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ON THE SETS OF SEQUENCES THAT ARE STRONGLY $\alpha-$BOUNDED AND $\alpha-$CONVERGENT TO NAUGHT WITH INDEX p.

Abstract. In this paper we deal with sets of sequences generalizing the well known spaces $w_{\infty}^{p}(\lambda) = \left\{X / C(\lambda)\left(|X|^{p}\right) \in l_{\infty}\right\}$ and $c_{\infty}(\lambda) = (w_{\infty}(\lambda))_{\Delta(\lambda)}$. We consider the set $\left(w_{\alpha}^{p}(\lambda)\right)_{\Delta(\mu)}$ and the cases when the operators $C(\lambda)$ and $\Delta(\mu)$ are replaced by their transposes. These results generalize in a certain sense those given in [4, 10, 11, 13, 14, 16].

1. Notations and preliminary results.

For a given infinite matrix $A = (a_{nm})_{n,m \geq 1}$ we define the operators A_n for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm} x_m$$

where $X = (x_m)_{m \geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the infinite linear system

$$A_n(X) = b_n \quad n = 1, 2, \ldots \tag{1}$$

where $B = (b_n)_{n \geq 1}$ is a one-column matrix and X the unknown, see [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Equation (1) can be written in the form $AX = B$, where $AX = (A_n(X))_{n \geq 1}$. In this paper we shall also consider A as an operator from a sequence space into another sequence space.

A Banach space E of complex sequences with the norm $\|\|_E$ is a BK space if each projection $P_n : X \to P_n X = x_n$ is continuous. A BK space E is said to have AK, (see [17]), if for every $B = (b_m)_{m \geq 1} \in E$, $B = \sum_{m=1}^{\infty} b_m e_m$, where $e_m = (0, \ldots, 1, 0, \ldots)$, 1 being in the m-th position, i.e.

$$\left\| \sum_{m=N+1}^{\infty} b_m e_m \right\|_E \to 0 \quad (n \to \infty).$$

s, c_0, c, l_{∞} are the sets of all sequences, the set of sequences that converge to zero, that are convergent and that are bounded respectively. cs and l_1 are the sets of convergent and absolutely convergent series respectively. We shall use the set

$$U^{+*} = \left\{ (u_n)_{n\geq 1} \in s \ / \quad u_n > 0 \ \forall n \right\}.$$

Using Wilansky's notations [17], we define for any sequence $\alpha = (\alpha_n)_{n\geq 1} \in U^{+*}$ and for any set of sequences E, the set

$$\alpha * E = \left\{ (x_n)_{n\geq 1} \in s \ / \quad \left(\frac{x_n}{\alpha_n} \right)_n \in E \right\}.$$

Writing

$$\alpha * E = \begin{cases} s_\alpha \text{ if } E = l_\infty, \\ s_\alpha^\circ \text{ if } E = c_0, \\ s_\alpha^\bullet \text{ if } E = c, \end{cases}$$

we have for instance

$$\alpha * c_0 = s_\alpha^\circ = \left\{ (x_n)_{n\geq 1} \in s \ / \quad x_n = o(\alpha_n) \quad n \to \infty \right\}.$$

Each of the spaces $\alpha * E$, where $E \in \{l_\infty, c_0, c\}$, is a BK space normed by

$$\|X\|_{s_\alpha} = \sup_{n\geq 1} \left(\frac{|x_n|}{\alpha_n} \right), \tag{2}$$

and s_α° has AK, see [10].

Now let $\alpha = (\alpha_n)_{n\geq 1}$ and $\beta = (\beta_n)_{n\geq 1} \in U^{+*}$. By $S_{\alpha,\beta}$ we denote the set of infinite matrices $A = (a_{nm})_{n,m\geq 1}$ such that

$$(a_{nm}\alpha_m)_{m\geq 1} \in l_1 \text{ for all } n \geq 1 \text{ and } \sum_{m=1}^{\infty} |a_{nm}| \alpha_m = O(\beta_n) \ \ (n \to \infty).$$

$S_{\alpha,\beta}$ is a Banach space with the norm

$$\|A\|_{S_{\alpha,\beta}} = \sup_{\nu\geq 1} \left(\sum_{m=1}^{\infty} |a_{nm}| \frac{\alpha_m}{\beta_n} \right).$$

Let E and F be any subsets of s. When A maps E into F we shall write $A \in (E, F)$, see [2]. So for every $X \in E$, $AX \in F$, ($AX \in F$ will mean that for each $n \geq 1$ the series defined by $y_n = \sum_{m=1}^{\infty} a_{nm}x_m$ is convergent and $(y_n)_{n\geq 1} \in F$). It has been proved in [13] that $A \in (s_\alpha, s_\beta)$ iff $A \in S_{\alpha,\beta}$. So we can write that $(s_\alpha, s_\beta) = S_{\alpha,\beta}$.

When $s_\alpha = s_\beta$ we obtain the Banach algebra with identity $S_{\alpha,\beta} = S_\alpha$, (see [1, 4, 5]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$.

We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$. If $\|I - A\|_{S_\alpha} < 1$, we shall say that $A \in \Gamma_\alpha$. Since S_α is a Banach algebra with identity, we have the useful result: if $A \in \Gamma_\alpha$, A is bijective from s_α into itself.

If $\alpha = (r^n)_{n\geq 1}$, Γ_α, S_α, s_α, s_α° and $s_\alpha^\bullet$ are replaced by Γ_r, S_r, s_r, s_r° and $s_r^\bullet$ respectively (see [1, 4, 5, 6, 7, 8]). When $r = 1$, we obtain $s_1 = l_\infty$, $s_1^\circ = c_0$ and $s_1^\bullet = c$, and putting $e = (1, 1, ...)$ we have $S_1 = S_e$. It is well known, see [2] that

$$(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1.$$

For any subset E of s, we put

$$AE = \{Y \in s \ / \ \exists X \in E \quad Y = AX\}.$$

If F is a subset of s, we shall denote

$$F(A) = F_A = \{X \in s \ / \ Y = AX \in F\}.$$

We can see that $F(A) = A^{-1}F$.

2. Some properties of the operators $\Delta(\lambda)$, $\Delta^+(\lambda)$ and Σ^+ relative to the sets s_α, s_α° and $s_\alpha^\bullet$.

Here we shall deal with the operators represented by $C(\lambda)$, $C^+(\lambda)$, $\Delta(\lambda)$ and $\Delta^+(\lambda)$.

Let $U = \left\{(u_n)_{n\geq 1} \in s \ / \quad u_n \neq 0 \ \forall n\right\}$. We define $C(\lambda) = (c_{nm})_{n,m\geq 1}$ for $\lambda = (\lambda_n)_{n\geq 1} \in U$, by

$$c_{nm} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

So, we put $C^+(\lambda) = C(\lambda)^t$. It can be proved that the matrix $\Delta(\lambda) = \left(c'_{nm}\right)_{n,m\geq 1}$ with

$$c'_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

is the inverse of $C(\lambda)$, see [13]. Similarly we put $\Delta^+(\lambda) = \Delta(\lambda)^t$. If $\lambda = e$ we get the well known operator of first difference represented by $\Delta(e) = \Delta$ and it is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$ and Δ and Σ belong to any given space S_R with $R > 1$. Writing $D_\lambda = (\lambda_n \delta_{nm})_{n,m\geq 1}$, (where $\delta_{nm} = 0$ for $n \neq m$ and $\delta_{nn} = 1$ otherwise), we have $\Delta^+(\lambda) = D_\lambda \Delta^+$. So for any given $\alpha \in U^{+*}$, we see that if $\frac{\alpha_{n-1}}{\alpha_n}\left|\frac{\lambda_n}{\lambda_{n-1}}\right| = O(1)$, then $\Delta^+(\lambda) = D_\lambda \Delta^+ \in \left(s_{\left(\frac{\alpha}{|\lambda|}\right)}, s_\alpha\right)$. Since $Ker\,\Delta^+(\lambda) \neq 0$, we are led to define the set

$$s_\alpha^*\left(\Delta^+(\lambda)\right) = s_\alpha\left(\Delta^+(\lambda)\right) \bigcap s_{\left(\frac{\alpha}{|\lambda|}\right)} = \left\{X = (x_n)_{n\geq 1} \in s_{\left(\frac{\alpha}{|\lambda|}\right)} \ / \ \Delta^+(\lambda)\, X \in s_\alpha\right\}.$$

It can be easily seen that

$$s^*_{\left(\frac{\alpha}{|\lambda|}\right)}\left(\Delta^+(e)\right) = s^*_{\left(\frac{\alpha}{|\lambda|}\right)}\left(\Delta^+\right) = s_\alpha^*\left(\Delta^+(\lambda)\right).$$

We obtain similar results with the set $s_\alpha^{\circ *}\left(\Delta^+(\lambda)\right) = s_\alpha^\circ\left(\Delta^+(\lambda)\right) \bigcap s^\circ_{\left(\frac{\alpha}{|\lambda|}\right)}$.

2.1. Properties of the sequence $C(\alpha)\alpha$.

We shall use the following sets

$$\begin{aligned}
\widehat{C_1} &= \left\{\alpha \in U^{+*} / \quad \frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right) = O(1) \ \ (n \to \infty)\right\},\\
\widehat{C} &= \left\{\alpha \in U^{+*} / \quad \left(\frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right)\right)_{n\geq 1} \in c\right\},\\
\widehat{C_1^+} &= \left\{\alpha \in U^{+*} \bigcap cs \ / \quad \frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty}\alpha_k\right) = O(1) \ \ (n \to \infty)\right\},\\
\Gamma &= \left\{\alpha \in U^{+*} \ / \ \overline{\lim}_{n\to\infty}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1\right\}
\end{aligned}$$

and

$$\Gamma^+ = \left\{\alpha \in U^{+*} \ / \ \overline{\lim}_{n\to\infty}\left(\frac{\alpha_{n+1}}{\alpha_n}\right) < 1\right\}.$$

Note that $\alpha \in \Gamma^+$ if and only if $\frac{1}{\alpha} \in \Gamma$. We shall see in Proposition 1 that if $\alpha \in \widehat{C_1}$, α tends to infinity. On the other hand we see that $\Delta \in \Gamma_\alpha$ implies $\alpha \in \Gamma$. We also have $\alpha \in \Gamma$ if and only if there is an integer $q \geq 1$ such that

$$\gamma_q(\alpha) = \sup_{n\geq q+1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) < 1.$$

We obtain the following results in which we put $[C(\alpha)\alpha]_n = \dfrac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right)$.

PROPOSITION 1. *Let $\alpha \in U^{+*}$. Then*

i) $\dfrac{\alpha_{n-1}}{\alpha_n} \to 0$ *if and only if* $[C(\alpha)\alpha]_n \to 1$.

ii) $[C(\alpha)\alpha]_n \to l$ *implies that* $\dfrac{\alpha_{n-1}}{\alpha_n} \to 1 - \dfrac{1}{l}$.

iii) If $\alpha \in \widehat{C_1}$ then there are $K > 0$ and $\gamma > 1$ such that $\alpha_n \geq K\gamma^n$ for all n.

iv) $\alpha \in \Gamma$ implies that $\alpha \in \widehat{C_1}$ and there exist a real $b > 0$ and an integer q, such that

$$[C(\alpha)\alpha]_n \leq \frac{1}{1-\chi} + b\chi^n \quad \text{for } n \geq q+1 \text{ and } \chi = \gamma_q(\alpha) \in]0,1[.$$

v) $\alpha \in \Gamma^+$ *implies* $\alpha \in \widehat{C_1^+}$.

Proof. i), ii), iii) and iv) have been proved in [10].

Assertion v). If $\alpha \in \Gamma^{+}$, there are $\chi' \in]0,1[$ and an integer $q' \geq 1$ such that

$$\frac{\alpha_k}{\alpha_{k-1}} \leq \chi' \quad \text{for } k \geq q'.$$

Then we have for every $n \geq q'$

$$\frac{1}{\alpha_n}\left(\sum_{k=n}^{\infty}\alpha_k\right) = \sum_{k=n}^{\infty}\left(\frac{\alpha_k}{\alpha_n}\right) \leq 1 + \sum_{k=n+1}^{\infty}\left[\prod_{i=0}^{k-n-1}\left(\frac{\alpha_{k-i}}{\alpha_{k-i-1}}\right)\right] \leq \sum_{k=n}^{\infty}\chi'^{k-n} = O(1).$$

This gives the conclusion.

□

REMARK 1. Note that as a direct consequence of Proposition 2.1, we have

$$\widehat{C} \subset \Gamma \subset \widehat{C_1}.$$

We also have $\widehat{C} \neq \Gamma$, see [4]. On the other hand we see that $\widehat{C_1} \bigcap \widehat{C_1^{+}} = \Gamma \bigcap \Gamma^{+} = \phi$.

2.2. The spaces $w_{\alpha}^{p}(\lambda)$, $w_{\alpha}^{\circ p}(\lambda)$ and $w_{\alpha}^{\bullet p}(\lambda)$ for $p > 0$.

In this subsection we recall some results on the sets that generalize the sets $w_{\infty}^{p}(\lambda)$, $w_{0}^{p}(\lambda)$ and $w^{p}(\lambda)$ for given real $p > 0$.

For any given real $p > 0$ and every sequence $X = (x_n)_{n\geq 1}$, we put $|X|^{p} = \left(\left|x_n^{p}\right|\right)_n$ and

$$\begin{aligned}
w_{\alpha}^{p}(\lambda) &= \left\{X \in s \,/\, C(\lambda)\left(|X|^{p}\right) \in s_{\alpha}\right\},\\
w_{\alpha}^{\circ p}(\lambda) &= \left\{X \in s / \, C(\lambda)\left(|X|^{p}\right) \in s_{\alpha}^{\circ}\right\},\\
w_{\alpha}^{\bullet p}(\lambda) &= \left\{X \in s \,/\, X - le^{t} \in w_{\alpha}^{\circ p}(\lambda) \quad \text{for some } l \in C\right\}.
\end{aligned}$$

For instance we see that

$$w_{\alpha}^{p}(\lambda) = \left\{X = (x_n)_n \in s \,/\, \sup_{n\geq 1}\left(\frac{1}{|\lambda_n|\,\alpha_n}\sum_{k=1}^{n}|x_k|^{p}\right) < \infty\right\}.$$

If there exist A and $B > 0$, such that $A < \alpha_n < B$ for all n, we get the well known spaces $w_{\alpha}^{p}(\lambda) = w_{\infty}^{p}(\lambda)$, $w_{\alpha}^{\circ p}(\lambda) = w_{0}^{p}(\lambda)$ and $w_{\alpha}^{\bullet p}(\lambda) = w^{p}(\lambda)$, see [14, 15]. In the case when $\lambda = (n)_{n\geq 1}$, the previous sets have been introduced in [3] by Maddox and it is written $w_{\infty}^{p}(\lambda) = w_{\infty}^{p}$, $w_{0}^{p}(\lambda) = w_{0}^{p}$ and $w^{p}(\lambda) = w^{p}$. It is proved that each of the sets w_{0}^{p} and w_{∞}^{p} is a $p-$normed FK space for $0 < p < 1$, (that is a complete linear metric space in which each projection P_n is continuous), and a BK space for

$1 \leq p < \infty$ with respect to the norm

$$\|X\| = \begin{cases} \sup_{\nu \geq 1} \left(\frac{1}{2^\nu} \left(\sum_{n=2^\nu}^{2^{\nu+1}-1} |x_n|^p \right) \right) & \text{if } 0 < p < 1, \\ \sup_{\nu \geq 1} \left(\frac{1}{2^\nu} \left(\sum_{n=2^\nu}^{2^{\nu+1}-1} |x_n|^p \right) \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty. \end{cases}$$

w_0^p has the property AK, and every sequence $X = (x_n)_{n \geq 1} \in w^p \left((n)_n \right)$ has a unique representation

$$X = le^t + \sum_{n=1}^{\infty} (x_n - l)\, e_n^t, \text{where } l \in C \text{ is such that } X - le^t \in w_0^p,$$

When $p = 1$, we omit the index p and write $w_\alpha^p(\lambda) = w_\alpha(\lambda)$, $\overset{\circ}{w}{}_\alpha^p(\lambda) = \overset{\circ}{w}_\alpha(\lambda)$ and $w_\alpha^{\bullet p}(\lambda) = w_\alpha(\lambda)$. It has been proved in [14], that if λ is a strictly increasing sequence of reals tending to infinity then $w_0(\lambda)$ and $w_\infty(\lambda)$ are BK spaces and $w_0(\lambda)$ has AK, with respect to the norm

$$\|X\| = \|C(\lambda)(|X|)\|_{l^\infty} = \sup_n \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} |x_k| \right).$$

Recall the next results given in [10].

THEOREM 1. *Let α and λ be any sequences of U^{+*}.*

i) Consider the following properties

- *a)* $\frac{\alpha_{n-1}\lambda_{n-1}}{\alpha_n \lambda_n} \to 0$*;*
- *b)* $s_\alpha^\bullet(C(\lambda)) = s_{\alpha\lambda}^\bullet$.
- *c)* $\alpha\lambda \in \widehat{C_1}$*;*
- *d)* $w_\alpha(\lambda) = s_{\alpha\lambda}$*;*
- *e)* $\overset{\circ}{w}_\alpha(\lambda) = \overset{\circ}{s}_{\alpha\lambda}$*;*
- *f)* $w_\alpha^\bullet(\lambda) = \overset{\circ}{s}_{\alpha\lambda}$.

We have a)⇒b), c)⇔d) and c)⇒e) and f).

ii) If $\alpha\lambda \in \widehat{C_1}$, $w_\alpha(\lambda)$, $\overset{\circ}{w}_\alpha(\lambda)$ and $w_\alpha^\bullet(\lambda)$ are BK spaces with respect to the norm

$$\|X\|_{s_{\alpha\lambda}} = \sup_{n \geq 1} \left(\frac{|x_n|}{\alpha_n \lambda_n} \right),$$

and $\overset{\circ}{w}_\alpha(\lambda) = w_\alpha^\bullet(\lambda)$ has AK.

2.3. Properties of some new sets of sequences.

In this subsection we shall characterize the sets $E(\Delta(\mu))$, $E\left(\Delta^{+}(\mu)\right)$ for $E \in \left\{s_\alpha, s_\alpha^\circ, s_\alpha^\bullet\right\}$, and the sets $w_\alpha^p(\lambda)$, $w_\alpha^{+p}(\lambda)$ and $w_\alpha^{\circ+p}(\lambda)$.

In order to state some new results we need the following lemmas. First recall the well known result.

LEMMA 1. *$A \in (c_0, c_0)$ if and only if*

$$\begin{cases} A \in S_1, \\ \lim_n a_{nm} = 0 \quad \textit{for each } m \geq 1. \end{cases}$$

The next result has been shown in [11].

LEMMA 2. *If Δ^+ is bijective from s_α into itself, then $\alpha \in cs$.*

We also need to state the following elementary result.

LEMMA 3. *We have*

$$\Sigma^{+}\left(\Delta^{+}X\right) = X \quad \forall X \in c_0 \quad \textit{and} \quad \Delta^{+}\left(\Sigma^{+}X\right) = X \quad \forall X \in cs.$$

Put now

$$\begin{aligned} w_\alpha^{+p}(\lambda) &= \left\{X \in s \;/\; C^{+}(\lambda)\left(|X|^p\right) \in s_\alpha\right\}, \\ w_\alpha^{\circ+p}(\lambda) &= \left\{X \in s \;/\; C^{+}(\lambda)\left(|X|^p\right) \in s_\alpha^\circ\right\}, \end{aligned}$$

see [11]. Letting $\beta^- = (\beta_{n-1})_{n\geq 1}$, with $\beta_0 = 1$, for any $\beta = (\beta_n)_{n \geq 1} \in U^{+*}$, we can state the following results.

THEOREM 2. *Let $\alpha \in U^{+*}$, $\lambda, \mu \in U$ and $p > 0$. We successively have*

i) a) $s_\alpha(\Delta(\mu)) = s_{\left(\frac{\alpha}{|\mu|}\right)}$ *if and only if* $\alpha \in \widehat{C_1}$;

b) $s_\alpha^\circ(\Delta(\mu)) = s^\circ_{\left(\frac{\alpha}{|\mu|}\right)}$ *if and only if* $\alpha \in \widehat{C_1}$;

c) $s_\alpha^\bullet(\Delta(\mu)) = s^\bullet_{\left(\frac{\alpha}{|\mu|}\right)}$ *if and only if* $\alpha \in \widehat{C}$.

ii) a) $s_\alpha\left(\Delta^{+}(\mu)\right) = s_{\left(\frac{\alpha}{|\mu|}\right)^-}$ *if and only if* $\dfrac{\alpha}{|\mu|} \in \widehat{C_1}$;

b) $\dfrac{\alpha_{n-1}}{\alpha_n}\dfrac{\mu_n}{\mu_{n-1}} = o(1)$ *implies* $s_\alpha^\circ\left(\Delta^{+}(\mu)\right) = s^\circ_{\left(\frac{\alpha}{|\mu|}\right)^-}$;

c) $\dfrac{\alpha}{|\mu|} \in \widehat{C_1^+}$ *if and only if* $s_\alpha^*\left(\Delta^{+}(\mu)\right) = s_{\left(\frac{\alpha}{|\mu|}\right)}$;

d) $\dfrac{\alpha}{|\mu|} \in \widehat{C_1^+}$ *if and only if* $s_\alpha^{\circ*}\left(\Delta^{+}(\mu)\right) = s^\circ_{\left(\frac{\alpha}{|\mu|}\right)}$.

iii) a) $s_\alpha\left(\Sigma^+\right)=s_\alpha$ *if and only if* $\alpha\in\widehat{C_1^+}$ *and* $s_\alpha^\circ\left(\Sigma^+\right)=s_\alpha^\circ$ *if and only if* $\alpha\in\widehat{C_1^+}$.

b) $\alpha\in\widehat{C_1^+}$ *if and only if* $w_\alpha^{+p}(\lambda)=s_{(\alpha|\lambda|)^{\frac{1}{p}}}$,

c) if $\alpha\in\widehat{C_1^+}$, *then* $w_\alpha^{\circ +p}(\lambda)=s^\circ_{(\alpha|\lambda|)^{\frac{1}{p}}}$,

d) $\alpha|\lambda|\in\widehat{C_1}$ *if and only if* $w_\alpha^p(\lambda)=s_{(\alpha|\lambda|)^{\frac{1}{p}}}$.

e) If $\alpha|\lambda|\in\widehat{C_1}$, *then* $w_\alpha^{\circ p}(\lambda)=s^\circ_{(\alpha|\lambda|)^{\frac{1}{p}}}$.

Proof. Assertion i) has been proved in [10]. Throughout the proof of part ii) we shall put $\beta=\frac{\alpha}{|\mu|}$.

Assertion ii) a). First we have $s_\alpha\left(\Delta^+(\mu)\right)=s_\beta\left(\Delta^+\right)$. Indeed,

$$X\in s_\alpha\left(\Delta^+(\mu)\right)\Leftrightarrow D_\mu\Delta^+X\in s_\alpha\Leftrightarrow\Delta^+X\in s_\beta\Leftrightarrow X\in s_\beta\left(\Delta^+\right).$$

To get a), it is enough to show that $\beta\in\widehat{C_1}$ if and only if $s_\beta\left(\Delta^+\right)=s_{\beta^-}$. We assume that $\beta\in\widehat{C_1}$. From the inequality

$$\frac{\beta_{n-1}}{\beta_n}\leq\frac{1}{\beta_n}\left(\sum_{k=1}^n\beta_k\right)=O(1),$$

we deduce that $\dfrac{\beta_{n-1}}{\beta_n}=O(1)$ and $\Delta^+\in\left(s_{\beta^-},s_\beta\right)$. Then for any given $B\in s_\beta$ the solutions of the equation $\Delta^+X=B$ are given by $x_1=-u$ and

$$-x_n=u+\sum_{k=1}^{n-1}b_k,\text{ for }n\geq 2,\tag{3}$$

where u is an arbitrary scalar. So there exists a real $K>0$, such that

$$\frac{|x_n|}{\beta_{n-1}}=\frac{\left|u+\sum_{k=1}^{n-1}b_k\right|}{\beta_{n-1}}\leq\frac{|u|+K\left(\sum_{k=1}^{n-1}\beta_k\right)}{\beta_{n-1}}=O(1),$$

since iii) in Proposition 1 implies $\frac{|u|}{\beta_{n-1}}=O(1)$. So $X\in s_\alpha$ and we conclude that Δ^+ is surjective from s_{β^-} into s_β. Then $\beta=\frac{\alpha}{|\mu|}\in\widehat{C_1}$ implies

$$s_\alpha\left(\Delta^+(\mu)\right)=s_{\left(\frac{\alpha}{|\mu|}\right)^-}.$$

Conversely, assume that $s_\alpha\left(\Delta^+(\mu)\right)=s_{\left(\frac{\alpha}{|\mu|}\right)^-}$. If we take $B=\beta$, we get $x_n=x_1-\sum_{k=1}^{n-1}\beta_k$, where x_1 is an arbitrary scalar and

$$\frac{x_n}{\beta_{n-1}}=\frac{x_1}{\beta_{n-1}}-\frac{1}{\beta_{n-1}}\left(\sum_{k=1}^{n-1}\beta_k\right)=O(1).$$

Putting $x_1 = 0$, we conclude that $\beta \in \widehat{C_1}$.

ii) b) First we have $\Delta^+ \in \left(\overset{\circ}{s}_{\beta^-}, \overset{\circ}{s}_{\beta}\right)$ because $\dfrac{\beta_{n-1}}{\beta_n} = O(1)$. Let us show that Δ^+ is surjective from $\overset{\circ}{s}_{\beta^-}$ into $\overset{\circ}{s}_{\beta}$. For this, let $B = (b_n)_{n\geq 1} \in \overset{\circ}{s}_{\beta}$. The solutions $X = (x_n)_{n\geq 1}$ of the equation $\Delta^+ X = B$ are given by (3). We have

$$\frac{x_n}{\beta_{n-1}} = o(1) - \frac{\sum_{k=1}^{n-1} b_k}{\beta_{n-1}},$$

because from Proposition 2.1, the condition $\dfrac{\beta_{n-1}}{\beta_n} = o(1)$ implies $\beta \in \widehat{C_1}$ and $\beta \to \infty$. Since $B \in \overset{\circ}{s}_{\beta}$ there is a sequence $\nu = (\nu_n)_{n\geq 1} \in c_0$, such that $b_n = \beta_n \nu_n$. Then we have for a real $M > 0$

$$\frac{\left|\sum_{k=1}^{n-1} b_k\right|}{\beta_{n-1}} \leq \frac{1}{\beta_{n-1}} \left(\sum_{k=1}^{n-1} \beta_k \nu_k\right) \quad \text{for all } n \geq 2.$$

It remains to show that $\frac{1}{\beta_{n-1}} \sum_{k=1}^{n-1} \beta_k \nu_k = o(1)$. For this consider any given $\varepsilon > 0$. Since $\beta \to \infty$ there is an integer N such that

$$S_n = \frac{1}{\beta_{n-1}} \left|\sum_{k=1}^{N} \beta_k \nu_k\right| \leq \frac{\varepsilon}{2}$$

for $n > N$, and

$$\sup_{k\geq N+1} (|\nu_k|) \leq \frac{\varepsilon}{2 \sup_{n\geq 2} \left([C(\beta)\beta]_{n-1}\right)}.$$

Writing $R_n = \dfrac{1}{\beta_{n-1}} \left|\sum_{k=N+1}^{n-1} \beta_k \nu_k\right|$ for $n > N+2$, we deduce that

$$R_n \leq \left(\sup_{N+1\leq k\leq n-1} (|\nu_k|)\right) [C(\beta)\beta]_{n-1} \leq \frac{\varepsilon}{2}.$$

Finally, we obtain

$$\frac{|x_n|}{\beta_{n-1}} = \left|\frac{1}{\beta_{n-1}} \left(\sum_{k=1}^{N} \beta_k \nu_k\right) + \frac{1}{\beta_{n-1}} \left(\sum_{k=N+1}^{n-1} \beta_k \nu_k\right)\right| \leq S_n + R_n \leq \varepsilon \quad \text{for} \quad n \geq N,$$

and $X \in \overset{\circ}{s}_{\beta-}$. So we have proved ii) b).

Assertion ii) c). Necessity. Assume that $\beta = \frac{\alpha}{|\mu|} \in \widehat{C_1^+}$. Since we have $s_\alpha^*\left(\Delta^+(\mu)\right) = s_\beta^*\left(\Delta^+\right) = s_\beta$, it is enough to show that Δ^+ is bijective from s_β to s_β. We can write that $\Delta^+ \in \left(s_\beta, s_\beta\right)$, since

$$\frac{\beta_{n+1}}{\beta_n} \leq \frac{1}{\beta_n}\left(\sum_{k=n}^{\infty}\beta_k\right) = O(1) \quad (n \to \infty). \tag{4}$$

Further, from $s_\beta \subset cs$, we deduce using Lemma 3 that for any given $B \in s_\beta$, $\Delta^+\left(\Sigma^+ B\right) = B$. On the other hand $\Sigma^+ B = \left(\sum_{k=n}^{\infty} b_k\right)_{n\geq 1} \in s_\beta$, since $\beta \in \widehat{C_1^+}$. So Δ^+ is surjective from s_β into s_β. Finally, Δ^+ is injective because the equation

$$\Delta^+ X = O$$

admits the unique solution $X = O$ in s_β, since $Ker\,\Delta^+ = \left\{ue^t / \quad u \in C\right\}$ and $e^t \notin s_\beta$.

Sufficiency. For every $B \in s_\beta$ the equation $\Delta^+ X = B$ admits a unique solution in s_β. Then from Lemma 2, $\beta \in cs$ and since $s_\beta \subset cs$ we deduce from Lemma 3 that $X = \Sigma^+ B \in s_\beta$ is the unique solution of $\Delta^+ X = B$. Taking $B = \beta$, we get $\Sigma^+\beta \in s_\alpha$ that is $\beta \in \widehat{C_1^+}$.

As above to prove ii) d) it is enough to verify that $\beta = \frac{\alpha}{|\mu|} \in \widehat{C_1^+}$ if and only if $s_\beta^{\circ *}\left(\Delta^+\right) = s_\beta$. If $\beta \in \widehat{C_1^+}$, Δ^+ is bijective from s_β° into itself. Indeed, we have $D_{\frac{1}{\beta}}\Delta^+ D_\beta \in (c_0, c_0)$ from (4) and Lemma 1. Furthermore, since $\beta \in \widehat{C_1^+}$ we have $s_\beta^\circ \subset cs$ and for every $B \in s_\beta^\circ$, $\Delta^+\left(\Sigma^+ B\right) = B$. From Lemma 1, we have $\Sigma^+ \in \left(s_\beta^\circ, s_\beta^\circ\right)$, so the equation $\Delta^+ X = B$ admits in s_β° the solution $X_0 = \Sigma^+ B$ and we have proved that Δ^+ is surjective from s_β° into itself. Finally, $\beta \in \widehat{C_1^+}$ implies that $e^t \notin s_\beta^\circ$, so $Ker\,\Delta^+ \bigcap s_\beta^\circ = \{0\}$ and we conclude that Δ^+ is bijective from s_β° into itself.

iii) a) comes from ii), since $\alpha \in \widehat{C_1^+}$ if and only if Δ^+ is bijective from s_α into itself and is also bijective from s_α° into itself, and

$$\Sigma^+\left(\Delta^+ X\right) = \Delta^+\left(\Sigma^+ X\right) = X \quad \text{for all } X \in s_\alpha.$$

b) Assume that $\alpha \in \widehat{C_1^+}$. Since $C^+(\lambda) = \Sigma^+ D_{\frac{1}{\lambda}}$, we have

$$w_\alpha^{+p}(\lambda) = \left\{X \in s \;/\; \left(\Sigma^+ D_{\frac{1}{\lambda}}\right)\left(|X|^p\right) \in s_\alpha\right\} = \left\{X \;/\; D_{\frac{1}{\lambda}}\left(|X|^p\right) \in s_\alpha\left(\Sigma^+\right)\right\};$$

and since $\alpha \in \widehat{C_1^+}$ implies $s_\alpha\left(\Sigma^+\right) = s_\alpha$, we conclude that

$$w_\alpha^{+p}(\lambda) = \left\{X \in s \;/\; |X|^p \in D_\lambda s_\alpha = s_{\alpha|\lambda|}\right\} = s_{(\alpha|\lambda|)^{\frac{1}{p}}}.$$

Conversely, we have $(\alpha\,|\lambda|)^{\frac{1}{p}} \in s_{(\alpha|\lambda|)^{\frac{1}{p}}} = w_{\alpha}^{+p}(\lambda)$. So

$$C^{+}(\lambda)\left[(\alpha\,|\lambda|)^{\frac{1}{p}}\right]^{p} = \left(\sum_{k=n}^{\infty}\frac{\alpha_k\,|\lambda_k|}{|\lambda_k|}\right)_{n\geq 1} \in s_{\alpha},$$

i.e. $\alpha \in \widehat{C_1^{+}}$ and we have proved i). We get iii) c) reasoning as above.

iii) d) has been proved in [4]. iii) e) Assume that $\alpha\,|\lambda| \in \widehat{C_1}$. Then

$$w_{\alpha}^{\circ p}(\lambda) = \left\{X \in s \;/\quad |X|^{p} \in \Delta(\lambda)\, s_{\alpha}^{\circ}\right\}.$$

Since $\Delta(\lambda) = \Delta D_{\lambda}$, we get $\Delta(\lambda)\, s_{\alpha}^{\circ} = \Delta s_{\alpha|\lambda|}^{\circ}$. Now, from i) b) we deduce that $\alpha\,|\lambda| \in \widehat{C_1}$ implies that Δ is bijective from $s_{\alpha|\lambda|}^{\circ}$ into itself and $w_{\alpha}(\lambda) = s^{\circ}_{(\alpha|\lambda|)^{\frac{1}{p}}}$. We get e) reasoning as above.

□

As a direct consequence of Theorem 2 we obtain the following results given in [11].

COROLLARY 1. *Let $r > 0$ be any real. We get*

$$r > 1 \Leftrightarrow s_r(\Delta) = s_r \Leftrightarrow s_r^{\circ}(\Delta) = s_r^{\circ} \Leftrightarrow s_r\left(\Delta^{+}\right) = s_r.$$

We deduce from the previous section the following.

3. Sets of sequences that are strongly $\alpha-$bounded and $\alpha-$convergent to zero with index p and generalizations.

In this section we deal with sets generalizing the well known sets of sequences that are strongly bounded and convergent to zero.

First we recall some results given in [10].

3.1. Sets $c_{\alpha}(\lambda,\mu)$, $c_{\alpha}^{\circ}(\lambda,\mu)$ and $c_{\alpha}^{\bullet}(\lambda,\mu)$.

If $\alpha = (\alpha_n)_n \in U^{+*}$ is a given sequence, we consider now for $\lambda \in U$, $\mu \in s$ the space

$$c_{\alpha}(\lambda,\mu) = (w_{\alpha}(\lambda))_{\Delta(\mu)} = \{X \in s \;/\; \Delta(\mu)\, X \in w_{\alpha}(\lambda)\}$$

It is easy to see that

$$c_{\alpha}(\lambda,\mu) = \{X \in s \;/\; C(\lambda)\,(|\Delta(\mu)\, X|) \in s_{\alpha}\},$$

that is

$$c_{\alpha}(\lambda,\mu) = \left\{X = (x_n)_n \in s \;/\; \sup_{n\geq 2}\left(\frac{1}{|\lambda_n|\,\alpha_n}\sum_{k=2}^{n}|\mu_k x_k - \mu_{k-1}x_{k-1}|\right) < \infty\right\}.$$

See [10, 11, 13]. Similarly we define the following sets

$$\begin{aligned} \overset{\circ}{c}_{\alpha}(\lambda,\mu) &= \left\{X\in s \ / \ C(\lambda)(|\Delta(\mu)X|)\in \overset{\circ}{s}_{\alpha}\right\}, \\ c_{\alpha}^{\bullet}(\lambda,\mu) &= \left\{X\in s \ / \ X-le^{t}\in \overset{\circ}{c}_{\alpha}(\lambda,\mu) \quad \text{for some } l\in C\right\} \end{aligned}$$

Recall that if $\lambda=\mu$ it is written that $c_0(\lambda)=(w_0(\lambda))_{\Delta(\lambda)}$,

$$c(\lambda)=\left\{X\in s \ / \ X-le^{t}\in c_0(\lambda) \ \text{ for some } l\in C\right\},$$

and $c_\infty(\lambda)=(w_\infty(\lambda))_{\Delta(\lambda)}$, see [16]. It can be easily seen that

$$c_0(\lambda)=\overset{\circ}{c}_{e}(\lambda,\lambda),\ c_\infty(\lambda)=c_e(\lambda,\lambda) \ \text{ and } c(\lambda)=c_e^{\bullet}(\lambda,\lambda).$$

These sets are called sets of sequences that are strongly bounded, strongly convergent to 0 and strongly convergent. If $\lambda\in U^{+*}$ is a sequence strictly increasing to infinity, $c(\lambda)$ is a Banach space with respect to

$$\|X\|_{c_\infty(\lambda)}=\sup_{n\geq 1}\left(\frac{1}{\lambda_n}\sum_{k=1}^{n}|\lambda_k x_k-\lambda_{k-1}x_{k-1}|\right)$$

with the convention $x_0=0$. Each of the spaces $c_0(\lambda)$, $c(\lambda)$ and $c_\infty(\lambda)$ is a BK space, with respect to the previous norm (see [14]). $c_0(\lambda)$ has AK and every $X\in c(\lambda)$ has a unique representation given by

$$X=le^{t}+\sum_{k=1}^{\infty}(x_k-l)\,e_k^{t}, \tag{5}$$

where $X-le^{t}\in c_0$. The number l is called the strong $c(\lambda)$-limit of the sequence X.

We obtain the next result given in [10]:

THEOREM 3. *Let α, λ and μ be sequences of U^{+*}.*

i) Consider the following properties

a) $\alpha\lambda\in\widehat{C_1}$;

b) $c_\alpha(\lambda,\mu)=s_{\alpha\frac{\lambda}{\mu}}$;

c) $\overset{\circ}{c}_{\alpha}(\lambda,\mu)=\overset{\circ}{s}_{\alpha\frac{\lambda}{\mu}}$;

d) $c_\alpha^{\bullet}(\lambda,\mu)=\left\{X\in s \ / \ X-le^{t}\in \overset{\circ}{s}_{\alpha\frac{\lambda}{\mu}} \quad \textit{for some } l\in C\right\}$.

We have a)⇔b) and a)⇒c) and d).

ii) If $\alpha\lambda \in \widehat{C_1}$, then $c_\alpha(\lambda,\mu)$, $\overset{\circ}{c}_\alpha(\lambda,\mu)$ and $c^\bullet_\alpha(\lambda,\mu)$ are BK spaces with respect to the norm

$$\|X\|_{s_{\alpha\frac{\lambda}{\mu}}} = \sup_{n\geq 1}\left(\mu_n \frac{|x_n|}{\alpha_n\lambda_n}\right).$$

$\overset{\circ}{c}_\alpha(\lambda,\mu)$ has AK and every $X \in c^\bullet_\alpha(\lambda,\mu)$ has a unique representation given by (5), where $X - le \in \overset{\circ}{s}_{\alpha\frac{\lambda}{\mu}}$.

We immediatly deduce the following:

COROLLARY 2. *Assume that α, λ and $\mu \in U^{+*}$.*

i) If $\alpha\lambda \in \widehat{C_1}$ and $\mu \in l_\infty$, then

$$c^\bullet_\alpha(\lambda,\mu) = \overset{\circ}{s}_{\alpha\frac{\lambda}{\mu}}. \tag{6}$$

ii) $\lambda \in \Gamma \Rightarrow \lambda \in \widehat{C_1} \Rightarrow c_0(\lambda) = \overset{\circ}{s}_\lambda$ and $c_\infty(\lambda) = s_\lambda$.

3.2. Generalization

In this subsection we consider spaces generalizing the well known spaces of sequences $c_\infty(\lambda)$ and $c_0(\lambda)$ that are strongly bounded and convergent to naught.

For given real $p > 0$, let us put

$$\begin{aligned}
c^p_\alpha(\lambda,\mu) &= \left(w^p_\alpha(\lambda)\right)_{\Delta(\mu)} = \left\{X \;/\; C(\lambda)\left(|\Delta(\mu)X|^p\right) \in s_\alpha\right\},\\
c^{\cdot+p}_\alpha(\lambda,\mu) &= \left(w^p_\alpha(\lambda)\right)_{\Delta^+(\mu)} = \left\{X \;/\; C(\lambda)\left(\left|\Delta^+(\mu)X\right|^p\right) \in s_\alpha\right\},\\
c^{+\cdot p}_\alpha(\lambda,\mu) &= \left(w^{+p}_\alpha(\lambda)\right)_{\Delta(\mu)} = \left\{X \;/\; C^+(\lambda)\left(|\Delta(\mu)X|^p\right) \in s_\alpha\right\},\\
c^{+p}_\alpha(\lambda,\mu) &= \left(w^{+p}_\alpha(\lambda)\right)_{\Delta^+(\mu)} = \left\{X \;/\; C^+(\lambda)\left(\left|\Delta^+(\mu)X\right|^p\right) \in s_\alpha\right\}.
\end{aligned}$$

When s_α is replaced by $\overset{\circ}{s}_\alpha$ in the previous definitions, we shall write $\widetilde{c^p_\alpha}(\lambda,\mu)$, $\widetilde{c^{\cdot+p}_\alpha}(\lambda,\mu)$, $\widetilde{c^{+\cdot p}_\alpha}(\lambda,\mu)$ and $\widetilde{c^{+p}_\alpha}(\lambda,\mu)$, instead of $c^p_\alpha(\lambda,\mu)$, $c^{\cdot+p}_\alpha(\lambda,\mu)$, $c^{+\cdot p}_\alpha(\lambda,\mu)$ and $c^{+p}_\alpha(\lambda,\mu)$. For instance, it can be easily seen that

$$c^p_\alpha(\lambda,\mu) = \left\{X = (x_n)_{n\geq 1} \;/\; \sup_{n\geq 1}\left[\frac{1}{|\lambda_n|\alpha_n}\left(\sum_{k=1}^{n}|\mu_k x_k - \mu_{k-1}x_{k-1}|^p\right)\right] < \infty\right\},$$

$$c^{\cdot+p}_\alpha(\lambda,\mu) = \left\{X = (x_n)_{n\geq 1} \;/\; \sup_{n\geq 1}\left[\frac{1}{|\lambda_n|\alpha_n}\left(\sum_{k=1}^{n}|\mu_k x_k - \mu_{k+1}x_{k+1}|^p\right)\right] < \infty\right\},$$

$$c^{+\cdot p}_\alpha(\lambda,\mu) = \left\{X = (x_n)_{n\geq 1} \;/\; \sup_{n\geq 1}\left[\frac{1}{\alpha_n}\sum_{k=n}^{\infty}\left(\frac{1}{|\lambda_k|}|\mu_k x_k - \mu_{k-1}x_{k-1}|^p\right)\right] < \infty\right\},$$

$$\widetilde{c^{+p}_\alpha}(\lambda,\mu) = \left\{X = (x_n)_{n\geq 1} \;/\; \lim_{n\to\infty}\left[\frac{1}{\alpha_n}\sum_{k=n}^{\infty}\left(\frac{1}{|\lambda_k|}|\mu_k x_k - \mu_{k+1}x_{k+1}|^p\right)\right] = 0\right\},$$

with the convention $x_0 = 0$. We shall say that $c^p_\alpha(\lambda,\mu)$ and $\widetilde{c^p_\alpha}(\lambda,\mu)$ are the sets of sequences that are strongly $\alpha-$bounded and $\alpha-$convergent to 0 with index p. If

$\lambda = \mu$, $\alpha = e$ and $p = 1$, then $c_\alpha^p(\lambda, \mu) = c_\infty(\lambda)$ and $\widetilde{c_\alpha^p}(\lambda, \mu) = c_0(\lambda)$ are the sets of sequences that are strongly bounded and strongly convergent to zero.

Now we shall put $\zeta_p = \frac{(\alpha|\lambda|)^{\frac{1}{p}}}{|\mu|} = \left(\frac{(\alpha_n|\lambda_n|)^{\frac{1}{p}}}{|\mu_n|}\right)_{n\geq 1}$, $\zeta_p^- = \left(\frac{(\alpha_{n-1}|\lambda_{n-1}|)^{\frac{1}{p}}}{|\mu_{n-1}|}\right)_{n\geq 1}$ with $\frac{(\alpha_0|\lambda_0|)^{\frac{1}{p}}}{|\mu_0|} = 1$ and $\kappa = \left(\left(\frac{\alpha_{n-1}}{\alpha_n}\left|\frac{\lambda_{n-1}}{\lambda_n}\right|\right)^{\frac{1}{p}}\left|\frac{\mu_n}{\mu_{n-1}}\right|\right)_{n\geq 2}$. From the results of Section 2 we obtain

THEOREM 4. *i) If $\alpha|\lambda| \in \widehat{C_1}$ and $(\alpha|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$, then*

$$c_\alpha^p(\lambda, \mu) = s_{\zeta_p} \quad \text{and} \quad \widetilde{c_\alpha^p}(\lambda, \mu) = s_{\zeta_p}^\circ. \tag{7}$$

ii) Assume that $\alpha|\lambda| \in \widehat{C_1}$.

a) If $\zeta_p = \dfrac{(\alpha|\lambda|)^{\frac{1}{p}}}{\mu} \in \widehat{C_1}$, then $c_\alpha^{\cdot+p}(\lambda, \mu) = s_{\zeta_p^-}$;

b) if $\kappa = 0(1)$, then $\widetilde{c_\alpha^{+\cdot p}}(\lambda, \mu) = s_{\zeta_p}^\circ$.

iii) Assume that $\alpha|\lambda| \in \widehat{C_1}$.

a) $(\alpha|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ implies

$$c_\alpha^{+\cdot p}(\lambda, \mu) = s_{\zeta_p} \quad \text{and} \quad \widetilde{c_\alpha^{+\cdot p}}(\lambda, \mu) = s_{\zeta_p}^\circ.$$

iv) Assume that $\alpha \in \widehat{C_1^+}$.

a) if $\zeta_p \in \widehat{C_1}$ then $c_\alpha^{+p}(\lambda, \mu) = s_{\zeta_p^-}$;

b) if $\kappa = o(1)$ then $\widetilde{c_\alpha^{+p}}(\lambda, \mu) = s_{\zeta_p^-}^\circ$.

Proof. Assertion i). First, we have

$$c_\alpha^p(\lambda, \mu) = \left\{X \,/\, \Delta(\mu) X \in w_\alpha^p(\lambda)\right\};$$

and since $\alpha|\lambda| \in \widehat{C_1}$, we get from iii) d) in Theorem 2, $w_\alpha^p(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$. Thus, using the identities $\Delta(\mu)^{-1} = C(\mu) = D_{\frac{1}{\mu}}\Sigma$ we get $c_\alpha^p(\lambda, \mu) = D_{\frac{1}{\mu}}\Sigma s_{(\alpha|\lambda|)^{\frac{1}{p}}}$; and since $(\alpha|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$, we deduce that Δ is bijective from $s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ into itself, i.e. $\Sigma s_{(\alpha|\lambda|)^{\frac{1}{p}}} = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ and we conclude that $c_\alpha^p(\lambda, \mu) = s_{\zeta_p}$. By a similar reasoning we obtain $\widetilde{c_\alpha^p}(\lambda, \mu) = s_{\zeta_p}^\circ$.

Assertion ii) a). Here we get

$$c_\alpha^{\cdot+p}(\lambda, \mu) = \left\{X \,/\, \Delta^+(\mu) X \in w_\alpha^p(\lambda)\right\};$$

and since $\alpha\,|\lambda| \in \widehat{C_1}$, we have $w_\alpha^p(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$. So

$$c_\alpha^{\cdot +p}(\lambda,\mu) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}\left(\Delta^+(\mu)\right),$$

and from ii) a) in Theorem 2, we get

$$s_{(\alpha|\lambda|)^{\frac{1}{p}}}\left(\Delta^+(\mu)\right) = s_{\zeta_p^-} \text{ if } \zeta_p \in \widehat{C_1}. \tag{8}$$

This gives the conclusion.

Statement ii) b). As above we obtain using ii) b) in Theorem 2

$$\widetilde{c_\alpha^{+p}}(\lambda,\mu) = s^\circ_{(\alpha|\lambda|)^{\frac{1}{p}}}\left(\Delta^+(\mu)\right) = s^\circ_{\zeta_p^-},$$

since $\kappa = o(1)$.

iii) We have

$$c_\alpha^{+\cdot p}(\lambda,\mu) = \left\{X \ / \ \Delta(\mu)\,X \in w_\alpha^{+p}(\lambda)\right\}.$$

If $\alpha \in \widehat{C_1^+}$, then $w_\alpha^{+p}(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ and

$$c_\alpha^{+\cdot p}(\lambda,\mu) = C(\mu)\,s_{(\alpha|\lambda|)^{\frac{1}{p}}} = D_{\frac{1}{\mu}}\Sigma s_{(\alpha|\lambda|)^{\frac{1}{p}}}.$$

From i) a) in Theorem 2, $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ implies $\Sigma s_{(\alpha|\lambda|)^{\frac{1}{p}}} = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ and we conclude that $c_\alpha^{+\cdot p}(\lambda,\mu) = s_{\zeta_p}$. We get $\widetilde{c_\alpha^{+\cdot p}}(\lambda,\mu) = s^\circ_{\zeta_p}$ reasoning as above.

iv) a) Since $w_\alpha^{+p}(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}$ for $\alpha \in \widehat{C_1^+}$, we deduce that

$$c_\alpha^{+p}(\lambda,\mu) = \left\{X \ / \ \Delta^+(\mu)\,X \in w_\alpha^{+p}(\lambda) = s_{(\alpha|\lambda|)^{\frac{1}{p}}}\right\} = s_{(\alpha|\lambda|)^{\frac{1}{p}}}\left(\Delta^+(\mu)\right);$$

and we conclude using (8). b) can be obtained reasoning as in ii) b).

□

REMARK 2. Note that the previous sets are BK spaces and we can write for instance that if $\alpha\,|\lambda| \in \widehat{C_1}$ and $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$, then $c_\alpha^{\cdot +p}(\lambda,\mu)$ is a BK space with respect to the norm $\|\,\|_{s_{\xi_p}}$ and $\widetilde{c_\alpha^{\cdot +p}}(\lambda,\mu)$ has AK.

COROLLARY 3. *Assume that* $\alpha\,|\lambda| \in \Gamma$. *Then*

i) $c_\alpha^p(\lambda,\mu) = c_\alpha^{+\cdot p}(\lambda,\mu) = s_{\zeta_p}$;

ii) $\widetilde{c_\alpha^p}(\lambda,\mu) = s^\circ_{\zeta_p}$ *and* $\widetilde{c_\alpha^{+\cdot p}}(\lambda,\mu) = s^\circ_{\zeta_p}$.

Proof. Since $\Gamma \subset \widehat{C_1}$, it is enough to show that $\alpha\,|\lambda| \in \Gamma$ if and only if $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \Gamma$ and apply i) and ii) in Theorem (4). So put $q = 1/p > 0$, $\xi = \alpha\,|\lambda|$ and show that $\xi \in \Gamma$ if and only if $\xi^q \in \Gamma$. If $\xi \in \Gamma$ there is an integer N such that $\sup_{n\geq N+1}\left(\frac{\xi_{n-1}}{\xi_n}\right) < 1$, then

$$\left(\frac{\xi_{n-1}}{\xi_n}\right)^q \leq \left[\sup_{n\geq N+1}\left(\frac{\xi_{n-1}}{\xi_n}\right)\right]^q < 1 \text{ for all } n \geq N+1,$$

and $\xi^q \in \Gamma$. Conversely, assume that $\xi^q \in \Gamma$, that is $\limsup_{n\to\infty}\left(\frac{\xi_{n-1}}{\xi_n}\right)^q < 1$. By a similar reasoning we get

$$\frac{\xi_{n-1}}{\xi_n} \leq \left[\sup_{n\geq N+1}\left(\frac{\xi_{n-1}}{\xi_n}\right)^q\right]^{\frac{1}{q}} < 1 \text{ for all } n \geq N+1,$$

and $\xi \in \Gamma$. We conclude applying Theorem (4).

□

In order to assert the next corollary, we need the following elementary lemma.

LEMMA 4. *Let $q > 0$ be any real and $\alpha \in U^{+*}$ a nondecreasing sequence. Then*

i) $\alpha \in \widehat{C_1}$ implies $\alpha^q \in \widehat{C_1}$, for $q \geq 1$,

ii) $\alpha^q \in \widehat{C_1}$ implies $\alpha \in \widehat{C_1}$, for $0 < q < 1$.

Proof. Let $q \geq 1$. Since α is nondecreasing we see immediatly that for any given integer $n \geq 1$: $\alpha_k \sum_{k=1}^{n}\left(\alpha_n^{q-1} - \alpha_k^{q-1}\right) = \sum_{k=1}^{n}\left(\alpha_n^{q-1}\alpha_k - \alpha_k^q\right) \geq 0$, and

$$\frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right) \geq \frac{1}{\alpha_n^q}\left(\sum_{k=1}^{n}\alpha_k^q\right). \tag{9}$$

Since $\alpha \in \widehat{C_1}$ implies $\frac{1}{\alpha_n}\left(\sum_{k=1}^{n}\alpha_k\right) = O(1)$, we obtain i) using the inequality (9). Now, writing $\beta = \alpha^q \in \widehat{C_1}$ and applying i), we get $\alpha = \beta^{\frac{1}{q}} \in \widehat{C_1}$ for $0 < q < 1$. This permits us to conclude for ii).

□

COROLLARY 4. *Assume that $\alpha, \lambda \in U^{+*}$ and $\alpha\,|\lambda|$ is a nondecreasing sequence.*

i) If $p > 1$, then $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ implies

$$c_\alpha^p(\lambda,\mu) = s_{\zeta_p} \quad \text{and} \quad \widetilde{c_\alpha^p}(\lambda,\mu) = s_{\zeta_p}^{\circ}; \tag{10}$$

ii) if $0 < p \leq 1$, $\alpha\,|\lambda| \in \widehat{C_1}$ if and only if $c_\alpha^p(\lambda,\mu) = s_{\zeta_p}$.

iii) If $\alpha\,|\lambda| \in \Gamma$, *then (10) holds.*

Proof. i). If $p > 1$, $0 < \frac{1}{p} < 1$ and from Lemma 4, $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ implies $\alpha\,|\lambda| \in \widehat{C_1}$. So we conclude using i) in Theorem 4.

Assertion ii). The necessity comes from i) in Theorem (4). Sufficiency. First, put

$$\widetilde{\alpha} = \left((-1)^n \frac{(\alpha_n\,|\lambda_n|)^{\frac{1}{p}}}{\mu_n} \right)_{n \geq 1}.$$

We have $\widetilde{\alpha} \in c_\alpha^p(\lambda, \mu) = s_{\zeta_p}$ and using the convention $\alpha_0 = 0$, we can write

$$|\Delta(\mu)\,\widetilde{\alpha}| = \left(\left| \mu_n (-1)^n \frac{(\alpha_n\,|\lambda_n|)^{\frac{1}{p}}}{\mu_n} - \mu_{n-1} (-1)^{n-1} \frac{(\alpha_{n-1}\,|\lambda_{n-1}|)^{\frac{1}{p}}}{\mu_{n-1}} \right| \right)_{n \geq 1}.$$

So

$$|\Delta(\mu)\,\widetilde{\alpha}|^p = \left(\left((\alpha_n\,|\lambda_n|)^{\frac{1}{p}} + (\alpha_{n-1}\,|\lambda_{n-1}|)^{\frac{1}{p}} \right)^p \right)_{n \geq 1}.$$

Then the condition $\Sigma\,|\Delta(\mu)\,\widetilde{\alpha}|^p \in s_{\alpha|\lambda|}$ implies that there is a real $M > 0$ such that for every n:

$$\frac{1}{\alpha_n\,|\lambda_n|} \left(\sum_{k=1}^{n} \alpha_k\,|\lambda_k| \right) \leq \frac{1}{\alpha_n\,|\lambda_n|} \left(\sum_{k=1}^{n} \left((\alpha_k\,|\lambda_k|)^{\frac{1}{p}} + (\alpha_{k-1}\,|\lambda_{k-1}|)^{\frac{1}{p}} \right)^p \right) \leq M.$$

We conclude that $\alpha\,|\lambda| \in \widehat{C_1}$. So ii) can be deduced from Theorem (4).

□

In the next result we shall denote by $c_\alpha^p(\lambda)$ the set $c_\alpha^p(\lambda, \lambda)$.

Consider now the following identities.

$$c_\alpha^p(\lambda) = s_{\left(\alpha^{\frac{1}{p}}\,|\lambda|^{\frac{1}{p}-1}\right)} \tag{11}$$

$$\widetilde{c_\alpha^p}(\lambda) = s^{\circ}_{\left(\alpha^{\frac{1}{p}}\,|\lambda|^{\frac{1}{p}-1}\right)} \tag{12}$$

COROLLARY 5. *Assume that* $\alpha\,|\lambda|$ *is nondecreasing.*

i) If $0 < p \leq 1$, *then*

- *a)* $\alpha\,|\lambda| \in \widehat{C_1}$ *if and only if (11) holds.*
- *b)* $\alpha\,|\lambda| \in \widehat{C_1}$ *implies that (12) holds.*

ii) If $p > 1$, *the condition* $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$ *implies that (11) and (12) hold.*

iii) $\alpha\,|\lambda| \in \Gamma$ *implies (11) and (12).*

Proof. i) a) comes from ii) in Corollary 4, where $\lambda = \mu$. The proof of i) b) comes from Lemma 4 and i) in Theorem (4). ii) comes from i) in Corollary 4. iii) comes from Corollary 3.

□

Now we can give an application which can be considered as corollary.

COROLLARY 6. *i)* $c_\infty^p(\lambda) \neq l_\infty$ *in the following cases:*

a) $0 < p < 1$ *and* $|\lambda| \in \widehat{C_1}$*;*

b) $p > 1$ *and* $|\lambda|^{\frac{1}{p}} \in \widehat{C_1}$.

ii) $c_\infty(\lambda) = l_\infty$ *if and only if* $|\lambda| \in \widehat{C_1}$.

iii) Assume that $\alpha \to \infty$.

a) Let $p > 1$. *If* $(\alpha\,|\lambda|)^{\frac{1}{p}} \in \widehat{C_1}$, *then*

$$c_\alpha^p(\lambda) = l_\infty \quad \textit{implies} \quad |\lambda_n| \to \infty \quad \textit{as } n \to \infty.$$

b) If $0 < p < 1$ *and* $\alpha\,|\lambda| \in \widehat{C_1}$, *then*

$$c_\alpha^p(\lambda) = l_\infty \quad \textit{implies} \quad \lambda \in c_0.$$

Proof. Case a). Since $|\lambda| \in \widehat{C_1}$, we have $c_\infty^p(\lambda) = s_{|\lambda|^{\frac{1}{p}-1}}$. So the identity $c_\infty^p(\lambda) = l_\infty$ implies that there are K_1 and $K_2 > 0$ such that

$$K_1 \leq |\lambda_n|^{\frac{1}{p}-1} \leq K_2 \quad \text{for all } n. \tag{13}$$

Since $\dfrac{1}{p} - 1 > 0$ and $|\lambda| \in \widehat{C_1}$, we deduce that $|\lambda_n|^{\frac{1}{p}-1} \to \infty$ as $n \to \infty$, which is contradictory.

Case b). Here we get $|\lambda_n|^{\frac{1}{p}-1} = o(1)$ and (13) cannot be satisfied. ii) comes from the equivalence a)⇔b) in i) of Theorem 3 in which we put $\alpha = e$ and $\lambda = \mu$.

Assertion iii). Condition a) implies $c_\alpha^p(\lambda) = s_{\left(\alpha^{\frac{1}{p}}|\lambda|^{\frac{1}{p}-1}\right)}$. From the identity $c_\alpha^p(\lambda) = l_\infty$, there exist K_1 and $K_2 > 0$ such that

$$K_1 \leq \alpha_n^{\frac{1}{p}}\,|\lambda_n|^{\frac{1}{p}-1} \leq K_2 \quad \text{and} \quad \frac{K_1}{\alpha_n^{\frac{1}{p}}} \leq |\lambda_n|^{\frac{1}{p}-1} \leq \frac{K_2}{\alpha_n^{\frac{1}{p}}} \quad \text{for all } n \geq 1.$$

Since $\frac{1}{p} - 1 < 0$ we conclude that $|\lambda_n| \to \infty$ as $n \to \infty$. b) can be obtained by a similar reasoning.

□

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