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ON THE ENERGY OF SECTIONS OF TRIVIALIZABLE SPHERE BUNDLES

Abstract. Let $E \to M$ be a vector bundle with a metric connection over a Riemannian manifold M and consider on E the Sasaki metric. We find a condition for a section of the associated sphere bundle to be a critical point of the energy among all smooth unit sections. We apply the criterion to some particular cases where M is parallelizable, for instance $M = S^7$ or a compact simple Lie group G with a bi-invariant metric, and E is the trivial vector bundle with a connection induced by octonian multiplication or an irreducible real orthogonal representation of G, respectively. Generically, these bundles have no parallel unit sections.

1. Introduction

Beginning with G. Wiegmink and C. M. Wood [5, 6], critical points of the energy of unit tangent fields have been extensively studied (see for instance in [1] the abundant bibliography on the subject). We are interested in a natural generalization, namely, critical points of the energy of sections of sphere bundles.

Let $\pi : E \to M$ be a vector bundle with a metric connection ∇ over an oriented Riemannian manifold, that is, each fiber has an inner product depending smoothly on the base point and

$$Z \langle V, W\rangle = \langle \nabla_Z V, W\rangle + \langle V, \nabla_Z W\rangle$$

for all vector fields Z on M and all smooth sections V, W of E.

On E one can define the canonical Sasaki metric associated with ∇ in such a way that the map

$$(d\pi, \mathcal{K})_\xi : T_\xi E \to T_q M \times E_q$$

is a linear isometry for each $\xi \in E$ (here $q = \pi(\xi)$ and $\mathcal{K}$ is the connection operator associated with ∇).

Let $\pi : E \to M$ be as before and denote by $E^1 = \{\xi \in E \mid \|\xi\| = 1\}$ the associated sphere bundle. Let N be a relatively compact open subset of M with smooth (possibly empty) boundary. Given a smooth section $V : M \to E^1$, the total bending of V on N is defined by

$$\mathcal{B}_N(V) = \int_N \|\nabla V\|^2,$$

*Partially supported by foncyt, ciem (conicet) and secyt (unc).

where $(\nabla V)_p : T_pM \to E_p$, $\|\nabla V\|^2 = \operatorname{tr}(\nabla V)^*(\nabla V)$ and integration is taken with respect to the volume associated to the Riemannian metric of M.

Consider on E the Sasaki metric. As in the case of vector fields, there exist constants c_1 and c_2, depending only on the dimension and the volume of N, such that the energy $\mathcal{E}_N$ of the section V, thought of as map $V : N \to E$, is given by

$$\mathcal{E}_N(V) = c_1 + c_2\,\mathcal{B}_N(V).$$

In the following we refer to the energy of the section instead of the bending, since that is a subject more commonly studied. In every example we will be concerned with the nonexistence of parallel unit sections, since they are trivial minima of the functional.

DEFINITION 1. *A smooth section $V : M \to E^1$ is said to be a harmonic section if for every relatively compact open subset N of M with smooth (possibly empty) boundary, V is a critical point of the functional $\mathcal{B}_N$ (or equivalently, of the energy $\mathcal{E}_N$) applied to smooth sections W of M satisfying $W|_{\partial N} = V|_{\partial N}$.*

Notice that a harmonic section may be not a harmonic map from M to E^1 (see for example [2, 3], where $E = TM$).

The rough Laplacian Δ acts on smooth sections of E as follows:

$$(\Delta V)(p) = \sum_{i=1}^{n} \left(\nabla_{Z^i}\nabla_{Z^i}V\right)(p),$$

where $\left\{Z^i \mid i = 1, \dots, n\right\}$ is any section of orthonormal frames on a neighborhood of p in M satisfying $\left(\nabla_{Z^i}Z^j\right)(p) = 0$ for all i, j.

THEOREM 1. *Let $\pi : E \to M$ be a vector bundle with a metric connection over an oriented Riemannian manifold and consider on E the associated Sasaki metric. The section $V : M \to E^1$ is a harmonic section if and only if there is a smooth real function f on M such that*

$$\Delta V = fV.$$

REMARK 1. This condition was proved for the particular case where E is the tangent bundle, by Wiegmink [5] and Wood [6] for compact manifolds and by Gil-Medrano [1] for general (not necessarily compact) manifolds (with a different presentation). Their proofs can be adapted to the present more general case.

2. Applications

Let M be a parallelizable manifold with a fixed parallelization $\left\{X^1, \dots, X^n\right\}$. Let $\mathcal{V}$ be a finite dimensional vector space with an inner product and $\mathfrak{o}(\mathcal{V})$ the set of all skew-symmetric endomorphisms of $\mathcal{V}$. Let $E = M \times \mathcal{V} \to M$ be the trivial vector bundle. For $v \in \mathcal{V}$, let $L_v : M \to E$ be the "constant" section $L_v(p) = (p, v)$.

PROPOSITION 1. *Given a map* $\theta : \left\{X^1, \dots, X^n\right\} \to \mathfrak{o}(\mathcal{V})$, *there exists a unique connection* ∇ *on* $E \to M$ *such that*

$$(\nabla_{X^i} L_v)(p) = L_{\theta(X^i)v}(p) \tag{1}$$

for all $p \in M$ *and all* $i = 1, \dots, n$. *Moreover, the connection is metric.*

Proof. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of $\mathcal{V}$. Let $X \in T_pM$ and $\sigma : M \to E$ be a smooth section. Then

$$X = \sum_{i=1}^{m} a_i X^i(p) \qquad \text{and} \qquad \sigma = \sum_{j=1}^{n} f_j L_{v_j}$$

for some numbers a_i and smooth functions $f_j : M \to \mathbb{R}$. A standard computation shows that

$$(\nabla_X \sigma)(p) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i \left(X^i_p (f_j) L_{v_j}(p) + f_j(p) \left(L_{\theta(X^i)v_j} \right)(p) \right)$$

defines a connection on E satisfying condition (1), which is metric since $\theta\left(X^i\right)$ is skew-symmetric for all i.

□

EXAMPLE 1. The Levi-Civita connection of a Lie group G with a left invariant Riemannian metric may be obtained in this way: Let $\mathfrak{g}$ be the Lie algebra of G endowed with an arbitrary inner product. Let ∇ be the connection on $E = G \times \mathfrak{g} \to G$ induced by $\theta : \mathfrak{g} \to \mathfrak{o}(\mathfrak{g})$ given by

$$\theta(X) Y = \frac{1}{2} \left(\mathrm{ad}_X Y - (\mathrm{ad}_X)^* Y - (\mathrm{ad}_Y)^* X \right),$$

and any left invariant parallelization of G, where * means transpose with respect to the inner product at the identity. In this case the map

$$F : E \to TG, \quad F(g, v) = d\ell_g(v) \tag{2}$$

(ℓ_g denotes left multiplication by g) is an affine vector bundle isomorphism, and moreover an isometry if E and TG carry the corresponding Sasaki metrics.

EXAMPLE 2. A particular case of Example 1 is the following: If the metric on G is bi-invariant, or equivalently the inner product is $\mathrm{Ad}(G)$-invariant, we have

$$\theta(X) Y = \frac{1}{2} [X, Y].$$

EXAMPLE 3. Let G be a compact connected Lie group and $(\mathcal{V}, \rho)$ a real orthogonal representation of G. Proposition 1 provides a connection ∇ on $E = G \times \mathcal{V} \to G$ induced by any left invariant parallelization and $\theta = \lambda\, d\rho$, for some $\lambda \in \mathbb{R}$.

Let $E = G \times \mathcal{V} \to G$ as in Example 3. For $v \in \mathcal{V}$, let R_v the section of E defined by

$$R_v(g) = \left(g, \rho\left(g^{-1}\right)v\right). \tag{3}$$

The sections L_v and R_v are called left and right invariant, respectively, since in the particular case where $\mathcal{V} = \mathfrak{g}$, $\rho = \mathrm{Ad}$ they correspond to left and right invariant vector fields, respectively, via the isomorphism (2).

REMARK 2. Although the vector bundles $E \to G$ of Example 3 are topologically trivial (as for instance the tangent spaces of parallelizable manifolds are) in most cases they are not geometrically trivial, as shown in (b) of the following Theorem.

THEOREM 2. *Let G be a compact connected simple Lie group endowed with a bi-invariant Riemannian metric. Let $(\mathcal{V}, \rho)$ be an irreducible real orthogonal representation of G and let $E = G \times \mathcal{V}$ with the Sasaki metric induced by the connection associated to any left invariant parallelization of G and $\theta = \lambda d\rho$, for some $\lambda \in \mathbb{R}$. The following assertions are true:*

(a) The left and right invariant unit sections are harmonic sections of $E^1 \to G$.

(b) If $\lambda = 0$ or $\lambda = 1$, then L_v or R_v, respectively, are parallel sections for all $v \in \mathcal{V}$. If $0 \neq \lambda \neq 1$, then the bundle $E \to G$ has no parallel unit sections.

REMARK 3. (a) The result is still valid if G is semisimple and the metric of G is a negative multiple of the Killing form.
(b) If $(\mathcal{V}, \rho) = (\mathfrak{g}, \mathrm{Ad})$ and $\lambda = 1/2$, we have the well-known fact that the unit left invariant vector fields on G are harmonic sections of $T^1G \to G$, since they are Killing vector fields and G is Einstein [5] (see in [3, Section 4] the case where the bi-invariant metric is not Einstein).

We need the following Lemma to prove the Theorem.

LEMMA 1. *Let ∇ be the connection on the bundle $E \to G$ as in the hypothesis of* Theorem 2. *If Z is a left invariant vector field on G, then*

$$(\nabla_Z \nabla_Z R_v)(g) = \left(g, (\lambda - 1)^2 d\rho(Z)^2 \rho\left(g^{-1}\right)v\right) \tag{4}$$

for all $g \in G$, $v \in \mathcal{V}$.

Proof. Let V be a smooth section of $E \to G$ and suppose that $V(h) = (h, u(h))$. Denote $w(h) = (d/dt)_0 u(h \exp(tZ))$ and $\gamma(t) = g \exp(tZ)$ for $t \sim 0$. We may assume that $Z \neq 0$, otherwise the assertion is trivial. A smooth section W such that

$$W(\gamma(t)) = (\cos t)\, L_{u(g)}(\gamma(t)) + (\sin t)\, L_{w(g)}(\gamma(t))$$

satisfies $W(g) = V(g)$ and $(W \circ \gamma)'(0) = (V \circ \gamma)'(0)$. Hence, $(\nabla_Z V)(g) = (\nabla_Z W)(g)$, which by (1) equals

$$L_{\lambda\, d\rho(Z) u(g)}(g) + L_{w(g)}(g) = (g, \lambda\, d\rho(Z)\, u(g) + w(g)).$$

Applying this procedure to $V = R_v$, that is, $u(h) = \rho\left(h^{-1}\right) v$ and $w(h) = -d\rho(Z)\,\rho\left(h^{-1}\right) v$, one obtains

$$(\nabla_Z R_v)(g) = \left(g, (\lambda - 1)\, d\rho(Z)\, \rho\left(g^{-1}\right) v\right). \tag{5}$$

Finally, applying again the procedure to the section $V = \nabla_Z R_v$, one obtains (4).

□

Proof of Theorem 2. (a) Let $\{Z_1, \ldots, Z_n\}$ be an orthonormal basis of $\mathfrak{g}$ and consider on G the associated left invariant parallelization. Given $v \in \mathcal{V}$, by (1) we compute

$$\begin{aligned}(\Delta L_v)(g) &= \sum_{i=1}^{n} \left(\nabla_{Z^i} \nabla_{Z^i} L_v\right)(g) = \sum_{i=1}^{n} L_{\lambda^2 d\rho(Z^i)^2 v}(g) \\ &= \left(g, \lambda^2 \sum_{i=1}^{n} d\rho\left(Z^i\right)^2 v\right) = \left(g, \lambda^2 \mathcal{C}_\rho(v)\right),\end{aligned}$$

where $\mathcal{C}_\rho$ is a multiple of the Casimir of the representation ρ (notice that the metric is a negative multiple of the Killing form). Now, the Casimir is a multiple of the identity, since ρ is irreducible (a direct application of Schur's Lemma). Hence, $\Delta L_v = \mu L_v$ for some μ and so L_v is a harmonic section of $E^1 \to G$ by Theorem 1. On the other hand, a straightforward computation shows that

$$d\rho(Z)\,\rho\left(g^{-1}\right) = \rho\left(g^{-1}\right) d\rho\,(\mathrm{Ad}\,(g)\, Z)$$

for all $g \in G$ and $Z \in \mathfrak{g}$. Hence, if we call $U^i = \mathrm{Ad}\,(g)\, Z^i$, we have by Lemma 1 that

$$\begin{aligned}(\Delta R_v)(g) &= \sum_{i=1}^{n} \left(g, (\lambda - 1)^2 \rho\left(g^{-1}\right) d\rho\left(U^i\right)^2 v\right) \\ &= \left(g, (\lambda - 1)^2 \rho\left(g^{-1}\right) \mathcal{C}_\rho(v)\right),\end{aligned}$$

since $\left\{U^i \mid i = 1, \ldots, n\right\}$ is an orthonormal basis of $\mathfrak{g}$ (the metric on G is bi-invariant). As before, $\mathcal{C}_\rho$ is a multiple $\bar{\mu}$ of the identity, hence $\Delta R_v = \bar{\mu}(\lambda - 1)^2 R_v$, which implies by Theorem 1 that R_v is a harmonic section of $E^1 \to G$.

(b) If $\lambda = 0$, clearly L_v is parallel by definition of the connection. If $\lambda = 1$, then R_v is parallel by (5). Suppose that a smooth unit section V with $V(e) = (e, v)$ is parallel. Then, for $X, Y \in \mathfrak{g}$ the curvature

$$\begin{aligned}R(X, Y)(e, v) &= \left(\nabla_X \nabla_Y L_v - \nabla_Y \nabla_X L_v - \nabla_{[X,Y]} L_v\right)(e) \\ &= (e, [\theta(X), \theta(Y)]\, v - \theta[X, Y]\, v) \\ &= \left(e, \lambda^2 [d\rho(X), d\rho(Y)]\, v - \lambda d\rho[X, Y]\, v\right) \\ &= (e, \lambda(\lambda - 1)\, d\rho[X, Y]\, v)\end{aligned}$$

vanishes. If G is semisimple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Hence, $0 \neq \lambda \neq 1$ implies that $d\rho\,(Z)\, v = 0$ for all $Z \in \mathfrak{g}$. This contradicts the fact that ρ is irreducible.

$\square$

Next we deal with an analogue of the particular case of Theorem 2 when $\mathcal{V} = \mathbb{H}$ is the algebra of quaternions, $G = S^3 = \{q \in \mathbb{H} \mid |q| = 1\}$ and $\rho\,(q)\, X = q.X$ (quaternion multiplication) for $X \in \operatorname{Im} \mathbb{H} = T_1 S^3$. (It is not a particular case of Theorem 2, since S^7 is not a Lie group.)

Let $\mathbb{O} \cong \mathbb{R}^8$ denote the octonians with the canonical inner product and let $S^7 = \{q \in \mathbb{O} \mid |q| = 1\}$ with the induced metric. The tangent space of S^7 at the identity may be identified with $\operatorname{Im} \mathbb{O}$, the purely imaginary octonians. Fix an orthonormal basis $\{x_1, \ldots, x_7\}$ of $\operatorname{Im} \mathbb{O}$ and consider the parallelization of S^7 consisting of the corresponding left invariant vector fields X^i's, that is, $X^i\,(q) = q.x^i \in q^\perp = T_q S^7$. By analogy with (3), given $v \in \mathbb{O}$, we define the section R_v of the trivial vector bundle $S^7 \times \mathbb{O} \to S^7$ by $R_v\,(q) = (q, \bar{q} v)$.

THEOREM 3. *Let $E = S^7 \times \mathbb{O} \to S^7$ be the trivial vector bundle with the connection ∇ induced by*

$$\theta : \left\{X^1, \ldots, X^7\right\} \to \mathfrak{o}\,(\mathbb{O}), \quad \theta\left(X^i\right) v = \lambda\, x_i v,$$

with $\lambda \in \mathbb{R}$, and consider on E the Sasaki metric induced by ∇. The connection is independent of the choice of the orthonormal basis of $\operatorname{Im}\mathbb{O}$. *If $v \in \mathbb{O}$ with $|v| = 1$, the following assertions are true for the sections L_v, R_v of the associated spherical bundle $E^1 \to S^7$.*

(a) If $\lambda = 0$, then L_v and R_v are harmonic sections. If $\lambda \neq 0$, then L_v is a harmonic section and R_v is a harmonic section if and only if $v = \pm 1$.

(b) If $0 \neq \lambda \neq 1$, then the bundle $E^1 \to S^7$ has no parallel sections. The section L_v is parallel if and only if $\lambda = 0$, and R_v is parallel if and only if $\lambda = 1$ and $v = \pm 1$.

Before proving the theorem we recall from Chapter 6 of [4] some facts about the octonians $\mathbb{O}$ (also called Cayley numbers), which are a non-associative normed algebra with identity, isomorphic to $\mathbb{R}^8$ as an inner product vector space. The algebra $\mathbb{O}$ is $\mathbb{H} \times \mathbb{H}$, with the multiplication given by

$$(a, b)\,(c, d) = \left(ac - \bar{d}b, da + b\bar{c}\right). \tag{6}$$

Setting $1 = (1, 0)$ and $e = (0, 1)$, one writes $(a, b) = a + be$. If $u = a + x$ with $a \in \mathbb{R}.1$ and $\langle x, 1\rangle = 0$, the conjugate of u is $\bar{u} = a - x$ and $\langle u, v\rangle = \operatorname{Re}\ (u\bar{v})$ holds for all $u, v \in \mathbb{O}$. If $x \in \operatorname{Im} \mathbb{O} = 1^\perp$ with $|x| = 1$, then

$$x^2 = -x\bar{x} = -\,|x|^2 = -1. \tag{7}$$

Moreover, if $\langle u, v\rangle = 0$, then

$$u\,(\bar{v}w) = -v\,(\bar{u}w) \tag{8}$$

for all w. From Lemma 6.11 of [4] and its proof we have that the associator

$$[u, v, w] = (uv)\,w - u\,(vw)$$

is an alternating 3-linear form which vanishes either if one of the arguments is real or if two consecutive arguments are conjugate. In particular, if $x \in \operatorname{Im}\mathbb{O}$ with $|x| = 1$, we have by (7) that for all v,

$$x\,(xv) = \left(x^2\right) v - [x, x, v] = -v + [x, \bar{x}, v] = -v. \tag{9}$$

LEMMA 2. *Let $z = x_\ell$ be an element of the basis of* $\operatorname{Im}\mathbb{O}$ *considered above and denote $Z = X_\ell$. Then for unit octonians v and q one has*

$$(\nabla_Z R_v)\,(q) = (q, \lambda z\,(\bar{q}v) - (z\bar{q})\,v) \tag{10}$$

and

$$(\nabla_Z \nabla_Z R_v)\,(q) = -\left(1+\lambda^2\right) R_v\,(q) - 2\lambda\,(q, z\,((z\bar{q})\,v))\,. \tag{11}$$

Proof. The assertions follow proceeding as in the proof of Lemma 1, setting $\rho\,(q)\,X = qX$ and $d\rho\,(z)\,X = zX$, taking into account that $\mathbb{O}$ is not associative and using (9). □

Proof of Theorem 3. (a) First we show that $\theta\left(X^i\right)$ is skew symmetric for all $i = 1, \ldots, 7$. Indeed, given $v \in \mathbb{O}$, since $x_i \in \operatorname{Im}\mathbb{O}$, then

$$\langle \lambda x_i v, v\rangle = \lambda \operatorname{Re}\,((x_i v)\,\bar{v}) = \lambda \operatorname{Re}\left([x_i, v, \bar{v}] - x_i\,|v|^2\right) = 0,$$

by one of the properties of the associator mentioned above. On the other hand, by definition of the connection and (9), we compute

$$\begin{aligned}(\Delta L_v)\,(q) &= \sum_{i=1}^{7} \left(\nabla_{X^i}\nabla_{X^i} L_v\right)(q) = \sum_{i=1}^{7} L_{\lambda^2 x^i\left(x^i v\right)}\,(q) = \\ &= \left(q, -\sum_{i=1}^{7}\lambda^2 v\right) = \left(q, -7\lambda^2 v\right) = -7\lambda^2 L_v\,(q)\,.\end{aligned}$$

By Theorem 1, L_v is a harmonic section of $E^1 \to S^7$ for any λ and using (11) and (9), R_v is a harmonic section if $\lambda = 0$ or $v = \pm 1$. Now we consider the case $\lambda \neq 0$. If R_v is a harmonic section, by Theorem 1 and (11) there exists a smooth function f on S^7 such that

$$\sum_{\ell=1}^{7} x_\ell\,((x_\ell \bar{q})\,v) = f\,(q)\,\bar{q}v \tag{12}$$

for all $q \in S^7$. By Proposition 6.40 in [4], based on a theorem of Artin, we may suppose without loss of generality that $v = a + bi$, with $a^2 + b^2 = 1$. We must show that $b = 0$. Take $\bar{q} = c + dj$ with $c^2 + d^2 = 1$ and suppose that $\{x_\ell \mid \ell = 1, \dots, 7\}$ is the canonical basis $\{i, j, k, e, ie, je, ke\}$. Now a straightforward computation using (6) and (9) yields that $\sum_{\ell=1}^{7} x_\ell\,((x_\ell j)\,i) = -k$. Setting $\xi = ac + cbi + adj$, equality (12) becomes

$$-7\xi - dbk = f\,(c - dj)\,(\xi - dbk)\,.$$

Suppose that $b \neq 0$. If $b = \pm 1$ (so $a = 0$), taking $c = d \neq 0$, one has $1 = f\,(c - dj) = -7$. If $b \neq \pm 1$ (so $a \neq 0$), taking $c = 0, d = 1$, one gets also a contradiction. Thus, $b = 0$ as desired.

(b) By definition of the connection, L_v is parallel if and only if $\lambda = 0$. Suppose that $0 \neq \lambda \neq 1$. As in the proof of Theorem 2 (b), we show that for any $v \in \mathbb{O}$, $v \neq 0$, there exist an orthonormal set $\{x, y\} \subset T_1 S^7 = \operatorname{Im}\mathbb{O}$ such that the curvature $R\,(x, y)\,v \neq 0$. Let X, Y be the left invariant vector fields on S^7 corresponding to x and y, respectively. By Proposition 6.40 of [4], based on a theorem of Artin, the span H of $\{1, x, y, xy\}$ is a normed subalgebra isomorphic to the quaternions. Hence, one can think of X, Y as left invariant vector fields on the Lie group $S^3 = H \cap S^7$. Therefore $[X, Y]\,(1) = xy - yx$. Using (8) we compute

$$\begin{aligned} R\,(x, y)\,v &= \left(\nabla_X \nabla_Y L_v - \nabla_Y \nabla_X L_v - \nabla_{[X,Y]} L_v\right)(1) \\ &= \lambda^2 x\,(yv) - \lambda^2 y\,(xv) - \lambda\,(xy - yx)\,v \\ &= 2\lambda\,(\lambda x\,(yv) - (xy)\,v) \\ &= 2\lambda\,((\lambda - 1)\,(xy)\,v - \lambda\,[x, y, v])\,. \end{aligned}$$

If $v = \pm 1$, for any orthonormal set $\{x, y\} \subset \operatorname{Im}\mathbb{O}$ one has clearly

$$R\,(x, y)\,v = \pm 2\lambda\,(\lambda - 1)\,xy \neq 0.$$

If $v \neq \pm 1$, then $u := \operatorname{Im} v \neq 0$ and taking an orthonormal set $\{x, y\}$ in $\operatorname{Im}\mathbb{O}$, with $y = \bar{u}/|u|$, by the properties of the associator given after (8), one has $R\,(x, y)\,v = 2\lambda\,(\lambda - 1)\,(xy)\,v \neq 0$. Finally, by (10), R_v is not parallel if $\lambda = 0$, and if $\lambda = 1$, then $(\nabla_Z R_v)\,(q) = (q, -[z, \bar{q}, v])$ for all $q \in S^7$, $\operatorname{Re} z = 0$. Similar arguments yield that in this case R_v is parallel if and only if $v = \pm 1$. This concludes the proof of (b).

□

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