}_{\substack{m=0,1,2,\ldots \\ k=1,\ldots,g_m}}$$

be an orthonormal system of spherical harmonics in $\mathbb{R}^N$, complete in $\mathcal{L}^2(\Sigma_N)$ ($m$ is the degree of the polynomial, $g_m$ is the dimension of the space of spherical harmonics of degree $m$ in $\mathbb{R}^N$). For any fixed $x \in \mathbb{R}^N$, $y \in \Sigma_N$, we can expand:

$$\Gamma_{ij}(x;y) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(x)\ Y_{km}(y) \quad \text{for } i, j = 1, \ldots, q. \tag{34}$$

We explicitly note that for $m = 0$ the coefficients in the above expansion are zero, because of the vanishing property $(f)$ of Theorem 11. Also, note that the integral of $Y_{km}(y)$ over $\Sigma_N$, for $m \geq 1$, is zero. If $y \in \mathbb{R}^N$, let $y' = D\left(\|y\|^{-1}\right) y$; recall that, by (ii) at page 393, $y' \in \Sigma_N$. By (34) and homogeneity of $\Gamma_{ij}(x;\cdot)$ we have

$$\Gamma_{ij}(x;y) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(x)\ \frac{Y_{km}\left(y'\right)}{\|y\|^Q} \quad \text{for } i, j = 1, \ldots, q.$$

Then

$$K_{ij}(f)(x) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c_{ij}^{km}(x)\ T_{km} f(x) \tag{35}$$

with

$$T_{km} f(x) = P.V. \int H_{km}(y^{-1} \circ x)\ f(y)\, dy \tag{36}$$

and

$$H_{km}(x) = \frac{Y_{km}\left(x'\right)}{\|x\|^Q}. \tag{37}$$

We will use the following bounds about spherical harmonics:

$$g_m \leq c(N) \cdot m^{N-2} \quad \text{for every } m = 1, 2, \ldots \tag{38}$$

$$\left| \left(\frac{\partial}{\partial x}\right)^{\beta} Y_{km}(x) \right| \leq c(N) \cdot m^{\left(\frac{N-2}{2} + |\beta|\right)} \tag{39}$$

for $x \in \Sigma_N$, $k = 1, \dots, g_m$, $m = 1, 2, \dots$.

Moreover, if $f \in C^\infty(\Sigma_N)$ and if $f(x) \sim \sum_{k,m} b_{km} Y_{km}(x)$ is the Fourier expansion of $f(x)$ with respect to $\left\{Y_{km}\right\}$, that is

$$b_{km} = \int_{\Sigma_N} f(x)\, Y_{km}(x)\, d\sigma(x)$$

then, for every positive integer $r$ there exists $c_r$ such that

$$|b_{km}| \leq c_r \cdot m^{-2r} \sup_{\substack{x\in\Sigma_N \\ |\beta|=2r}} \left| \left(\frac{\partial}{\partial x}\right)^{\beta} f(x) \right|. \tag{40}$$

In view of Theorem 12, we get from (40) the following bound on the coefficients $c_{ij}^{km}(x)$ appearing in the expansion (34): for every positive integer $r$ there exists a constant $c = c(r, G, \mu)$ such that

$$\sup_{x\in\mathbb{R}^N} \left| c_{ij}^{km}(x) \right| \leq c(r, G, \mu) \cdot m^{-2r} \tag{41}$$

for every $m = 1, 2, \dots$; $k = 1, \dots, g_m$; $i, j = 1, \dots, q$.

### 3.4. Estimates on singular integrals of convolution type and their commutators, and convergence of the series

We now focus our attention on the singular integrals of convolution type defined by (36), (37) and their commutators. Our goal is to prove, for these operators, bounds of the kind (31), (32); moreover, we need to know explicitly the dependence of the constants on the indexes $k, m$, appearing in the series (35). To this aim, we apply some abstract results about singular integrals in spaces of homogeneous type, proved by Bramanti-Cerutti in [4]. To state precisely these results, we recall the following:

DEFINITION 1. *Let $X$ be a set and $d: X \times X \to [0, \infty)$. We say that $d$ is a quasidistance if it satisfies properties (9), (10), (11). The balls defined by $d$ induce a topology in $X$; we assume that the balls are open sets, in this topology. Moreover, we assume there exists a regular Borel measure $\mu$ on $X$, such that the "doubling condition" is satisfied:*

$$\mu\left(B_{2r}(x)\right) \leq c \cdot \mu\left(B_r(x)\right)$$

*for every $r > 0$, $x \in X$, some constant $c$. Then we say that $(X, d, \mu)$ is a space of homogenous type.*

*Let $(X, d, \mu)$ be an unbounded space of homogenous type. For every $x \in X$, define*

$$r_x = \sup\{r > 0 : B_r(x) = \{x\}\}$$

*(here $\sup\emptyset = 0$). We say that $(X, d, \mu)$ satisfies a reverse doubling condition if there exist $c' > 1$, $M > 1$ such that for every $x \in X$, $r > r_x$*

$$\mu(B_{Mr}(x)) \geq c' \cdot \mu(B_r(x)).$$

THEOREM 16. (See [4]). *Let $(X, d, \mu)$ be a space of homogenous type and, if $X$ is unbounded, assume that the reverse doubling condition holds. Let $k : X \times X \setminus \{x = y\} \to \mathbb{R}$ be a kernel satisfying:*

*i) the growth condition:*

$$|k(x, y)| \le \frac{c_1}{\mu\left(B(x, d(x, y))\right)} \quad \text{for every } x, y \in X, \text{ some constant } c_1 \tag{42}$$

*ii) the "Hörmander inequality": there exist constants $c_2 > 0$, $\beta > 0$, $M > 1$ such that for every $x_0 \in X$, $r > 0$, $x \in B_r(x_0)$, $y \notin B_{Mr}(x_0)$,*

$$|k(x_0,y) - k(x, y)| + |k(y, x_0) - k(y, x)| \le$$

$$\le \frac{c_2}{\mu\left(B(x_0, d(x_0, y))\right)} \cdot \frac{d(x_0,x)^\beta}{d(x_0,y)^\beta}\,; \tag{43}$$

*iii) the cancellation property: there exists $c_3 > 0$ such that for every $r$, $R$, $0 < r < R < \infty$, a.e. $x$*

$$\left|\int_{r<d(x,y)<R} k(x, y)\, d\mu(y)\right| + \left|\int_{r<d(x,z)<R} k(z, x)\, d\mu(z)\right| \le c_3. \tag{44}$$

*iv) the following condition: for a.e. $x \in X$ there exists*

$$\lim_{\varepsilon\to 0} \int_{\varepsilon<d(x,y)<1} k(x, y)\, d\mu(y). \tag{45}$$

*For $f \in \mathcal{L}^p$, $p \in (1, \infty)$, set*

$$K_\varepsilon f(x) = \int_{\varepsilon<d(x,y)<1/\varepsilon} k(x, y)\, f(y)\, d\mu(y).$$

*Then $K_\varepsilon f$ converges (strongly) in $\mathcal{L}^p$ for $\varepsilon \to 0$ to an operator $K f$ satisfying*

$$\|K f\|_p \le c\,\|f\|_p \quad \text{for every } f \in \mathcal{L}^p, \tag{46}$$

*where the constant $c$ depends on $X$, $p$ and all the constants involved in the assumptions.*

*Finally, for the operator $K$ the commutator estimate holds:*

$$\|C[K, a]\, f\|_p \le c\,\|a\|_*\,\|f\|_p \tag{47}$$

*for every $f \in \mathcal{L}^p$, $a \in BMO$, and $c$ the same constant as in* (46).

REMARK 3. The constant $c$ in (46), (47) has the following form:

$$c(p, X, \beta, M) \cdot (c_1 + c_2 + c_3).$$

*Proof.* To see this, note that if $k$ satisfies (42), (43), (44) with constants $c_1$, $c_2$, $c_3$, then $k' \equiv k/(c_1 + c_2 + c_3)$ satisfies (42), (43), (44) with constants 1, 1, 1, so that for the kernel $k'$ $c = c(p, X, \beta, M)$.

□

Let us apply Theorem 16 to our case. By (12), our space satisfies also the reverse doubling condition. Consider the kernels:

$$k(x,y) = H_{km}\left(y^{-1}\circ x\right) \text{ with } H_{km}(x) = \frac{Y_{km}\left(x'\right)}{\|x\|^{Q}}.$$

By homogeneity, $k$ satisfies (42) with

$$c_1 = c(G)\cdot \sup_{x\in\Sigma_N} |Y_{km}(x)|. \tag{48}$$

To check condition (43) we need the following:

PROPOSITION 1. *Let $f \in C^1\left(\mathbb{R}^N\setminus\{0\}\right)$ be homogeneous of degree $\lambda < 1$. There exist $c = c(G, f) > 0$, $M = M(G) > 1$ such that*

$$|f(x\circ y) - f(x)| + |f(y\circ x) - f(x)| \leq c\,\|y\|\,\|x\|^{\lambda-1} \tag{49}$$

*for every $x, y$ such that $\|x\| \geq M\,\|y\|$. Moreover*

$$c = c(G)\cdot \sup_{z\in\Sigma_N} |\nabla f(z)|.$$

*Proof.* This proposition is essentially proved in [11], apart from the explicit form of the constant $c$.

Choose $M > 1$ such that if $\|x\| = 1$ and $\|y\| \leq 1/M$ then $\|x\circ y\| \geq 1/2$. Set:

$$F(x,y) = f(x\circ y),\quad L(x,y) = x\circ y$$

and

$$K \equiv \left\{(x,y) : \|x\| = 1 \text{ and } \|y\| \leq 1/M\right\}.$$

By homogeneity, it is enough to prove (49) for $(x,y)\in K$. Since $f(z)$ is smooth for $\|z\| \geq 1/2$ and $L$ is smooth (everywhere), by the mean value theorem

$$|f(x\circ y) - f(x)| = |F(x,y) - F(x,0)| \leq |y|\cdot\left|\nabla F\left(x, y^*\right)\right|$$

with $\left(x, y^*\right)\in K$. But:

$$\sup_{(x,y)\in K}\left|\frac{\partial F}{\partial x_i}(x,y)\right| \leq \sum_{j=1}^{N} \sup_{(x,y)\in K}\left|\frac{\partial L_j}{\partial x_i}(x,y)\right|\cdot \sup_{\|z\|\geq\frac{1}{2}}\left|\frac{\partial f}{\partial z_j}(z)\right| \leq$$

$$\leq c(G)\cdot \sup_{z\in\Sigma_N}|\nabla f(z)|,$$

and the same holds for $\sup_{(x,y)\in K}\left|\frac{\partial F}{\partial y_i}(x,y)\right|$.

Recalling that $|y| \leq c(G)\,\|y\|$ when $\|y\| \leq 1$ (see (8)), and repeating the argument for $|f(y\circ x) - f(x)|$, we get the result.

□

By Proposition 1, $k$ satisfies (43), with $\beta = 1$, $M = M(G)$, and

(50) $$c_2 = c(G) \cdot \sup_{x \in \Sigma_N} |\nabla Y_{km}(x)| .$$

As to (44), the left hand side equals:

$$\left| \int_{r<\|y\|<R} H_{km}(y)\, dy \right| + \left| \int_{r<\|y^{-1}\|<R} H_{km}(y)\, dy \right| .$$

The first term is a multiple of

$$\left| \int_{\Sigma_N} H_{km}(y)\, d\sigma\,(y) \right|$$

and therefore vanishes; the second term, by (42), (48) and (10) can be seen to be bounded by $c(G) \cdot c_1$ (see for instance Remark 4.6 in [4]). Hence $c_3$ has the same form of $c_1$. Moreover, (45) is trivially satisfied, by the vanishing property of $H_{km}$.

Finally, combining (39) with (48), (50) we get, by Theorem 16 and Remark 3, the following:

THEOREM 17. *For every $p \in (1,\infty)$ there exists a constant $c$ such that for every $a \in BMO$, $f \in \mathcal{L}^p\left(\mathbb{R}^N\right)$, $m = 1, 2, \ldots$; $k = 1, \ldots, g_m$*

$$\|T_{km}(f)\|_{\mathcal{L}^p\left(\mathbb{R}^N\right)} \leq c\, \|f\|_{\mathcal{L}^p(\mathbb{R}^N)}$$

$$\left\| C\left[T_{km}, a\right](f)\right\|_{\mathcal{L}^p(\mathbb{R}^N)} \leq c\, \|a\|_* \, \|f\|_{\mathcal{L}^p(\mathbb{R}^N)} .$$

*Explicitly, $c = c(p, G) \cdot m^{N/2}$.*

We now turn to the expansion (35). Combining Theorem 17 with the uniform bound (41) on the coefficients in the expansion (which crucially depends on Theorem 12), and using (38), we get Theorem 14, where the constant in (31), (32) is $c(p, G, \mu)$.

## 4. Uniform bounds for the derivatives of fundamental solutions of families of operators

In this section we prove Theorem 12; this will complete the proof of Theorem 2. The proof of Theorem 12 is carried out repeating an argument by Rothschild-Stein (contained in §6 of [28]); this, in turn, is based on several results proved by Kohn in [21]. We will not repeat the whole proof, but will state its steps, pointing out the necessary changes to adapt the argument to our case. As in the previous section, it will be enough to write the proof for Case B.

Let $\mathcal{A}_\mu$ be the set of $q \times q$ constant matrices $A = \{a_{ij}\}$, satisfying:

$$\mu^2 |\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}\, \xi_i \xi_j \leq \mu^{-2} |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^q,$$

where $\mu$ is the same as in (18), (19). Throughout this section we will consider the operator

$$\mathcal{L}_A = \sum_{i,j=1}^{q} a_{ij} X_i X_j + X_0$$

where $A = \{a_{ij}\} \in \mathcal{A}_\mu$ and the fields $X_i$ satisfy the assumptions stated in §2.1. Let $\Gamma^A$ be the fundamental solution for $\mathcal{L}_A$, homogeneous of degree $(2 - Q)$ (see §3.1), and let:

$$T_A : f \mapsto f * \Gamma^A.$$

By Theorem 9, $T_A \mathcal{L}_A = \mathcal{L}_A T_A =$identity. We will prove that:

$$\sup_{\|x\|=1} \left| \left( \frac{\partial}{\partial x} \right)^\beta \Gamma^A(x) \right| \leq c(\beta, G, \mu). \tag{51}$$

Note that, if $\mathcal{L}_0$ is the frozen operator defined in (24), $\mathcal{L}_0 = a_0(x_0) \cdot \mathcal{L}_A$ with $A = \left\{ a_{ij}(x_0)/a_0(x_0) \right\}_{i,j=1}^q \in \mathcal{A}_\mu$ and $\Gamma(x_0, \cdot) = a_0(x_0) \cdot \Gamma^A$. This shows that (26) follows from (51).

The operator $T_A$ can be regarded as a fractional integral operator, for which the following estimates hold:

THEOREM 18. *a) If* $1 < p < \frac{Q}{2}$ *and* $\frac{1}{s} = \frac{1}{p} - \frac{2}{Q}$, *then*

$$\|T_A f\|_s \leq c \, \|f\|_p$$

*with*

$$c = c(p, G) \cdot \sup_{\Sigma_N} \left| \Gamma^A \right|.$$

*b) If* $\frac{Q}{2} < p < Q$ *and* $\beta = 2 - \frac{Q}{p}$ *(hence* $\beta \in (0, 1)$*), then for every* $f \in C_0^\infty$ *and every* $x_1, x_2 \in \mathbb{R}^N$

$$|(T_A f)(x_1) - (T_A f)(x_2)| \leq c \, \|f\|_p \, d(x_1, x_2)^\beta$$

*with*

$$c = c(p, G) \cdot \left\{ \sup_{\Sigma_N} \left| \Gamma^A \right| + \sup_{\Sigma_N} \left| \nabla \Gamma^A \right| \right\}.$$

*c) If* $p > \frac{Q}{2}$ *and sprt* $f \subseteq B_r(x_0)$ *for some* $r > 0$, $x_0 \in \mathbb{R}^N$

$$\|T_A f\|_{\mathcal{L}^\infty(B_r)} \leq c \, \|f\|_{\mathcal{L}^p(B_r)}$$

*with*

$$c = c(p, G, r) \cdot \sup_{\Sigma_N} \left| \Gamma^A \right|.$$

Note: parts *b*-*c* of Theorem 18 will be used only in §6(proof of Theorem 4).

*Proof.* Part *a*) follows from Proposition 1.11 in [11], or also from results about fractional integrals on general spaces of homogeneous type, see [15]. The form of the constant $c$ depends on the bound:

$$\left| \Gamma^A(x) \right| \leq \sup_{\Sigma_N} \left| \Gamma^A \right| \cdot \frac{1}{\|x\|^{Q-2}}. \tag{52}$$

Part *b*) could also be proved as a consequence of results in [15], but it is easier to prove it directly. Let $M$ be the same number as in Proposition 1; let us write:

$$|(T_A f)(x_1) - (T_A f)(x_2)| \leq \int_{\mathbb{R}^N} \left| \left[ \Gamma^A(y^{-1} \circ x_1) - \Gamma^A(y^{-1} \circ x_2) \right] f(y) \right| dy \leq$$

$$\leq \int_{\|y^{-1}\circ x_1\|\geq M\left\|x_2^{-1}\circ x_1\right\|} \ldots\ dy + \int_{\|y^{-1}\circ x_1\|\leq M\left\|x_2^{-1}\circ x_1\right\|} \ldots\ dy = I + II.$$

By Proposition 1,

$$I \leq c(p,G)\cdot \sup_{\Sigma_N}\left|\nabla\Gamma^A\right| \left\|x_2^{-1}\circ x_1\right\| \int_{\|y^{-1}\circ x_1\|\geq M\left\|x_2^{-1}\circ x_1\right\|} \frac{|f(y)|}{\left\|y^{-1}\circ x_1\right\|^{Q-1}}\, dy.$$

Let $p$, $p'$ be conjugate exponents; by Hölder's inequality and a change of variables

$$I \leq c\left\|x_2^{-1}\circ x_1\right\| \|f\|_p \left(\int_{\|y\|\geq M\left\|x_2^{-1}\circ x_1\right\|} \frac{1}{\|y\|^{(Q-1)p'}}\, dy\right)^{1/p'} \leq$$

computing the integral, under the assumption $p < Q$,

$$\leq c\left\|x_2^{-1}\circ x_1\right\|^{\beta} \|f\|_p$$

where $\beta = 1 - \frac{1}{q}\,[(Q-1)(q-1)-1] = 2 - \frac{Q}{p} \in (0,1)$.

By (52),

$$II \leq \sup_{\Sigma_N}\left|\Gamma^A\right|\cdot$$

$$\cdot\int_{\|y^{-1}\circ x_1\|\leq M\left\|x_2^{-1}\circ x_1\right\|} |f(y)| \left\{\frac{1}{\left\|y^{-1}\circ x_1\right\|^{Q-2}} + \frac{1}{\left\|y^{-1}\circ x_2\right\|^{Q-2}}\right\} dy =$$

$$= II' + II''.$$

By Hölder's inequality and reasoning as above, we get, if $p > Q/2$,

$$II' \leq c\left\|x_2^{-1}\circ x_1\right\|^{\beta} \|f\|_p .$$

As to $II''$, if $\left\|y^{-1}\circ x_1\right\| \leq M\left\|x_2^{-1}\circ x_1\right\|$, then $\left\|y^{-1}\circ x_2\right\| \leq c\left\|x_2^{-1}\circ x_1\right\|$ and therefore $II''$ can be handled as $II'$.

As to $c$), noting that $x, y \in B_r(x_0) \Rightarrow y^{-1}\circ x \in B_{Kr}(0)$ for some $K = K(G)$, we can write, by Hölder's inequality (let $p'$ be the conjugate exponent of $p$):

$$\|T_A f\|_{\mathcal{L}^\infty(B_r(x_0))} \leq \|f\|_{\mathcal{L}^p(B_r(x_0))}\cdot\left\|\Gamma^A\right\|_{\mathcal{L}^{p'}(B_{Kr}(0))} \leq \quad \text{(by (52))}$$

$$\leq \|f\|_{\mathcal{L}^p(B_r(x_0))}\cdot c(p,G)\cdot\sup_{\Sigma_N}\left|\Gamma^A\right|\cdot r^{2-Q/p},$$

which proves the result, assuming $p > Q/2$.

□

Now, let

$$S_{ij}^A f = X_i X_j T_A f.$$

By ($c$) of Theorem 11, setting $\Gamma_{ij}^A = X_i X_j \Gamma^A$, we can write

$$S_{ij}^A f = P.V.\left(\Gamma_{ij}^A * f\right) + \alpha_{ij}(A)\cdot f. \tag{53}$$

Let us apply Theorem 16 and Remark 3 to the kernel $\Gamma_{ij}^A$. By the properties ($d$), ($e$), ($f$) listed in Theorem 11 and Proposition 1, we get:

THEOREM 19. *For every* $p \in (1,\infty)$, $f \in C_0^\infty\left(\mathbb{R}^N\right)$,

$$\left\| P.V.\left(\Gamma_{ij}^A * f\right)\right\|_p \ \leq\ c\,\|f\|_p$$

*with*

$$c = c(p,G)\cdot\left\{\sup_{\Sigma_N}\left|\Gamma^A\right| + \sup_{\Sigma_N}\left|\nabla\Gamma^A\right|\right\}.$$

LEMMA 2. *For every* $p \in (1,\infty)$ *and for every* $A_0 \in \mathcal{A}_\mu$*, there exists* $\varepsilon > 0$ *such that if* $|A - A_0| < \varepsilon$*, then*

$$\|E_A f\|_p \leq \frac{1}{2}\,\|f\|_p$$

*where* $|A|$ *denotes the Euclidean norm of the matrix* $A$ *and*

$$E_A = \left(\mathcal{L}_{A_o} - \mathcal{L}_A\right)\,T_{A_o}.$$

This Lemma is proved in [28] (Lemma 6.5), for a different class of operators.

*Proof.* Let us write

$$\mathcal{L}_{A_o} - \mathcal{L}_A = \sum_{i,j=1}^{q}\left(a_{ij}^o - a_{ij}\right)\,X_i X_j.$$

Then

$$E_A f = \sum_{i,j=1}^{q}\left(a_{ij}^o - a_{ij}\right)\,X_i X_j\,T_{A_o} f.$$

By (53) and Theorem 19 we get the result.

□

LEMMA 3. *Let* $p \in (1, Q/2)$ *and let* $\frac{1}{s} = \frac{1}{p} - \frac{2}{Q}$*. There exists* $c = c(G,\mu,p)$ *such that for every* $A \in \mathcal{A}_\mu$

$$\|T_A f\|_s \leq c\,\|f\|_p\,. \tag{54}$$

This Lemma is an adjustment of Lemma 6.7 proved in [28], which contains a minor mistake (it implicitly assumes $Q > 4$).

*Proof.* Let $A_0 \in \mathcal{A}_\mu$ and let $E_A$ and $\varepsilon$ be as in the previous lemma. For every $f \in \mathcal{L}^p$, if $|A - A_0| < \varepsilon$, then for every $p \in (1,\infty)$, $\|E_A f\|_p \leq \frac{1}{2}\,\|f\|_p$ , so that we can write

$$\sum_{n=0}^{\infty}\left(E_A^n\right) f = (I - E_A)^{-1} f \equiv g. \tag{55}$$

Therefore

$$f = (I - E_A)g = g - \mathcal{L}_{A_o}T_{A_o}g + \mathcal{L}_A T_{A_o}g = \mathcal{L}_A T_{A_o}g,$$

that is

$$T_A f = T_{A_o}g.$$

Again from (55) we have

$$\|g\|_p \leq \sum_{n=0}^{\infty} \|E_A\|^n \|f\|_p = 2\|f\|_p ,$$

hence by Theorem 18.*a*, if $p, s$ are as in the statement of the theorem,

$$\|T_A f\|_s = \left\|T_{A_o} g\right\|_s \leq c(G, p, A_0)\|g\|_p \leq 2c\|f\|_p .$$

Since this is true for every fixed $A_0 \in \mathcal{A}_\mu$ and any matrix $A$ such that $|A - A_0| < \varepsilon$, by the compactness of $\mathcal{A}_\mu$ in $\mathbb{R}^{q^2}$ we can choose a constant $c = c(G, p, \mu)$ such that (54) holds for every $A \in \mathcal{A}_\mu$.

□

THEOREM 20. *Let $\varphi, \varphi_1 \in C_0^\infty\left(\mathbb{R}^N\right)$ with $\varphi_1 = 1$ on sprt$\varphi$. There exists $\varepsilon = \varepsilon(G)$ and, for every $t \in \mathbb{R}$, there exists $c = c(t, \mu, G, \varphi, \varphi_1)$ such that for every $A \in \mathcal{A}_\mu$ and every $u \in C_0^\infty\left(\mathbb{R}^N\right)$*

$$\|\varphi u\|_{H^{t+\varepsilon,2}} \leq c\left\{\|\varphi_1 \mathcal{L}_A u\|_{H^{t,2}} + \|\varphi_1 u\|_{\mathcal{L}^2}\right\}.$$

*(Recall that the norm of $H^{t,2}\left(\mathbb{R}^N\right)$ has been defined is §2.3).*

This Theorem is proved in [21] for a different class of operators and without taking into account the exact dependence of the constant on the parameters. To point out the slight modifications which are necessary to adapt the proof to our case, we will state the main steps of the proof of Theorem 20. Before doing this, however, we show how from Lemma 3 and Theorem 20, the uniform bound (26) follows. This, again, is an argument contained in [28], which we include, for convenience of the reader, to make more readable the exposition. Moreover, a minor correction is needed here to the proof of [28].

*Proof of (26) from Lemma 3 and Theorem 20.* Throughout the proof, $B_r$ will be a ball centered at the origin. Let $g \in C_0^\infty(B_2 \setminus B_1)$ such that $\|g\|_2 \leq 1$, and let $\varphi, \varphi_1 \in C_0^\infty\left(\mathbb{R}^N\right)$ such that: $\varphi = 1$ in $B_{1/4}$, sprt$\varphi \subseteq B_{1/2}$, $\varphi_1 = 1$ in $B_{1/2}$, sprt$\varphi_1 \subseteq B_1$. Let $f = T_A g$. Since $\mathcal{L}_A f = g = 0$ in $B_1$ and $\mathcal{L}_A$ is hypoelliptic, $f \in C^\infty(B_1)$.

Pick a positive number $p$ such that $\max\left(\frac{1}{2}, \frac{2}{Q}\right) < \frac{1}{p} < \min\left(\frac{1}{2} + \frac{2}{Q}, 1\right)$ and let $s$ be as in Lemma 3. Note that $1 < p < Q/2$ and $p < 2 < s$. Then, by Lemma 3:

$$\|\varphi_1 f\|_2 \leq c(\varphi_1)\|f\|_s \leq c(\varphi_1, G, \mu, p)\|g\|_p \leq c(\varphi_1, G, \mu)\|g\|_2 \leq c(\varphi_1, G, \mu).$$

Applying Theorem 20 to $\varphi, \varphi_1, f$, since $\mathcal{L}_A f = 0$ on sprt$\varphi_1$, we get:

$$\|\varphi f\|_{H^{t+\varepsilon,2}} \leq c\,\|\varphi_1 f\|_2 \leq c(t, \varphi, \varphi_1, G, \mu)$$

for every $t \in \mathbb{R}$. Therefore, by the standard Sobolev embedding Theorems, we can bound any (isotropic) Hölder norm $C^{h,\alpha}$ of $f$ on $B_{1/4}$ with a constant $c(h, G, \mu)$; in particular, for every differential operator $P$:

$$|Pf(0)| \leq c(P, G, \mu). \tag{56}$$

Now, recall that $f = g * \Gamma^A$. If $P$ is any left invariant differential operator,

$$Pf(0) = \int P\,\Gamma^A(y^{-1})\,g(y)\,dy. \tag{57}$$

Since (56) holds for every $g \in C_0^\infty(B_2 \setminus B_1)$ such that $\|g\|_2 \leq 1$, from (57) we get

$$\left\| P\,\Gamma^A(y^{-1}) \right\|_{\mathcal{L}^2(B_2\setminus B_1)} \leq c(P, G, \mu). \tag{58}$$

Now, writing any differential operator $\left(\frac{\partial}{\partial x}\right)^\beta$ in terms of left invariant vector fields, (58) gives us a bound on every $H^{k,2}$-norm of $\Gamma^A$ on $B_2 \setminus B_1$, and therefore, reasoning as above, on every Hölder norm $C^{h,\alpha}$ of $\Gamma^A$ on a smaller spherical shell $C \equiv B_{7/4} \setminus B_{5/4}$. In particular, we get

$$\sup_{x\in C} \left| \left(\frac{\partial}{\partial x}\right)^\beta \Gamma^A(x) \right| \leq c(\beta, G, \mu),$$

from which (51) follows, by homogeneity of $\Gamma^A$.

□

Now we come to Theorem 20, which is proved by Kohn in [21] for an operator of the kind

$$Pu \equiv \sum_{i=1}^q X_i^2 u + X_0 u + cu,$$

where the fields $X_i$ $(i = 0, 1, \ldots, q)$ satisfy Hörmander's condition. Reading carefully the paper [21], one can check that the whole proof can be repeated replacing the operator $P$ with $\mathcal{L}_A$; moreover, the constants depend on the matrix $A$ only through the number $\mu$. Actually, the matrix $A$ is involved in the proof only through the boundedness of its coefficients and the following elementary inequality:

$$|(\mathcal{L}_A u, u)| \geq \mu \sum_{i=1}^q \|X_i u\|^2, \quad \text{for every } u \in C_0^\infty\left(\mathbb{R}^N\right).$$

We can rephrase as follows the steps of the proof of Theorem 20 given in [21]:

(i) There exist $\varepsilon = \varepsilon(G)$, $c = c(G, \mu)$ such that for every $u \in C_0^\infty\left(\mathbb{R}^N\right)$ and every $A \in \mathcal{A}_\mu$

$$\|u\|_{H^{\varepsilon,2}} \leq c\left\{\|\mathcal{L}_A u\|_2 + \|u\|_2\right\}.$$

(ii) For every $t \in \mathbb{R}$, $M > 0$, there exists $c = c(t, M, \mu, G)$ such that for every $u \in C_0^\infty\left(\mathbb{R}^N\right)$ and every $A \in \mathcal{A}_\mu$

$$\|u\|_{H^{t+\varepsilon,2}} \leq c\left\{\|\mathcal{L}_A u\|_{H^{t,2}} + \|u\|_{H^{-M,2}}\right\},$$

where $\varepsilon$ is the same of (i).

(iii) (Localization of the above estimate).

$$\|\varphi u\|_{H^{t+\varepsilon,2}} \leq c\left\{\|\varphi_1 \mathcal{L}_A u\|_{H^{t,2}} + \|\varphi_1 u\|_{H^{-M,2}}\right\},$$

where $\varphi, \varphi_1 \in C_0^\infty\left(\mathbb{R}^N\right)$ with $\varphi_1 = 1$ on sprt $\varphi$, $c = c(\varphi, \varphi_1, t, M, \mu, G)$.

Since $\|\cdot\|_{H^{-M,2}} \leq \|\cdot\|_2$, from point (iii) we get Theorem 20.

REMARK 4 (AN ALGEBRA OF PSEUDODIFFERENTIAL OPERATORS ADAPTED TO THE FIELDS $X_i$).
For the reader who is interested in reviewing the proof of [21], we point out that, under our assumptions, many of the arguments of [21] can be simplified and made more self-contained by the following remark. We can precisely define an algebra of pseudodifferential operators, acting on the Schwarz' space $\mathcal{S}$ of smooth functions with fast decay at infinity. Consider the following kinds of operators:

(a) multiplication by a polynomial;

(b) $X_i$ $(i = 0, 1, \dots, q)$;

(c) for $t \in \mathbb{R}$, $\Lambda^t$ defined by $\widehat{(\Lambda^t u)}(\xi) = \left(1 + |\xi|^2\right)^{t/2} \widehat{u}(\xi)$.

By general properties of homogeneous groups (see [31], p. 621), the vector fields $X_i$ are linear combinations of $\partial/\partial x_i$ with polynomial coefficients (and, by Hörmander's condition, the $\partial/\partial x_i$'s are linear combinations of the $X_i$'s and their commutators, with polynomial coefficients). Therefore $X_i$ maps $\mathcal{S}$ into itself, while the same is true for the operators (a) and (c). The transpose of an operator of kind (a), (c) is the operator itself, while, since the fields $X_i$ are translation invariant, the transpose of $X_i$ is $-X_i$. (This fact also simplifies many of the arguments in [21]; in particular, note that $(X_0 u, u) = 0$). Let $\mathcal{P}$ be the algebra generated by operators (a), (b), (c) under sums, composition and transpose. This algebra is the suitable context where the whole proof can be carried out. On the contrary, in [21] some technical problems arise, since $X_i$ are defined only on $C_0^\infty\left(\mathbb{R}^N\right)$.

To complete the proof of Theorem 12 we have now to prove estimate (27). We actually prove a more general result which will be useful in §6.

Let $\left\{k_\gamma\right\}_{\gamma\in\Lambda}$ be a family of kernels such that $k_\gamma$ is homogeneous of degree $h - Q$ for some $h > 0$ and $k_\gamma \in C_0^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$. Let $T_\gamma$ be the distribution associated to $k_\gamma$ and let $P^h$ be a left invariant differential operator homogeneous of degree $h$. Then, Theorem 9 states that

$$P^h T_\gamma = P.V.\left(P^h k_\gamma\right) + \alpha_\gamma \,\delta. \tag{59}$$

(Observe that (25) is a particular case of (59)). With these notations, we can prove the following:

LEMMA 4. *If $k_\gamma$ satisfies a uniform bound like (26), that is, for every multiindex $\beta$*

$$\sup_{\gamma\in\Lambda} \sup_{\|y\|=1} \left| \left(\frac{\partial}{\partial y}\right)^\beta k_\gamma(y) \right| \le c(\beta),$$

*then*

$$\sup_{\gamma\in\Lambda} \left|\alpha_\gamma\right| \le c.$$

*Proof.* Let $u$ be a test function with $u(0) \neq 0$, sprt$u \subseteq B_1(0)$. By (59)

$$\alpha_\gamma \, u(0) = \langle (P^h)^T u, T_\gamma \rangle - \langle u, P.V.\left(P^h k_\gamma\right) \rangle =$$

$$= \int (P^h)^T u(x)\, k_\gamma(x)\, dx - \lim_{\varepsilon\to 0} \int_{\|x\|>\varepsilon} P^h k_\gamma(x)\, u(x)\, dx$$

(here $(\cdot)^T$ denotes transposition). Since $k_\gamma$ is locally integrable, the first integral is bounded, uniformly in $\gamma$ by (59). As to the second term, by the vanishing property of the kernel $P^h k_\gamma$ (see Lemma 1), homogeneity and (59) we can write

$$\left| \int_{\|x\|>\varepsilon} P^h k_\gamma(x)\, u(x)\, dx \right| = \left| \int_{\varepsilon<\|x\|<1} P^h k_\gamma(x)\, [u(x)-u(0)]\, dx \right| \le$$

$$\le \sup_{\|y\|\le 1} |\nabla u(y)| \int_{\varepsilon<\|x\|<1} \frac{c}{\|x\|^Q} \cdot |x|\, dx \le c.$$

(The convergence of the last integral follows from (8)).

□

## 5. Some properties of the Sobolev spaces $S^{k,p}$

We start pointing out the following interpolation inequality for Sobolev norms:

PROPOSITION 2. *Let $X$ be a left invariant vector field, homogeneous of degree $\alpha > 0$. Then for every $\varepsilon > 0$, $u \in S^p\left(\mathbb{R}^N\right)$, $p \in [1,\infty)$*

$$\|Xu\|_p \le \varepsilon \left\|X^2 u\right\|_p + \frac{2}{\varepsilon} \|u\|_p \,. \tag{60}$$

*Proof.* The following argument is taken from the proof of Theorem 9.4 in [13]. Let $\gamma(t)$ be the integral curve of $X$ with $\gamma(0) = 0$. Then, applying Taylor's theorem to the function $F(t) = u(x\circ\gamma(t))$,

$$u(x\circ\gamma(1)) = u(x) + Xu(x) + \int_0^1 (1-t)\; X^2 u(x\circ\gamma(t))dt.$$

Using the translation invariance of $\|\cdot\|_p$ and Minkowski's inequality, we get

$$\|Xu\|_p \le \left\|X^2 u\right\|_p + 2\,\|u\|_p \,.$$

Since $X$ is homogeneous, by a dilation argument we get the result.

□

We will need a version of (60) for functions defined on a ball (not necessarily vanishing at the boundary). For standard Sobolev norms, this result follows from the analog of (60) using an extension theorem (see for instance [14], pp. 169-173). However, it seems not easy to construct a continuous extension operator $E\colon S^p(B_r) \to S_0^p(B_{2r})$. We are going to show how to bypass this difficulty.

First we construct a suitable family of cutoff functions. Given two balls $B_{r_1}$, $B_{r_2}$ and a function $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$, let us write $B_{r_1} \prec \varphi \prec B_{r_2}$ to say that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_{r_1}$ and sprt$\varphi \subseteq B_{r_2}$.

LEMMA 5 (RADIAL CUTOFF FUNCTIONS). *For any $\sigma \in (0,1)$, $r > 0$, $k$ positive integer, there exists $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$ with the following properties:*

$$B_{\sigma r} \prec \varphi \prec B_{\sigma' r} \quad \textit{with } \sigma' = (1+\sigma)/2;$$

$$\left|P^j\varphi\right| \leq \frac{c(G,j)}{\sigma^{j-1}(1-\sigma)^j r^j} \quad \text{for } 1 \leq j \leq k,$$

*where $P^j$ is any left invariant differential monomial homogeneous of degree $j$.*

*Proof.* For simplicity, we prove the assertion for $k = 2$. The general case is similar. Pick a function $f : [0, r) \to [0, 1]$ such that:

$$f \equiv 1 \text{ in } [0, \sigma r),\ f \equiv 0 \text{ in } [\sigma' r, r),\ f \in C^\infty(0, r),$$
$$\left|f'\right| \leq \frac{c}{(1-\sigma)r}, \left|f''\right| \leq \frac{c}{(1-\sigma)^2 r^2}.$$

Setting $\varphi(x) = f(\|x\|)$, we can compute:

$$X_i\varphi(x) = f'(\|x\|)X_i(\|x\|);$$

$$X_iX_j\varphi(x) = f''(\|x\|)X_i(\|x\|)X_j(\|x\|) + f'(\|x\|)X_iX_j(\|x\|).$$

Since $X_i(\|x\|)$ is homogeneous of degree zero for $i = 1, \dots, q$, $X_iX_j(\|x\|)$ (for $i, j = 1, \dots, q$) and $X_0(\|x\|)$ are homogeneous of degree $-1$ and $f'(\|x\|) \neq 0$ for $\|x\| > \sigma r$, we get the result.

□

Another tool we need in this context is an approximation result by suitable mollifiers. For a fixed cutoff function $\varphi$, $B_1(0) \prec \varphi \prec B_2(0)$, set, for every $\varepsilon > 0$,

$$\varphi_\varepsilon(x) = c \cdot \varepsilon^{-Q}\varphi\left(D\left(\frac{1}{\varepsilon}\right)x\right)$$

with $c = \left(\int_{\mathbb{R}^N}\varphi(x)\,dx\right)^{-1}$. Then

LEMMA 6. *For $u \in S^{k,p}\left(\mathbb{R}^N\right)$ ($k$ nonnegative integer, $1 \leq p < \infty$) and $\varphi_\varepsilon$ as above, define $u_\varepsilon = \varphi_\varepsilon * u$. Then $u_\varepsilon \in C^\infty$ and $u_\varepsilon \to u$ in $S^{k,p}$ for $\varepsilon \to 0$.*

*Proof.* The proof follows the same line as in the Euclidean case. We just point out the following facts:

(i) convergence in $\mathcal{L}^p$ is established first for $u \in \Lambda^\beta(G,\mathbb{R}^N)$, $u$ with bounded support. The density of this space in $\mathcal{L}^p$ can be proved in a general space of homogeneous type (see for instance [4]);

(ii) since $X_i$ is right invariant, $X_i(u_\varepsilon) = \left(X_iu\right)_\varepsilon$: from this remark and convergence in $\mathcal{L}^p$ we get convergence in $S^{k,p}$;

(iii) to see that $u_\varepsilon \in C^\infty$, one has to consider right invariant vector fields $X_i^R$, write $X_i^R(\varphi_\varepsilon * u) = \left(X_i^R\varphi_\varepsilon\right) * u$, and iterate. The possibility of representing any Euclidean derivative in terms of right invariant vector fields (see §2.5) provides the conclusion.

□

The above Lemma is useful for us mainly in view of the following

COROLLARY 1. *If $u \in S^{k,p}(\Omega)$ ($1 \leq p < \infty$, $k \geq 1$) and $\varphi \in C_0^\infty(\Omega)$, then $u\varphi \in S_0^{k,p}(\Omega)$.*

*Proof.* The function $u\varphi$ is compactly supported in $\Omega$ and, extended to zero outside $\Omega$, belongs to $S^{k,p}\left(\mathbb{R}^N\right)$. Then $(u\varphi)_\varepsilon$ converges to $u\varphi$ in $S^{k,p}$. Since, for $\varepsilon$ small enough, $(u\varphi)_\varepsilon$ is compactly supported in $\Omega$, $(u\varphi)_\varepsilon \in C_0^\infty(\Omega)$ and $u\varphi \in\in S_0^{k,p}(\Omega)$.

□

THEOREM 21 (INTERPOLATION INEQUALITY IN CASE A). *Assume we are in Case A. For any $u \in S^{H,p}(B_r)$, $p \in [1,\infty)$, $H \geq 2$, $r > 0$, define the following quantities:*

$$\Phi_k = \sup_{\frac{1}{2}<\sigma<1}\left((1-\sigma)^k r^k \left\|D^k u\right\|_{\mathcal{L}^p(B_{r\sigma})}\right) \text{ for } k = 0, 1, 2, \ldots, H.$$

*Then for every integer $j$, $1 \leq j \leq H-1$, there exist positive constants $c$, $\delta_0$ depending on $G, j, H$ such that for every $\delta \in (0,\delta_0)$ we have*

$$\Phi_j \leq \delta\,\Phi_H + \frac{c}{\delta^{j/(H-j)}}\,\Phi_0. \tag{61}$$

*Proof.* We proceed by induction on $H$. Let $H = 2$.

Let $u \in S^p(B_r)$ and $\varphi$ a cutoff function as in Lemma 5. By Corollary 1, $u\varphi \in S_0^p(B_r)$, hence by density we can apply Proposition 2, writing

$$\|X_i\,(u\varphi)\|_p \leq \varepsilon\left\{\|\varphi\, X_i X_i u\|_p + \|2\,X_i u\; X_i\varphi\|_p + \|u\; X_i X_i\varphi\|_p\right\} + \frac{2}{\varepsilon}\,\|\varphi u\|_p$$

for any $\varepsilon > 0$. Hence

$$\|Du\|_{\mathcal{L}^p(B_{\sigma r})} \leq \varepsilon\left\{\left\|D^2u\right\|_{\mathcal{L}^p(B_{\sigma' r})} + \frac{c(G)}{(1-\sigma)\,r}\,\|Du\|_{\mathcal{L}^p(B_{\sigma' r})} + \right.$$

$$\left.+\frac{c(G)}{\sigma\,(1-\sigma)^2\,r^2}\,\|u\|_{\mathcal{L}^p(B_{\sigma' r})}\right\} + \frac{2}{\varepsilon}\,\|u\|_{\mathcal{L}^p(B_{\sigma' r})}\,.$$

Multiplying both sides for $(1-\sigma)\,r$ and choosing $\varepsilon = \delta\sigma\,(1-\sigma)\,r$ we find

$$(1-\sigma)\,r\,\|Du\|_{\mathcal{L}^p(B_{\sigma r})} \leq \delta\sigma\,(1-\sigma)^2\,r^2\left\|D^2u\right\|_{\mathcal{L}^p(B_{\sigma' r})} +$$

$$+c(G)\delta\sigma\,(1-\sigma)\,r\,\|Du\|_{\mathcal{L}^p(B_{\sigma' r})} + c(G)\left(\delta + \frac{1}{\delta\sigma}\right)\|u\|_{\mathcal{L}^p(B_{\sigma' r})} \leq$$

$$\text{(noting that } (1-\sigma') = (1-\sigma)/2)$$

$$\leq 4\delta\Phi_2 + C(G)\delta\Phi_1 + c(G)\left(\delta + \frac{2}{\delta}\right)\Phi_0.$$

Therefore

$$\Phi_1 \leq \frac{4\delta}{1-c(G)\delta}\,\Phi_2 + \frac{c(G)\left(\delta + \frac{2}{\delta}\right)}{1-c(G)\delta}\,\Phi_0$$

which, for $\delta$ small enough, is equivalent to

$$\Phi_1 \leq \delta\Phi_2 + \frac{c}{\delta}\,\Phi_0. \tag{62}$$

Assume now that (61) holds for $H-1$. An argument similar to that used to obtain (62) applied to $D^{H-2}u$ yields

$$\Phi_{H-1} \leq \delta\Phi_H + \frac{c}{\delta}\,\Phi_{H-2}$$

while, by induction,

$$\Phi_{H-2} \leq \eta\Phi_{H-1} + \frac{c}{\eta^{H-2}}\,\Phi_0.$$

Therefore

$$\Phi_{H-1} \leq \delta\Phi_H + \frac{c}{\delta}\left(\eta\Phi_{H-1} + \frac{c}{\eta^{H-2}}\,\Phi_0\right)$$

and, choosing $\eta = \frac{\delta}{2c}$ we get

$$\Phi_{H-1} \leq 2\delta\Phi_H + \frac{c}{\delta^{H-1}}\,\Phi_0. \tag{63}$$

If $j = H-1$, this is exactly what we have to prove; if $j < H-1$, by induction

$$\Phi_j \leq \varepsilon\Phi_{H-1} + \frac{c}{\varepsilon^{j/(H-1-j)}}\,\Phi_0 \leq \quad \text{by (63)}$$

$$\leq \varepsilon\left(2\delta\Phi_H + \frac{c}{\delta^{H-1}}\,\Phi_0\right) + \frac{c}{\varepsilon^{j/(H-1-j)}}\,\Phi_0.$$

Choosing $2\delta = \eta^{1/(H-j)}$ and $\varepsilon = \eta^{1-1/(H-j)}$ we get the result.

□

REMARK 5. Note that the second part of the above proof does not hold in Case B since in that case $D^k u$ cannot be obtained as $D\left(D^{k-1}u\right)$. However, the proof for $H = 2$ holds also in Case B, since the field $X_0$ does not play any role in the definition of $Du$. We are going to prove an analogous interpolation inequality, in Case B, which will hold for $H$ even. This proof will be achieved in several steps.

Let

$$L \equiv \sum_{i=1}^{q} X_i^2 + X_0,$$

and let $\Gamma$ be the fundamental solution of $L$ homogeneous of degree two; recall that the transpose of $L$ is just

$$L^T \equiv \sum_{i=1}^{q} X_i^2 - X_0.$$

LEMMA 7. *Let $Q > 4$. For every integer $k \geq 2$ and any couple of left invariant differential monomials $P^{2k-1}$ and $P^{2k-2}$, homogeneous of degrees $2k-1$ and $2k-2$, respectively, we can determine two kernels $K^{(1)}$, $K^{(2)}$ (depending only on these monomials) which are smooth outside the origin and homogeneous of degrees $(1-Q)$, $(2-Q)$, respectively, such that for any test function u*

$$P^{2k-1}u(x) = \Big(\underbrace{(LL\ldots Lu)}_{k\ times} * K^{(1)}\Big)(x);$$

$$P^{2k-2}u(x) = \Big(\underbrace{(LL\ldots Lu)}_{k\ times} * K^{(2)}\Big)(x). \tag{64}$$

*Proof.* By induction on $k$. Let $k = 2$. By Lemma 1, we can write

$$u = Lu * \Gamma = (LLu * \Gamma) * \Gamma = LLu * K,$$

where $K = \Gamma * \Gamma$ is homogeneous of degree $(4 - Q)$. Hence

$$P^3 u = LLu * P^3 K = LLu * K^{(1)}; \quad P^2 u = LLu * P^2 K = LLu * K^{(2)},$$

with $K^{(1)}$, $K^{(2)}$ homogeneous of degree $(1 - Q)$, $(2 - Q)$, respectively.

Now, assume (64) holds for $k - 1$, that is

$$P^{2k-3} u(x) = \Bigg( \underbrace{(LL \dots Lu)}_{(k-1) \text{ times}} * K^{(1)} \Bigg)(x);$$

$$P^{2k-4} u(x) = \Bigg( \underbrace{(LL \dots Lu)}_{(k-1) \text{ times}} * K^{(2)} \Bigg)(x).$$

Any differential monomial $P^{2k-2}$ can be written either as $X_i \, P^{2k-3}$ (for some $i = 1, \dots, q$) or as $X_0 \, P^{2k-4}$. In the first case, we can write

$$\begin{aligned} P^{2k-2} u(x) &= X_i \Bigg( \underbrace{(LL \dots Lu)}_{(k-1) \text{ times}} * K^{(1)} \Bigg)(x) = \\ &= X_i \Bigg( \underbrace{(LL \dots Lu)}_{k \text{ times}} * \left( \Gamma * K^{(1)} \right) \Bigg)(x) = \\ &= \Bigg( \underbrace{(LL \dots Lu)}_{k \text{ times}} * X_i \left( \Gamma * K^{(1)} \right) \Bigg)(x) = \\ & \Bigg( \underbrace{(LL \dots Lu)}_{k \text{ times}} * \widetilde{K}^{(2)} \Bigg)(x), \end{aligned}$$

with $\widetilde{K}^{(2)}$ homogeneous of degree $((2 - Q) + (1 - Q) + Q) - 1 = 2 - Q$ (we have applied Lemma 1). In the second case,

$$\begin{aligned} P^{2k-2} u(x) &= X_0 \Bigg( \underbrace{(LL \dots Lu)}_{(k-1) \text{ times}} * K^{(2)} \Bigg)(x) = \\ &= \Bigg( \underbrace{(LL \dots Lu)}_{k \text{ times}} * X_0 \left( \Gamma * K^{(2)} \right) \Bigg)(x) = \\ & \Bigg( \underbrace{(LL \dots Lu)}_{k \text{ times}} * \widetilde{\widetilde{K}}^{(2)} \Bigg)(x), \end{aligned}$$

with, again, $\widetilde{\widetilde{K}}^{(2)}$ homogeneous of degree $(2 - Q)$.

Similarly, any differential monomial $P^{2k-1}$ can be written either as $P^2 \, P^{2k-3}$ (with $P^2 = X_0$ or $P^2 = X_i X_j$ for some $i, j = 1, \dots, q$) or as $P^3 \, P^{2k-4}$ (with $P^3 = X_i X_0$ for some $i = 1, \dots, q$). Reasoning as above we get the result for this case, too.

□

LEMMA 8. *For every integer $k \geq 2$ there exists a constant $c(G, k)$ such that for every $\varepsilon > 0$ and every test function $u$,*

$$\left\| D^{2k-j} u \right\|_p \leq \varepsilon \left\| D^{2k} u \right\|_p + \frac{c(G, k)}{\varepsilon^{2k-j}} \|u\|_p \quad \text{for } j = 1, 2, \, k \geq 2.$$

*Proof.* Let $K^{(1)}, K^{(2)}$ be as in Lemma 7. We split the kernel $K^{(1)}$ as

$$K^{(1)} = \varphi K^{(1)} + (1-\varphi)\,K^{(1)} \equiv K_0^{(1)} + K_\infty^{(1)},$$

where $\varphi$ is a cutoff function, $B_1(0) \prec \varphi \prec B_2(0)$. Therefore $K_0^{(1)}$ is homogeneous of degree $(1-Q)$ near the origin and has compact support, hence it is integrable, while $K_\infty^{(1)}$ is homogeneous of degree $(1-Q)$ near infinity and vanishes near the origin. Writing $\widetilde{f}(x) = f(x^{-1})$, we can compute

$$\int K_\infty^{(1)}\left(y^{-1}\circ x\right) LL\ldots Lu(y)\,dy = \int \widetilde{K}_\infty^{(1)}\left(x^{-1}\circ y\right) LL\ldots Lu(y)\,dy =$$

$$= \int L^T \widetilde{K}_\infty^{(1)}(x^{-1}\circ y)\,\underbrace{(LL\ldots L)}_{k-1 \text{ times}}u(y)\,dy = \ldots$$

$$= \int \underbrace{\left(L^T L^T \ldots L^T\right)}_{k \text{ times}}\widetilde{K}_\infty^{(1)}\left(x^{-1}\circ y\right)u(y)\,dy =$$

$$= u * \left(\underbrace{\left(L^T L^T \ldots L^T\right)}_{k \text{ times}}\widetilde{K}_\infty^{(1)}\right)^{\sim}(x) \equiv \left(u * K_1^{(1)}\right)(x).$$

Therefore

$$P^{2k-1}u(x) = \Big(\underbrace{(LL\ldots L)}_{k \text{ times}}u * \left[K_0^{(1)} + K_\infty^{(1)}\right]\Big)(x) =$$

$$= (LL\ldots Lu) * K_0^{(1)} + u * K_1^{(1)},$$

where $K_1^{(1)}$ is homogeneous of degree $(1-Q-2k)$ near infinity and vanishes near the origin, hence it is integrable. Integrability of $K_0^{(1)}, K_1^{(1)}$ gives

$$\left\|D^{2k-1}u\right\|_p \le c(G,k)\left\{\left\|D^{2k}u\right\|_p + \|u\|_p\right\}.$$

The same reasoning applied to $K^{(2)}$ gives

$$\left\|D^{2k-2}u\right\|_p \le c(G,k)\left\{\left\|D^{2k}u\right\|_p + \|u\|_p\right\}.$$

The conclusions follows from the last two inequalities and a dilation argument.

□

THEOREM 22. *[Interpolation inequality in Case B] For any function $u \in S^{2k,p}(B_r)$ ($k \ge 1$, $p \in [1,\infty)$, $r > 0$), let $\Phi_h$ ($h = 0,1,\ldots,2k$) be the seminorms defined in Theorem 21. Then*

$$\Phi_j \le \varepsilon\Phi_{2k} + c(\varepsilon,k)\Phi_0$$

*for every integer $j$ with $1 \le j \le 2k-1$ and every $\varepsilon > 0$.*

*Proof.* Let $\varphi$ be a cutoff function as in Lemma 5. By Corollary 1 $u\varphi \in S_0^{2k,p}(B_r)$, hence, by density, we can apply Lemma 8 to $u\varphi$. By standard arguments (see the first part of the proof of Theorem 21) we get, for every $\delta > 0$,

$$\Phi_{2k-j} \le \delta\sum_{h=0}^{2k}\Phi_h + \frac{c}{\delta^{(2k-j)/j}}\,\Phi_0 \quad (j = 1,2).$$

The previous inequality clearly holds if $k$ is replaced by any integer $i$ with $2 \le i \le k$. Adding up these inequalities for $2 \le i \le k$, $j = 1, 2$, we get

$$\sum_{i=2}^{k} \left(\Phi_{2i-1} + \Phi_{2i-2}\right) = \sum_{i=2}^{2k-1} \Phi_i \le 2\delta \sum_{i=2}^{k} \left(\sum_{h=0}^{2i} \Phi_h\right) + c(\delta, k) \cdot \Phi_0 \le$$

$$\le 2\delta k \sum_{h=0}^{2k} \Phi_h + c(\delta, k) \cdot \Phi_0.$$

Adding also (62) (which holds in Case B, too, as noted in Remark 5):

$$\Phi_1 \le \delta \Phi_2 + \frac{c}{\delta} \Phi_0$$

we can write

$$\sum_{h=0}^{2k-1} \Phi_h \le 2\delta k \sum_{h=0}^{2k} \Phi_h + c(\delta, k) \cdot \Phi_0$$

and, finally, for every $\varepsilon > 0$,

$$\sum_{h=0}^{2k-1} \Phi_h \le \varepsilon \cdot \Phi_{2k} + c(\varepsilon, k) \cdot \Phi_0$$

which proves the result.

□

Next, we need the following Sobolev-type embedding theorem:

THEOREM 23. *Let $u \in S_0^p(B_r)$ for some $r > 0$. Then:*
*a) if $1 < p < \frac{Q}{2}$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{2}{Q}$, then*

$$\|u\|_{p^*} \le c\,(p, G) \left\| D^2 u \right\|_p ;$$

*b) if $\frac{Q}{2} < p < Q$ and $\beta = 2 - \frac{Q}{p}$, then*

$$\|u\|_{\Lambda^\beta(B_r)} \le c(p, G, r) \left\| D^2 u \right\|_{\mathcal{L}^p(B_r)} .$$

*Proof.* Let $L$, $\Gamma$ be as in the proof of Lemma 7. Then for any $u \in C_0^\infty(B_r)$ we can write

$$u = Lu * \Gamma.$$

Theorem 18 then gives the assertion.

□

The results contained in Theorem 23 have been proved by Folland in [11], where a more complete theory of Sobolev and Hölder spaces defined by the vector fields $X_i$ is developed (see §§4, 5 in [11], in particular Theorems 4.17 and 5.15). However, Folland's theory relies on a deep analysis of the sub-Laplacian on stratified groups, and therefore does not cover completely the cases we are considering here: remember that under our assumptions, the group $G$ is graded but not necessarily stratified.

## 6. Local estimates for solutions to the equation $\mathcal{L}u = f$ in a domain

In this section we will prove Theorems 3 to 7, as a consequence of the basic estimate contained in Theorem 2 and the properties of Sobolev spaces expounded in §5. For convenience of the reader, we recall the statement of each Theorem before its proof.

THEOREM 3 (LOCAL $\mathcal{L}^p$-ESTIMATES FOR SOLUTIONS TO THE EQUATION $\mathcal{L}u = f$ IN A DOMAIN). *Under the assumptions stated in §2.3, let $\Omega$ be a bounded domain of $\mathbb{R}^N$ and $\Omega' \subset\subset \Omega$. If $u \in S^p(\Omega)$, then*

$$\|u\|_{S^p(\Omega')} \leq c\left\{\|\mathcal{L}u\|_{\mathcal{L}^p(\Omega)} + \|u\|_{\mathcal{L}^p(\Omega)}\right\}$$

*where $c = c(p, G, \mu, \eta, \Omega, \Omega')$.*

*Proof of Theorem 3.* Let $u \in S^p(\Omega)$, $B_r \subset \Omega$, and fix $r_0 = r_0(p, G, \mu, \eta)$ in such a way that for $r < r_0$ Theorem 2 holds. Let $\varphi$ be a cutoff function as in Lemma 5. By Corollary 1, $u\varphi \in S_0^p(B_r)$; then, by density, we can apply Theorem 2 to $u\varphi$:

$$\left\|X_i X_j (u\varphi)\right\|_p \leq c \left\|\mathcal{L}(u\varphi)\right\|_p .$$

From the above inequality we get:

$$\left\|X_i X_j u\right\|_{\mathcal{L}^p(B_{\sigma r})} \leq c\left\{\|\mathcal{L}u\|_{\mathcal{L}^p(B_{\sigma' r})} + \frac{c(G)}{(1-\sigma)r}\|Du\|_{\mathcal{L}^p(B_{\sigma' r})} + \right.$$

$$\left. + \frac{c(G)}{\sigma(1-\sigma)^2 r^2}\|u\|_{\mathcal{L}^p(B_{\sigma' r})}\right\}.$$

Multiplying both sides for $(1-\sigma)^2 r^2$, adding to both sides $(1-\sigma) r \|Du\|_{\mathcal{L}^p(B_{\sigma r})}$, reasoning like in the first part of the proof of Theorem 21 and applying (62) we get

$$\Phi_2 + \Phi_1 \leq c\left\{r^2 \|\mathcal{L}u\|_{\mathcal{L}^p(B_r)} + \|u\|_{\mathcal{L}^p(B_r)}\right\}.$$

Hence

$$r^2\left\|D^2 u\right\|_{\mathcal{L}^p\left(B_{r/2}\right)} + r\|Du\|_{\mathcal{L}^p\left(B_{r/2}\right)} \leq c\left\{r^2\|\mathcal{L}u\|_{\mathcal{L}^p(B_r)} + \|u\|_{\mathcal{L}^p(B_r)}\right\},$$

that is

$$\|u\|_{S^p(B_{r/2})} \leq c\left\{\|\mathcal{L}u\|_{\mathcal{L}^p(B_r)} + \|u\|_{\mathcal{L}^p(B_r)}\right\},$$

with $c = c(p, G, \mu, \eta, r)$, $r < r_0(p, G, \mu, \eta)$. The last estimate and a covering argument give the result.

□

THEOREM 4 (LOCAL HÖLDER CONTINUITY FOR SOLUTIONS TO THE EQUATION $\mathcal{L}u = f$ IN A DOMAIN). *Under the assumptions of Theorem 3, if $u \in S^p(\Omega)$ for some $p \in (1, \infty)$ and $\mathcal{L}u \in \mathcal{L}^s(\Omega)$ for some $s > Q/2$, then*

$$\|u\|_{\Lambda^\alpha(\Omega')} \leq c\left\{\|\mathcal{L}u\|_{\mathcal{L}^r(\Omega)} + \|u\|_{\mathcal{L}^p(\Omega)}\right\}$$

*for $r = \max(p, s)$, $\alpha = \alpha(Q, p, s) \in (0, 1)$, $c = c(G, \mu, p, s, \Omega, \Omega')$.*

*Proof of Theorem 4.* Let $u \in S^p(B_{2R})$ (for some $p \in (1,\infty)$) be a solution to $\mathcal{L}u = f$ in a ball $B_{2R}$, with $f \in \mathcal{L}^s(B_{2R})$ for some $s > Q/2$. We can always assume $p, s < Q$.

Let $\varphi$ be a cutoff function, $B_{R/2} \prec \varphi \prec B_R$. By Corollary 1, $u\varphi \in S_0^p(B_R)$. If $p > Q/2$, then by (b) of Theorem 23, $u \in \Lambda^\alpha(B_{R/2})$. If $p \le Q/2$ (we can assume $p < Q/2$), then by (a) of Theorem 23, $u\varphi \in \mathcal{L}^{p^*}(B_R)$, and $u \in \mathcal{L}^{p^*}(B_{R/2})$.

Assume $p^* < s$. Then by Theorem 3, $\|u\|_{S^{p^*}(B_{R/4})}$ is bounded. Hence, we can fix $\varphi_1$ with $B_{R/8} \prec \varphi_1 \prec B_{R/4}$ and repeat the argument finding that $u \in S^{p^{**}}(B_{R/16})$, and so on. After a number $k$ of iterations depending only on $p, Q$, we find that $u \in S^{\bar{p}}\left(B_{R/4^k}\right)$ for some $\bar{p} > Q/2$: more precisely, $k$ must be the integer belonging to the interval $\left(\frac{Q}{2p} - 1, \frac{Q}{2p}\right)$. This $k$ exists provided $Q/2p$ is not an integer, what we can always assume, replacing $p$, if necessary, with a slightly smaller $p_1$. Therefore, by $(b)$ of Theorem 23, $u \in \Lambda^\alpha(B_{R/2^{2k+1}})$ for some $\alpha \in (0,1)$.

If $u \in \mathcal{L}^{p^*}(B_{R/2})$ with $p^* \ge s$, Theorem 3 implies that $u \in S^s\left(B_{R/4}\right)$ and since $s > Q/2$, $u \in \Lambda^\alpha\left(B_{R/8}\right)$ for some $\alpha \in (0,1)$.

In any case, the following estimate holds:

$$\|u\|_{\Lambda^\alpha\left(B_{R/K}\right)} \le c\left\{\|f\|_{\mathcal{L}^s(B_R)} + \|u\|_{S^p(B_R)}\right\} \le$$

by Theorem 3

$$\le c\left\{\|f\|_{\mathcal{L}^s(B_R)} + \|f\|_{\mathcal{L}^p(B_{2R})} + \|u\|_{\mathcal{L}^p(B_{2R})}\right\}.$$

By a covering argument, we get the result for a bounded domain $\Omega$.

□

THEOREM 5 (REGULARITY OF THE SOLUTION IN TERMS OF SOBOLEV SPACES). *Under the assumptions of Theorem 3, if $a_0, a_{ij} \in S^{k,\infty}(\Omega)$, $u \in S^p(\Omega)$ and $\mathcal{L}u \in S^{k,p}(\Omega)$ for some positive integer k (k even in Case B), $1 < p < \infty$, then*

$$\|u\|_{S^{k+2,p}(\Omega')} \le c_1\left\{\|\mathcal{L}u\|_{S^{k,p}(\Omega)} + c_2\|u\|_{\mathcal{L}^p(\Omega)}\right\}$$

*where $c_1 = c_1(p, G, \mu, \eta, \Omega, \Omega')$ and $c_2$ depends on the $S^{k,\infty}(\Omega)$ norms of the coefficients.*

REMARK 6. Similarly to the proof of the interpolation inequality contained in Theorems 21 and 22, also the proof of Theorem 5 is more difficult in Case B than in Case A. The reason of this appears clear if one tries to adapt our proof of Case A also to Case B. In doing so, the main difficulty is the presence of the field $X_0$, which has weight two but cannot be seen as composition of two fields of weight one. This is also why the definition of the space $S^{k,p}$ in Case B is not optimal when $k$ is odd, as already noted in [28]. Therefore our proof of Theorem 5 in Case B will follow a different line and will require the restriction $k$ even. This also (but not only) depends on the restriction $k$ even that appears in Theorem 22. We also note that in the commutative case (that is $\mathcal{L}$ uniformly elliptic, in Case A, or uniformly parabolic, in Case B) this Theorem follows immediately from Theorem 3 (just differentiating the equation $\mathcal{L}u = f$).

*Proof of Theorem 5 in Case A.* The proof is divided into two parts: first we prove an estimate for the derivatives of order $k+2$ of a test function $u$ supported in a small ball, in terms of $\mathcal{L}u$ (as in Theorem 2); then we derive from this result a local estimate for any function in $S^{k+2,p}(\Omega)$ (as in the proof of Theorem 3).

Part I. The idea of this proof, in the case of the Kohn-laplacian on the Heisenberg group, is contained in [31]. Recall that $X_i^R$ $(i = 1, \dots, N)$ are the right invariant vector fields which agree with $X_i$ (and therefore with $\partial/\partial x_i$) at the origin. Here we will use the properties of $X_i^R$ stated in §2.5. Moreover, since $X_1^R, \dots, X_q^R$ generate the Lie algebra of right invariant vector fields, for every $k > q$ we can write

$$X_k^R = \sum_{1 \le i_j \le q} \vartheta_{i_1 \dots i_{\omega_k}} X_{i_1}^R X_{i_2}^R \dots X_{i_{\omega_k}}^R \tag{65}$$

for suitable constants $\vartheta_{i_1 \dots i_{\omega_k}}$ depending only on $G$.

Let us consider a test function $u$. We are going to establish a representation formula for $X_i u$ in terms of $D(\mathcal{L}u)$. For a fixed point $x_0 \in \mathbb{R}^N$, let us write

$$u(x) = \int_{\mathbb{R}^N} \Gamma\left(x_0 ; y^{-1} \circ x\right) \mathcal{L}_0 u(y) dy.$$

Then, for every $i = 1, \dots, q$:

$$X_i u(x) = \int_{\mathbb{R}^N} X_i \Gamma\left(x_0 ; y^{-1} \circ x\right) \mathcal{L}_0 u(y) dy = \quad \text{by (22)}$$

$$= \int_{\mathbb{R}^N} \sum_{k=i}^{N} X_k^R \left(c_i^k(\cdot)\ \Gamma(x_0 ; \cdot)\right)\left(y^{-1} \circ x\right) \mathcal{L}_0 u(y)\, dy = \quad \text{by (65)}$$

$$= \int_{\mathbb{R}^N} \sum_{k=i}^{q} X_k^R \left(c_i^k(\cdot)\ \Gamma(x_0 ; \cdot)\right)\left(y^{-1} \circ x\right) \mathcal{L}_0 u(y)\, dy +$$

$$+ \int_{\mathbb{R}^N} \sum_{k=q+1}^{N} \sum_{1 \le i_j \le q} \vartheta^k_{i_1 \dots i_{\omega_k}} X_{i_1}^R X_{i_2}^R \dots X_{i_{\omega_k}}^R \left(c_i^k(\cdot)\ \Gamma(x_0 ; \cdot)\right)\left(y^{-1} \circ x\right) \mathcal{L}_0 u(y)\, dy =$$

by (21)

$$= \int_{\mathbb{R}^N} \sum_{k=i}^{q} \left(c_i^k(\cdot)\, \Gamma(x_0 ; \cdot)\right)\left(y^{-1} \circ x\right) X_k\, \mathcal{L}_0 u(y)\, dy +$$

$$+ \int_{\mathbb{R}^N} \sum_{k=q+1}^{N} \sum_{1 \le i_j \le q} \vartheta^k_{i_1 \dots i_{\omega_k}} X_{i_2}^R \dots X_{i_{\omega_k}}^R \left(c_i^k(\cdot)\, \Gamma(x_0 ; \cdot)\right)\left(y^{-1} \circ x\right) X_{i_1} \mathcal{L}_0 u(y)\, dy =$$

$$= \int_{\mathbb{R}^N} \sum_{k=1}^{q} \widetilde{\Gamma}^{k,i}\left(x_0 ; y^{-1} \circ x\right) X_k\, \mathcal{L}_0 u(y)\, dy,$$

where the kernels $\widetilde{\Gamma}^{k,i}(x_0 ; \cdot)$ satisfy properties analogous to those of $\Gamma(x_0 ; \cdot)$:

- $\widetilde{\Gamma}^{k,i}(x_0 ; \cdot)$ is homogeneous of degree $(2 - Q)$, since $c_i^k(\cdot)$ is homogeneous of degree $\omega_k - \omega_i$ and $\omega_i = 1$ for $i \le q$;
- $\widetilde{\Gamma}^{k,i}(x_0 ; \cdot)$ is smooth outside the origin;
- the derivatives of any order of $\widetilde{\Gamma}^{k,i}(x_0 ; \cdot)$ satisfy the uniform bound expressed by Theorem 12, because the functions $c_i^k(\cdot)$ are smooth and do not depend on $x_0$.

The whole previous reasoning can be iterated, getting, for every positive integer $k$ and any left invariant differential monomial homogeneous of degree $k$,

$$P^k u(x) = \int_{\mathbb{R}^N} \sum_{i_1,i_2,\dots,i_k=1}^{q} \widetilde{\Gamma}^{\,i_1,i_2,\dots,i_k}\left(x_0\,;y^{-1}\circ x\right) X_{i_1} X_{i_2}\dots X_{i_k}\mathcal{L}_0 u(y)\,dy, \tag{66}$$

where the kernels $\widetilde{\Gamma}^{\,i_1,i_2,\dots,i_k}(x_0\,;\cdot)$ satisfy the same properties of $\widetilde{\Gamma}^{\,k,i}(x_0\,;\cdot)$.

Differentiating twice the representation formula (66), applying Theorem 9, writing $\mathcal{L}_0 = (\mathcal{L}_0 - \mathcal{L}) + \mathcal{L}$ and letting finally $x = x_0$ we get

$$X_j X_h P^k u(x) = \sum_{i_1,i_2,\dots,i_k=1}^{q} \Bigg( P.V.\int X_j X_h \widetilde{\Gamma}^{\,i_1,i_2,\dots,i_k}\left(x\,;y^{-1}\circ x\right)\cdot$$

$$\left\{ X_{i_1}X_{i_2}\dots X_{i_k}\mathcal{L}u(y) + X_{i_1}X_{i_2}\dots X_{i_k}\left(\sum_{r,s=1}^{q}\left[a_{rs}(x)-a_{rs}(y)\right]X_rX_s\,u\,(y)\right)\right\} dy$$

$$+\alpha^{\,i_1,i_2,\dots,i_k}(x)\,X_{i_1}X_{i_2}\dots X_{i_k}\mathcal{L}u(x)\Bigg)$$

for $j,h\in\{1,\dots,q\}$, where the functions $\alpha^{\,i_1,i_2,\dots,i_k}(x)$ are uniformly bounded, by Lemma 4.

Reasoning like in the proof of Theorem 2 and setting

$$\left\|D^h a\right\|_\infty = \max_{r,s}\left\|D^h a_{rs}\right\|_\infty$$

we conclude that

$$\left\|D^{k+2}u\right\|_p \le c\left\{\left\|D^k(\mathcal{L}u)\right\|_p + \sum_{j=1}^{k}\left\|D^{k+2-j}u\right\|_p\left\|D^j a\right\|_\infty\right\} \le$$

$$\le c\left\{\left\|D^k(\mathcal{L}u)\right\|_p + \|a\|_{S^{k,\infty}}\|u\|_{S^{k+1,p}}\right\} \tag{67}$$

with $c = c(p,G,\mu,\eta,k)$. By density, (67) holds for every $u\in S_0^{k+2,p}(B_r)$.

Part II. Now, let $u\in S^{k+2,p}(\Omega)$, $B_r\subset\Omega$ with $r$ small enough so that (67) holds for every function in $S_0^{k+2,p}(B_r)$. For any $\sigma\in\left(\frac{1}{2},1\right)$, pick a cutoff function $\varphi\in C_0^\infty\left(\mathbb{R}^N\right)$ like in Lemma 5, $B_{\sigma r}\prec\varphi\prec B_{\sigma' r}$ $(\sigma' = (1+\sigma)/2)$. Applying (67) to $u\varphi$ and using Lemma 5 we get, with some computation:

$$(1-\sigma)^{k+2}r^{k+2}\left\|D^{k+2}u\right\|_{\mathcal{L}^p(B_{\sigma r})} \le$$

$$\le c\left\{(1-\sigma)^2 r^2\|\mathcal{L}u\|_{S^{k,p}(B_{\sigma' r})} + \|a\|_{S^{k,\infty}(B_{\sigma' r})}\cdot\sum_{h=0}^{k+1}(1-\sigma)^h r^h\left\|D^h u\right\|_{\mathcal{L}^p(B_{\sigma' r})}\right\}.$$

Therefore:

$$\sum_{h=0}^{k+2}\Phi_h \le c\left\{r^2\|\mathcal{L}u\|_{S^{k,p}(B_r)} + \|a\|_{S^{k,\infty}(B_r)}\cdot\sum_{h=0}^{k+1}\Phi_h\right\}$$

with $c = c(p, G, \mu, \eta, k)$, $r < r_0(p, G, \mu, \eta)$. By Theorem 21, we get

$$\sum_{h=0}^{k+2} \Phi_h \le c_1 \left\{ r^2 \|\mathcal{L}u\|_{S^{k,p}(B_r)} + c_2 \cdot \Phi_0 \right\}$$

with $c_1 = c_1(p, G, \mu, \eta, k)$, $c_2 = c_2 \left( \|a\|_{S^{k,\infty}(B_r)} \right)$, and finally

$$\|u\|_{S^{k+2,p}(B_{r/2})} \le c_1 \left\{ r^2 \|\mathcal{L}u\|_{S^{k,p}(B_r)} + c_2 \|u\|_{\mathcal{L}^p(B_r)} \right\}$$

which implies the desired result, by a covering argument.

□

*Proof of Theorem 5 in Case B.* Again, the proof is divided in two parts. First we prove inequality (67) for any even integer $k$, in Case B. Then the Theorem follows by the same argument of the proof in Case A, Part II, applying the suitable interpolation inequality (Theorem 22).

In proving (67), we will consider in detail the case $k = 2$, and then we will briefly show how to iterate the argument.

Let $u$ be a test function, fix two points $x_0, y_0 \in \mathbb{R}^N$ and write $\mathcal{L}_{x_0}$, $\mathcal{L}_{y_0}$ to indicate the operators "frozen" at $x_0, y_0$, respectively (see §§3.1, 3.2). Let us write:

$$u(x) = \left( \mathcal{L}_{x_0} u * \Gamma(x_0, \cdot) \right)(x) = \left( \left[ \mathcal{L}_{y_0}(\mathcal{L}_{x_0} u) * \Gamma(y_0, \cdot) \right] * \Gamma(x_0, \cdot) \right)(x) =$$

$$= \left( \mathcal{L}_{y_0} \left( \mathcal{L}_{x_0} u \right) * K(x_0, y_0, \cdot) \right)(x),$$

where $K(x_0, y_0, z) = \left( \Gamma(y_0, \cdot) * \Gamma(x_0, \cdot) \right)(z)$ and we have applied Lemma 1 with $\alpha_1 = \alpha_2 = 2 - Q$. Note that condition $\alpha_1 + \alpha_2 < -Q$ in Lemma 1 gives, in our case, $Q > 4$. This condition certainly holds since the homogeneous group corresponding to the simplest noncommutative Lie algebra satisfying the assumptions of Case B is $Q = 6$ (see Example 2).

Now, let $P^4$ be any left invariant differential operator homogeneous of degree 4. By Lemma 1 and Theorem 9, $P^4 K(x_0, y_0, \cdot)$ is a singular kernel satisfying conditions $(d)$, $(e)$, $(f)$ of Theorem 11 and a uniform bound (23); reasoning like in §§3.1, 3.2 we can write

$$P^4 u(x) = P.V. \int_{\mathbb{R}^N} P^4 K \left( x_0, y_0, z^{-1} \circ x \right) \mathcal{L}_{y_0}(\mathcal{L}_{x_0} u)(z) dz +$$

$$+ \alpha(x_0, y_0) \cdot \mathcal{L}_{y_0}(\mathcal{L}_{x_0} u)(x) =$$

(writing $\mathcal{L}_{x_0} = (\mathcal{L}_{x_0} - \mathcal{L}) + \mathcal{L}$ and setting finally $x_0 = x$)

$$= P.V. \int_{\mathbb{R}^N} P^4 K \left( x, y_0, z^{-1} \circ x \right) \mathcal{L}_{y_0} \Bigg( \sum_{h,k=1}^{q} \left[ a_{hk}(x) - a_{hk}(z) \right] X_h X_k u(z) +$$

$$+ \left[ a_0(x) - a_0(z) \right] X_0 u(z) + \mathcal{L}u(z) \Bigg) dz + \alpha(x, y_0) \cdot \mathcal{L}_{y_0}(\mathcal{L}u)(x),$$

where $\alpha$ is a bounded function by Lemma 4. Now, if the support of $u$ is contained in a ball with radius $r$ small enough,

$$\left\| P^4 u \right\|_p \le c \left\{ \left\| \mathcal{L}_{y_0}(\mathcal{L}u) \right\|_p + \left\| D^2 a \right\|_\infty \left\| D^2 u \right\|_p + \|Da\|_\infty \left\| D^3 u \right\|_p \right\} + \frac{1}{2} \left\| D^4 u \right\|_p .$$

Therefore

$$\left\| D^4 u \right\|_p \leq c \left\{ \left\| D^2 (\mathcal{L}u) \right\|_p + \|a\|_{S^{2,\infty}} \|u\|_{S^{3,p}} \right\}$$

which is exactly (67), for $k = 2$. From now on Part II of the proof of Theorem 5 in Case A can be repeated, to get

$$\|u\|_{S^{4,p}(B_{r/2})} \leq c_1 \left\{ r^2 \|\mathcal{L}u\|_{S^{2,p}(B_r)} + c_2 \|u\|_{\mathcal{L}^p(B_r)} \right\}.$$

We now briefly discuss how to iterate the above proof for any even integer $k$. Assume, for instance that we want to bound the $\mathcal{L}^p$-norm of $P^6 u$. If we tried to write

$$u(x) = \left( \mathcal{L}_{y_0} \mathcal{L}_{y_0} \mathcal{L}_{x_0} u * \Gamma(y_0, \cdot) * \Gamma(y_0, \cdot) * \Gamma(x_0, \cdot) \right)(x)$$

the problem would be to assure the existence of the convolution of three fundamental solutions (this would lead to the restriction $Q > 6$, by Lemma 1). Instead, we must perform part of the differentiation $P^6$ before doing the third convolution. First of all, let us split $P^6$ as a composition of the kind $P^4 P^2$ or $P^3 P^3$ (note that the possible presence of $X_0$, which has weight two, implies that one of these cases occurs). If $P^6 = P^4 P^2$ we proceed as follows:

$$u(x) = \left( \mathcal{L}_{y_0} (\mathcal{L}_{x_0} u) * K(x_0, y_0, \cdot) \right)(x),$$

with $K$ as above;

$$P^2 u(x) = \left( \mathcal{L}_{y_0} (\mathcal{L}_{x_0} u) * P^2 K(x_0, y_0, \cdot) \right)(x),$$

where $P^2 K$ is homogeneous of degree $(2 - Q)$;

$$P^2 u(x) = \left( \mathcal{L}_{y_0} \mathcal{L}_{y_0} \mathcal{L}_{x_0} u * \left[ \Gamma(y_0, \cdot) * P^2 K(x_0, y_0, \cdot) \right] \right)(x) =$$

$$= \left( \mathcal{L}_{y_0} \mathcal{L}_{y_0} \mathcal{L}_{x_0} u * \widetilde{K}(x_0, y_0, \cdot) \right)(x),$$

where $\widetilde{K}$ is homogeneous of degree $(4 - Q)$; now applying $P^4$ to both sides of the last equality we can repeat the proof of the case $k = 2$. A similar reasoning applies to the case $P^6 = P^3 P^3$. This completes the proof of Theorem 5.

□

THEOREM 6 (REGULARITY OF THE SOLUTION IN TERMS OF HÖLDER SPACES). *Under the assumption of Theorem 3, if $a_0$, $a_{ij} \in S^{k,\infty}(\Omega)$, $u \in S^p(\Omega)$ and $\mathcal{L}u \in S^{k,s}(\Omega)$ for some positive integer k (k even in Case B), $1 < p < \infty$, $s > Q/2$, then*

$$\|u\|_{\Lambda^{k,\alpha}(\Omega')} \leq c_1 \left\{ \|\mathcal{L}u\|_{S^{k,r}(\Omega)} + c_2 \|u\|_{\mathcal{L}^p(\Omega)} \right\}$$

*where $r = \max(p, s)$, $\alpha = \alpha(Q, p, s) \in (0, 1)$, $c_1 = c_1(p, s, k, G, \mu, \eta, \Omega, \Omega')$ and $c_2$ depends on the $S^{k,\infty}(\Omega)$ norms of the coefficients.*

*Proof of Theorem 6.* Let us note that ($b$) of Theorem 23 implies the following embedding estimate:

let $u \in S_0^{k+2,p}(B_r)$ for some $r > 0$. Then if $\frac{Q}{2} < p < Q$ and $\beta = 2 - \frac{Q}{p}$,

$$\|u\|_{\Lambda^{k,\beta}(G, B_r)} \leq c(p, k, G, \mu, r) \|u\|_{S^{k+2,p}(B_r)}.$$

(It's enough to apply Theorem 23 to $D^k u$). Then, Theorem 6 follows from Theorem 5 as Theorem 4 follows from Theorem 3, with ($b$) of Theorem 23 replaced by the above inequality. We omit the details.

□

Finally, let us come to the proof of:

THEOREM 7 (OPERATORS WITH LOWER ORDER TERMS). *Consider an operator with lower order terms, of the following kind:*

$$\mathcal{L} \equiv \left( \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + a_0(x) X_0 \right) + \left( \sum_{i=1}^{q} c_i(x) X_i + c_0(x) \right) \equiv \mathcal{L}_2 + \mathcal{L}_1.$$

*i) If $c_i \in \mathcal{L}^\infty(\Omega)$ for $i = 0, 1, \dots, q$, then:*
*if the assumptions of Theorem 3 hold for $\mathcal{L}_2$, then the conclusions of Theorem 3 hold for $\mathcal{L}$;*
*if the assumptions of Theorem 4 hold for $\mathcal{L}_2$, then the conclusions of Theorem 4 hold for $\mathcal{L}$.*

*ii) If $c_i \in S^{k,\infty}(\Omega)$ for some positive integer $k$, $i = 0, 1, \dots, q$, then:*
*if the assumptions of Theorem 5 hold for $\mathcal{L}_2$, then the conclusions of Theorem 5 hold for $\mathcal{L}$;*
*if the assumptions of Theorem 6 hold for $\mathcal{L}_2$, then the conclusions of Theorem 6 hold for $\mathcal{L}$.*

*Proof.* Under the assumptions of Theorem 2, we can write, for every $u \in S_0^p(B_r)$, with $r$ small enough,

$$\left\| X_i X_j u \right\|_p \le c \left\{ \|\mathcal{L}u\|_p + \|\mathcal{L}_1 u\|_p \right\}.$$

If $c_i \in \mathcal{L}^\infty(\Omega)$ for $i = 0, 1, \dots, q$, then

$$\|\mathcal{L}_1 u\|_p \le c \|u\|_{S^{1,p}} \le \quad \text{by Proposition 2}$$

$$\le \varepsilon \left\| D^2 u \right\|_p + \frac{c}{\varepsilon} \|u\|_p .$$

Therefore

$$\left\| X_i X_j u \right\|_p \le c \left\{ \|\mathcal{L}u\|_p + \|u\|_p \right\}.$$

Using the last inequality instead of Theorem 2, we can repeat the proof of Theorem 3 for the complete operator $\mathcal{L}$. The same is true for Theorem 5, assuming $c_i \in S^{k,\infty}(\Omega)$. Finally, Theorems 4, 6 follow from Theorems 3, 5, respectively, without taking into account the form of the operator $\mathcal{L}$. We omit the details.

□

## References


[1] AGMON S., DOUGLIS A. AND NIRENBERG L., *Estimates near the boundary for solutions of elliptic partial differential eqautions satisfying general boundary conditions*, Comm. Pure Appl. Math., **12** (1959), 623–727 and **17** (1964), 35–92.

[2] BRAMANTI M. AND BRANDOLINI L., *$\mathcal{L}^p$-estimates for nonvariational hypoelliptic operators with VMO coefficients*, Trans. of the Amer. Math. Soc., **352** (2000), 781–822.

[3] BRAMANTI M. AND CERUTTI C., *$W_p^{1,2}$-solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients*, Comm. in Part. Diff. Eq., **18** 9-10 (1993), 1735–1763.

[4] BRAMANTI M. AND CERUTTI C., *Commutators of singular integrals in homogeneous spaces*, Boll. Unione Mat. Italiana, **7** 10B (1996), 843–883.

[5] BRAMANTI M., CERUTTI C. AND MANFREDINI M., *$\mathcal{L}^p$-estimates for some ultraparabolic operators with discontinuous coefficients*, J. Math. Anal. and Appl., **200** (1996), 332–354.

[6] CHIARENZA F., FRASCA M. AND LONGO P., *Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche di Mat. **XL** (1991), 149–168.

[7] CHIARENZA F., FRASCA M. AND LONGO P., *$W^{2,p}$-solvability of the Dirichlet problem for non divergence elliptic equations with VMO coefficients*, Trans. Am. Math. Soc. **336** 1 (1993), 841–853.

[8] COIFMAN R., ROCHBERG R. AND WEISS G., *Factorization theorems for Hardy spaces in several variables*, Annals Math. **103** (1976), 611–635.

[9] COIFMAN R. AND WEISS G., *Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes*, Lecture Notes in Mathematics **242**, Springer-Verlag, Berlin-Heidelberg-New York 1971.

[10] CALDERÓN A.P. AND ZYGMUND A., *Singular integral operators and differential equations*, Am. Journ. of Math. **79** (1957), 901–921.

[11] FOLLAND F.B., *Subelliptic estimates and function spaces on nilpotent Lie groups*, Arkiv for Math. **13** (1975), 161–207.

[12] FABES E.B. AND RIVIÈRE N., *Singular integrals with mixed homogeneity*, Studia Mathematica **27** (1966), 19–38.

[13] FOLLAND F.B. AND STEIN E.M., *Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure and Appl. Math. **XXVII** (1974), 429–522.

[14] GILBARG D. AND TRUDINGER N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin Heidelberg New York Tokio 1983 (2nd ed.).

[15] GATTO A.E. AND VÀGI S., *Fractional Integrals on Spaces of Homogeneous Type. Analysis and Partial Differential Equations*, ed. by Cora Sadosky, Lecture Notes in Pure and Applied Math. **122** (1990), 171–216.

[16] HÖRMANDER L., *Hypoelliptic second order differential equations*, Acta Mathematica, **119** (1967), 147–171.

[17] ILYIN V.A. AND POZNYAK E.G., *Linear Algebra*, MIR Publisher, Moscow 1986.

[18] JOHN F. AND NIRENBERG N., *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 175–188.

[19] JERISON D. AND SANCHEZ-CALLE A., *Estimates for the heat kernel for a sum of squares of vector fields*, Indiana Univ. Math. Jour. **35** 4 (1986), 835–854.

[20] KNAPP A.W. AND STEIN E.M., *Intertwining operators for semi-simple groups*, Ann. of Math. **93** (1971), 489–578.

[21] KOHN J.J., *Pseudo-differential operators and hypoellipticity*, Proc. Symp. Pure Math. Amer. Math. Soc. **23** (1973), 61–69.

[22] LANCONELLI E. AND LASCIALFARI F., *A boundary value problem for a class of quasilinear operators of Fokker-Planck type*, Ann. Univ. Ferrara Sez. VII **40** (1995), 65–84.

[23] LANCONELLI E. AND POLIDORO S., *On a class of hypoelliptic evolution operators*, Rend. Sem. Mat. Polit. Torino **51** 4 (1993), 137–171.

[24] MANFREDINI M., *The Dirichlet problem for a class of ultraparabolic equations*, Advances in Differential Equations **2** 5 (1997), 831–866.

[25] MANFREDINI M. AND POLIDORO S., *Interior regularity for weak solutions of ultraparabolic equations in divergence form with discontinuous coefficients*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **8** 1 (1998), no. 3, 651–675.

[26] A. NAGEL A., STEIN E.M. AND WAINGER S., *Balls and metrics defined by vector fields I: Basic properties*, Acta Mathematica **155** (1985), 130–147.

[27] MANFREDINI M. AND POLIDORO S., *A global lower bound for the fundamental solution of Kolmogorov-Fokker-Planck equations*, Arch. Rational Mech. Anal. **137** 4 (1997), 321–340.

[28] ROTHSCHILD L.P. AND STEIN E.M., *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.

[29] SANCHEZ-CALLE A., *Fundamental solutions and geometry of sum of squares of vector fields* Inv. Math. **78** (1984), 143–160.

[30] SARASON D., *Functions of vanishing mean oscillations* Trans. of Am. Math. Soc. **207** (1974), 391–405.

[31] STEIN E.M., *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press. Princeton, New Jersey 1993.

**AMS Subject Classification: 35H05, 35R05, 43A85**

Marco BRAMANTI
Dipartimento di Matematica
Politecnico di Milano
P.zza Leonardo da Vinci 32
20133 Milano, ITALIA
e-mail: `bramanti@unical.it`

Luca BRANDOLINI
Dipartimento di Ingegneria
Università degli Studi di Bergamo
viale Marconi 5
24044 Dalmine (BG), ITALIA
e-mail: `brandolini@unibg.it`

*Lavoro pervenuto in redazione il 20.03.1999 e in forma definitiva il 01.02.2000.*