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# ON THE DISCRETE WAVELET TRANSFORM OF STOCHASTIC PROCESSES

**Abstract.** We improve a result of Averkamp and Houndré concerning the characterization of second order processes with stationary increments via the discrete wavelet transform. Our result holds for a class of processes with a correlation function which is not twice continuously differentiable. Several examples are studied.

## 1. Introduction

The purpose of this paper is to continue the study of the stationarity of second–order processes via their discrete wavelet transform begun in [2]. We focus our attention on irregular processes, i.e., on a class of processes with a correlation function which is not twice continuously differentiable.

In the literature several papers ([1], [2], [4]) have been devoted to the analysis of stationarity of stochastic processes by using wavelets. The main result of [4] was the characterization of the second-order properties of a process via the corresponding properties of its continuous wavelet transform (CWT). In particular, in the case where the correlation function $\rho_X$ is of polynomial growth and $\psi$ is rapidly decreasing at infinity with exactly one vanishing moment, the authors showed that a process has (weakly) stationary increments if and only if its CWT is (weakly) stationary at all scales. Under stronger requirements on the analyzing wavelet, they also showed that a process has (weakly) stationary increments if and only if its CWT is (weakly) stationary at any fixed scales. These results were extended in [1] to random processes with polynomial growth and without any finite moments.

In the applications it is often convenient to use the discrete wavelet transform (DWT) instead of the CWT. In this direction Averkamp and Houndré ([2]) proved a version of the above results by using the DWT, under the hypothesis that the correlation function $\rho_X$ of the process $\{X_t\}_{t\in\mathbb{R}}$ is of class $\mathcal{C}^2(\mathbb{R}^2)$. This requirement appear to be fairly restrictive; for istance, the correlation function of the Brownian motion (or, more generally, of the fractional Brownian motion) and of the Ornstein–Uhlenbeck process are just continuous functions of $\mathbb{R}^2$. Finally, many authors show that the wavelets are a very interesting method to investigate an important and particular case of nonstationary processes as the fractional Brownian process (see [8], [11]).

In this paper we extend the main result of [2]. Our aim is twofold: on one hand, we give a theoretical basis to the study of the stationarity of irregular processes, i.e., of processes such that their covariance function $E(X_r\overline{X_s}) \notin \mathcal{C}^2(\mathbb{R}^2)$, via the discrete wavelet transform. On the other hand, we shall make use of generic wavelets with vanishing zero order moment; in particular, our results hold for wavelets that are not associated with a Multiresolution Analysis (MRA). Let us give some details.

An orthonormal *wavelet* is a function $\psi \in L^2(\mathbb{R})$ such that the set

$$\{\psi_{j,k}(x) \equiv 2^{j/2}\psi(2^j x - k) : \; j,k \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Two equations (see (2) below) characterize the set of the wavelets: it is well known that such set clearly includes the MRA wavelets properly. Given a generic wavelet $\psi$, the *Discrete Wavelet Transform (DWT)* of $X = \{X_t\}_{t\in\mathbb{R}}$ with respect to $\psi$ is defined to be the discrete random field $W = \{W(j,k)\}_{j,k\in\mathbb{Z}}$, where $W(j,k)$ is defined by

$$W(j,k) = \int_{\mathbb{R}} X_t \, \psi_{j,k}(t) \, \mathrm{d}t, \tag{1}$$

provided the path integral in (1) is defined with probability one.

We study the (weak) stationarity of $X$ via its discrete wavelet transform $W$. In particular we prove, under mild conditions on $\rho_X$, that $X$ has weakly stationary increments if and only if $W$ is weakly stationary at every scale. In our result a crucial role is played by a closed subspace $\mathcal{W}$ of $L^2(\mathbb{R}^2)$ constructed from the particular wavelet $\psi$ chosen to investigate the process. An orthonormal basis of $\mathcal{W}$ is given by the family $\{2^j\psi(2^j x - k_1)\psi(2^j y - k_2) : \; j, k_1, k_2 \in \mathbb{Z}\}$ and the projection of the covariance function on $\mathcal{W}$ characterizes the weak stationarity of the discrete wavelet transform of the process.

## 2. Notation and terminology

It is well known that $\psi$ is a wavelet if and only if

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = 1 && \text{a.e. } \xi \in \mathbb{R}, \\ &\sum_{j=0}^{\infty} \widehat{\psi}(2^j\xi)\,\overline{\widehat{\psi}(2^j(\xi + 2k\pi))} = 0 && \text{a.e. } \xi \in \mathbb{R},\ k \in 2\mathbb{Z}+1, \end{aligned} \tag{2}$$

and $\|\psi\|_2 \geq 1$, where $\widehat{\psi}$ denotes the Fourier transform of $\psi$ defined by

$$\widehat{\psi}(\xi) = \int_{\mathbb{R}} \psi(x) e^{-ix\xi} \, \mathrm{d}x.$$

See [7] for a general treatment on wavelets. We assume that $\psi$ is a generic wavelet, possibly not associated with a multiresolution analysis. An example of such type of wavelet is given by the Journé wavelet $\psi$, whose Fourier transform is given by $\widehat{\psi}(\xi) = \chi_I(|\xi|)$ where $I = [\frac{4\pi}{7}, \pi) \cup [4\pi, \frac{32\pi}{7})$ (see [9]).

A mild condition on the wavelet $\psi$ that we shall use in the sequel is the vanishing of its zero moment, i.e.,

$$\widehat{\psi}(0) = 0. \tag{3}$$

Let $(\Omega, \mathcal{B}, P)$ be a probability space and let $X = \{X_t\}_{t\in\mathbb{R}}$ be a second–order process, i.e. $X$ is jointly measurable and $X_t$ is square integrable for each $t \in \mathbb{R}$. The discrete wavelet transform of $X$, which is defined in (1) above, is a random field on $(\Omega, \mathcal{B}, P)$ that depends on $\psi$. Clearly, by the definition of DWT, we need that

$$\int_{\mathbb{R}} |X_t(\omega)\, \psi_{j,k}(t)|\, \mathrm{d}t < \infty \qquad \text{for a.e. } \omega \in \Omega.$$

Since $X$ has finite second order moments, a generic condition which ensures that (1) is well defined and is also a second-order sequence is that

$$\int_{\mathbb{R}} \sqrt{\rho_X(u,u)}\, \big|\psi\big(2^j u - k\big)\big|\, \mathrm{d}u < \infty, \tag{4}$$

for all $j, k \in \mathbb{Z}$ where $\rho_X(r,s) = E(X_r \overline{X_s})$, $r, s \in \mathbb{R}$. We assume throughout that (4) holds. Some authors often write $W^{\psi}$ to emphasize the fact the wavelet transform is taken with rispect to $\psi$. In the following, we denote by $W$ the DWT of a given process $X$ using a wavelet $\psi$.

An easy computation shows that the second moment function of $W$ is given by

$$\rho_W(j,k;l,m) = E\big(W(j,k)\overline{W(l,m)}\big) = \int_{\mathbb{R}^2} \rho_X(u,v)\psi_{j,k}(u)\overline{\psi_{l,m}(v)}\, \mathrm{d}u\, \mathrm{d}v.$$

A second-order process $\{X_t\}_{t\in\mathbb{R}}$ is said to be *(weakly) stationary* if for all $t, s, u \in \mathbb{R}$

$$E\big(X_{t+u}\overline{X_{s+u}}\big) = E\big(X_t \overline{X_s}\big).$$

We say that the process $\{X_t\}_{t\in\mathbb{R}}$ has *stationary increments* if $E\big(|X_t - X_s|^2\big)$ is finite and depends only on $t - s$. An elementary calculation shows that $\{X_t\}_{t\in\mathbb{R}}$ has *stationary increments* if and only if

$$E\big((X_{t+u} - X_{t'+u})(\overline{X_{s+u} - X_{s'+u}})\big) = E\big((X_t - X_{t'})(\overline{X_s - X_{s'}})\big), \tag{5}$$

for all $t, s, t', s', u \in \mathbb{R}$. Hence we may use (5) as definition of second–order process with stationary increments. Moreover, it is obvious that a weakly stationary process has stationary increments.

For discrete random processes $\{X_j\}_{j\in\mathbb{Z}}$, the definitions of (weakly) stationary random process and of random process with stationary increments are obtained from the corresponding definitions in the continuous case by obvious modifications. See [6] for more general information on stochastic processes.

Let us recall the main result of Averkamp and Houndré concerning the characterization of second order processes with stationary increments via the discrete wavelet transform ([2]).

THEOREM 1. *Let $\{X_t\}_{t\in\mathbb{R}}$ be a second-order process with $\rho_X$ twice continuously differentiable on $\mathbb{R}^2$, and such that its partial derivatives of order at most two have polynomial growth.*
*Let $\psi \in L^1(\mathbb{R})$ be a wavelet such that (3) holds, $\int_{\mathbb{R}} x\psi(x)\,\mathrm{d}x = 1$ and $\psi(x)(1+|x|)^N \in L^1(\mathbb{R})$, for all $N \in \mathbb{N}$.*
*Let also $W(j,k)$ be almost surely defined for all $j,k \in \mathbb{Z}$.*
*Then $\{X_t\}_{t\in\mathbb{R}}$ has weakly stationary increments if and only if, for all $j \in \mathbb{Z}$, $\{W(j,k)\}_{k\in\mathbb{Z}}$ is weakly stationary.*

Now, let us consider the following example.

EXAMPLE 1. The DWT of a fractional Brownian motion.
The *fractional Brownian motion (fBm)* process offers a convenient tool for modelling non stationary stochastic phenomena with long–term dependencies and $1/f$-type spectral behaviour over ranges of frequencies. We say that $\{B_t^H\}_{t\geq 0}$ is a fBm if it is a Gaussian, zero–mean, non stationary process such that

$$B_0^H = 0, \qquad B_{t+h}^H - B_t^H \sim N(0, \sigma_H^2 |h|^{2H}).$$

The parameter $H$, $0 < H < 1$, is called the “Hurst exponent”; if $H = 1/2$, then $\{B_t^H\}_{t\geq 0}$ is the classical Brownian motion. It is well known that

$$\begin{aligned} &E\big(B_t^H \overline{B_s^H}\big) = \frac{\sigma_H^2}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big),\\ &E\big((B_{t+u}^H - B_{t'+u}^H)\overline{(B_{s+u}^H - B_{s'+u}^H)}\big) =\\ &\qquad = \frac{\sigma_H^2}{2}\big(|t-s'|^{2H} + |t'-s|^{2H} - |t-s|^{2H} - |t'-s'|^{2H}\big). \end{aligned}$$

Hence $\{B_t^H\}$ has stationary increments but is not stationary. Note that $E(B_t^H \overline{B_s^H})$ is not of class $\mathcal{C}^2(\mathbb{R}^2)$ and that its degree of smothness decreases as $H$ tends to $0^+$.
Nevertheless, we show that, in this case, $\{W(j,k)\}_{k\in\mathbb{Z}}$ is weakly stationary, for all $j \in \mathbb{Z}$. Let $\psi$ be a wavelet that satisfies (3) and let $\{B_t^H\}_{t\geq 0}$ be a fBm. In order to do this, let us compute the autocorrelation function of the DWT of $\{B_t^H\}$ at every level $j$:

$$\begin{aligned} E\big(W(j,k)\overline{W(j,k')}\big) &= \frac{\sigma_H^2}{2}\int_{\mathbb{R}^2}\big(|u|^{2H} + |v|^{2H} - |u-v|^{2H}\big)\psi_{j,k}(u)\overline{\psi_{j,k'}(v)}\,\mathrm{d}u\,\mathrm{d}v\\ &= -\frac{\sigma_H^2}{2}\int_{\mathbb{R}^2}|u-v|^{2H}\psi_{j,k}(u)\overline{\psi_{j,k'}(v)}\,\mathrm{d}u\,\mathrm{d}v\\ &= -2^{j(3-2H)-1}\,\sigma_H^2\int_{\mathbb{R}}|t|^{2H}\int_{\mathbb{R}}\psi(s)\overline{\psi(s-t+k-k')}\,\mathrm{d}s\,\mathrm{d}t, \end{aligned}$$

for every $k, k' \in \mathbb{Z}$. Therefore, it is clear that

$$E\big(W(j,k)\overline{W(j,k')}\big) = E\big(W(j,k+n)\overline{W(j,k'+n)}\big),$$

for every $n \in \mathbb{N}$, so that $\{W(j,k)\}_{k\in\mathbb{Z}}$ is weakly stationary, for every scale $j$.

We note that the previous calculation (and hence the same result) holds for every wavelet $\psi$ satisfying (3). Moreover, it is obvious that the fBm processes do not satisfy the regularity conditions of the previous theorem. This is our starting point.

## 3. Discrete Wavelet Transform and stationarity

Let $\psi$ be a wavelet. For every $j \in \mathbb{Z}$, let $W_j$ be defined by

$$W_j = \overline{\text{span}\,\{\psi_{j,k} : \; k \in \mathbb{Z}\}}.$$

Set

$$\mathbf{W}_j = W_j \otimes W_j = \overline{\text{span}\,\{f(x,y) = f_1(x) f_2(y) : \; f_1, \; f_2 \in W_j\}}.$$

We know that $\{\psi_{j,k} : \; k \in \mathbb{Z}\}$ and $\{\Psi_{j,k,k'} : \; j, k, k' \in \mathbb{Z}\}$ are orthonormal bases for $W_j$ and $\mathbf{W}_j$ respectively, where $\Psi_{j,k,k'}(x,y) = \psi_{j,k}(x)\psi_{j,k'}(y)$. Moreover, let $\mathcal{W}$ be defined by

$$\mathcal{W} = \overline{\bigoplus_{j \in \mathbb{Z}} \mathbf{W}_j} = \overline{\text{span}\,\{\Psi_{j,k,k'} : \; j, k, k' \in \mathbb{Z}\}} \subseteq L^2(\mathbb{R}^2).$$

We recall that if $\psi$ is a wavelet, i.e., $\{\psi_{j,k} : \; j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, then $\{\Psi_{j,k,k'} : \; j, k, k' \in \mathbb{Z}\}$ in an orthonormal system of $L^2(\mathbb{R}^2)$ but it is not a basis, i.e., the function $\Psi_{0,0,0}$ is not a wavelet. Hence $\mathcal{W}$ is properly included in $L^2(\mathbb{R}^2)$.

In many cases the correlation function $\rho_X \notin L^2(\mathbb{R}^2)$. However, there are some cases in which $\rho_X \in L^2(\mathbb{R}^2)$. Now we make a digression and show how to characterize the weakly stationarity of the DWT of second order process in terms of $\rho_X$ in the case that $\rho_X \in L^2(\mathbb{R}^2)$.

In order to complete the system $\{\Psi_{j,k,k'} : \; j, k, k' \in \mathbb{Z}\}$, let us suppose that $\psi$ is a MRA wavelet, i.e. there exists a Multiresolution Analysis $\{V_j : \; j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ with scaling function $\varphi$ that generates $\psi$. For every $j, k, k' \in \mathbb{Z}$, let $\Phi^1_{j,k,k'}$ and $\Phi^2_{j,k,k'}$ be defined by

$$\Phi^1_{j,k,k'}(x,y) = \varphi_{j,k}(x)\psi_{j,k'}(y) \quad \text{and} \quad \Phi^2_{j,k,k'}(x,y) = \psi_{j,k}(x)\varphi_{j,k'}(y), \quad \forall x, y \in \mathbb{R}.$$

It is known that

$$\left\{\Psi_{j,k,k'}, \; \Phi^1_{j,k,k'}, \; \Phi^2_{j,k,k'} : \; j, k, k' \in \mathbb{Z}\right\}$$

is a basis of $L^2(\mathbb{R}^2)$ (see [5]).

PROPOSITION 1. *Let $\psi$ be a real, MRA wavelet that satisfies (3). Let $\{X_t\}_{t \in \mathbb{R}}$ be a second order process with a correlation function $\rho_X \in L^2(\mathbb{R}^2)$ and assume that (4) holds.*
*Then $\{W(j,k)\}_{k \in \mathbb{Z}}$ is weakly stationary for all $j \in \mathbb{Z}$ if and only if $\rho_X \in \mathcal{W}^{\perp}$.*

*Proof.* Let $\varphi$ be the scaling function of the MRA that generates $\psi$. Since $\rho_X \in L^2(\mathbb{R}^2)$ we may write

$$\rho_X = \sum_{i,l,l' \in \mathbb{Z}} \Big( c^1_{i,l,l'} \Phi^1_{i,l,l'} + c^2_{i,l,l'} \Phi^2_{i,l,l'} + d_{i,l,l'} \Psi_{i,l,l'} \Big),$$

for some constants $c^1_{i,l,l'},\ c^2_{i,l,l'},\ d_{i,l,l'}$. Hence

$$\begin{aligned} E\big(W(j,k)\overline{W(j,k')}\big) = \sum_{i,l,l' \in \mathbb{Z}} \Big( & c^1_{i,l,l'} \int_{\mathbb{R}} \varphi_{i,l}(u)\psi_{j,k}(u)\,\mathrm{d}u \int_{\mathbb{R}} \psi_{i,l'}(v)\overline{\psi_{j,k'}(v)}\,\mathrm{d}v \\ & + c^2_{i,l,l'} \int_{\mathbb{R}} \psi_{i,l}(u)\psi_{j,k}(u)\,\mathrm{d}u \int_{\mathbb{R}} \varphi_{i,l'}(v)\overline{\psi_{j,k'}(v)}\,\mathrm{d}v \\ & + d_{i,l,l'} \int_{\mathbb{R}} \psi_{i,l}(u)\psi_{j,k}(u)\,\mathrm{d}u \int_{\mathbb{R}} \psi_{i,l'}(v)\overline{\psi_{j,k'}(v)}\,\mathrm{d}v \Big) \\ = d_{j,k,k'}, \end{aligned} \tag{6}$$

for all integers $j, k, k'$.

If $\{W(j,k)\}_{k\in\mathbb{Z}}$ is weakly stationary, then for all integers $j, k, k', h$, with $h \neq 0$, we have, by (6),

$$\begin{aligned} 0 &= E\big(W(j,k)\overline{W(j,k')}\big) - E\big(W(j,k+h)\overline{W(j,k'+h)}\big) \\ &= d_{j,k,k'} - d_{j,k+h,k'+h}. \end{aligned} \tag{7}$$

We recall that $\rho_X \in L^2(\mathbb{R}^2)$ implies that $\sum_{j,k,k'} |d_{j,k,k'}|^2 < \infty$. Clearly from (7) we deduce that $d_{j,k,k'} = 0$, whence $\rho_X \in \mathcal{W}^\perp$.

The converse in obvious. If $\rho_X \in \mathcal{W}^\perp$, then $d_{j,k,k'} = 0$. Hence, for every integers $j, k, k'$, we have $E\big(W(j,k)\overline{W(j,k')}\big) = 0$. □

We remark that if $\rho_X \in L^2(\mathbb{R}^2) \cap \mathcal{W}^\perp$ and $\rho_X \neq 0$, then in order that $\{X_t\}_{t\in\mathbb{R}}$ has weakly stationary increments we need more information about $\rho_X$.

We remark that it is not trivial to remove the hypothesis that $\psi$ be real in Proposition 1. Observe that if $\psi$ is complex valued, then $\int_{\mathbb{R}} \psi_{i,l}(u)\psi_{j,k}(u)\,\mathrm{d}u$ it is not necessarily equal to $\delta_{i,j} \cdot \delta_{l,k}$. Observe also that the naive idea of defining $\widetilde{\Psi} = \overline{\psi} \cdot \psi$, $\widetilde{\Phi}^1 = \overline{\varphi} \cdot \psi$ and $\widetilde{\Phi}^2 = \overline{\psi} \cdot \varphi$, does not lead to a basis of $L^2(\mathbb{R}^2)$.

A similar remark applies also to the case where $\rho_X \notin L^2(\mathbb{R}^2)$.

Now we consider the general case where we do not assume that $\rho_X \in L^2(\mathbb{R}^2)$.

LEMMA 1. *Let $\psi$ be a wavelet that satisfies (3) and let $\{X_t\}_{t\in\mathbb{R}}$ be a second order process with a correlation function $\rho_X$ and assume that (4) holds.*
*Then, for a fixed integer $j$, $\{W(j,k)\}_{k\in\mathbb{Z}}$ is weakly stationary if and only if*

$$\begin{aligned} \int_{\mathbb{R}^2} \big(\rho_X(t,s) - \rho_X(t + 2^{-j}h, s + 2^{-j}h)\big) \cdot \\ \cdot \psi_{j,l}(t)\overline{\psi_{j,l'}(s)}\,\mathrm{d}t\,\mathrm{d}s = 0, \qquad \forall h, l, l' \in \mathbb{Z}. \end{aligned} \tag{8}$$

*Proof.* Since, for all $j, k, k', h \in \mathbb{Z}$,

$$E\big(W(j,k)\overline{W(j,k')}\big) - E\big(W(j,k+h)\overline{W(j,k'+h)}\big)$$
$$= \int_{\mathbb{R}^2} \big[\rho_X(t,s) - \rho_X(t+2^{-j}h, s+2^{-j}h)\big]\psi_{j,k}(t)\overline{\psi_{j,k'}(s)}\,\mathrm{d}t\,\mathrm{d}s,$$

the proof of the lemma is obvious. □

We remark that (8) tell us that for every fixed level $j$ the projection of the function $\rho_X(\cdot,\cdot) - \rho_X(\cdot + 2^{-j}h, \cdot + 2^{-j}h)$ on the space $\mathbf{W}_j$ is zero, for every integer $h$.

REMARK 1. Let $\psi$ and $X$ be as in Lemma 1. A sufficient (but not necessary) condition such that $\{W(j,k)\}_{k\in\mathbb{Z}}$ is weakly stationary at every level $j$ is given by

$$\rho_X(\cdot,\cdot) - \rho_X(\cdot + 2^{-j}h, \cdot + 2^{-j}h) \notin \big(\mathcal{W} \setminus \{0\}\big) \quad \forall j, h \in \mathbb{Z}. \tag{9}$$

The proof is an easy exercise. In the following example we apply Lemma 1 and show that (9) is not a necessary condition.

EXAMPLE 2. Let $\psi$ be a real valued wavelet and let

$$X_t = \sum_{n\in\mathbb{Z}} Z_n \psi(t-n),$$

where the $Z_n$ are i.i.d. random variables with mean 0 and variance 1. It is easy to show that the correlation function $\rho_X$ is given by

$$\rho_X(t,s) = \sum_{n\in\mathbb{Z}} \psi(t-n)\psi(s-n).$$

Hence, without other conditions, $\{X_t\}$ has not weakly stationary increments (see [2]). In order to show that the discrete wavelet transform is weakly stationary, let us prove (8). For every non positive integer $j$ and for all integers $h$, $n$ and $l$, we have that

$$\int_{\mathbb{R}} \psi(t-n+2^{-j}h)\,\overline{\psi_{j,l}(t)}\,\mathrm{d}t$$
$$= \frac{2^{-j/2}}{2\pi}\int_{\mathbb{R}} \widehat{\psi}(\xi)\,\overline{\widehat{\psi}(2^{-j}\xi)}e^{-i(n-2^{-j}(h+l))\xi}\,\mathrm{d}\xi$$
$$= \frac{2^{-j/2}}{2\pi}\int_0^{2\pi}\left(\sum_{s\in\mathbb{Z}} \widehat{\psi}(\xi+2s\pi)\,\overline{\widehat{\psi}(2^{-j}(\xi+2s\pi))}\right)e^{-i(n-2^{-j}(h+l))\xi}\,\mathrm{d}\xi$$
$$= \delta_{j,0}\delta_{0,n-h-l}, \tag{10}$$

because $\psi$ is a wavelet (see [7]). A similar computation shows that (10) holds for all positive integers $j$. Hence, by (10) we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \big(\rho_X(t,s) - \rho_X(t+2^{-j}h, s+2^{-j}h)\big)\psi_{j,l}(t)\,\overline{\psi_{j,l'}(s)}\ \mathrm{d}t\ \mathrm{d}s \\
&= \sum_{n\in\mathbb{Z}} \bigg( \int_{\mathbb{R}} \psi(t-n)\psi_{j,l}(t)\ \mathrm{d}t \int_{\mathbb{R}} \psi(s-n)\overline{\psi_{j,l'}(s)}\ \mathrm{d}s + \\
&\quad - \int_{\mathbb{R}} \psi(t+2^{-j}h-n)\psi_{j,l}(t)\ \mathrm{d}t \int_{\mathbb{R}} \psi(s+2^{-j}h-n)\overline{\psi_{j,l'}(s)}\ \mathrm{d}s \bigg) \\
&= \delta_{j,0} \sum_{n\in\mathbb{Z}} \big(\delta_{0,n-l}\,\delta_{0,n-l'} - \delta_{0,n-h-l}\,\delta_{0,n-h-l'}\big) \\
&= 0,
\end{aligned}$$

for every $j, l, l' \in \mathbb{Z}$. Hence condition (8) is satisfied.

Now we show that (9) is not a necessary condition for the weak stationarity of the DWT of $X$ at a fixed level. Let $\psi$ be the Haar wavelet and let $j = h = 1$ in (9). An easy computation gives that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \big(\rho_X(t,s) - \rho_X(t+2^{-1}, s+2^{-1})\big)\psi_{0,l}(t)\overline{\psi_{0,l}(s)}\ \mathrm{d}t\ \mathrm{d}s \\
&= \sum_{n\in\mathbb{Z}} \bigg( \int_{\mathbb{R}} \psi(t-n)\psi_{0,l}(t)\ \mathrm{d}t \int_{\mathbb{R}} \psi(s-n)\overline{\psi_{0,l}(s)}\ \mathrm{d}s + \\
&\quad - \int_{\mathbb{R}} \psi(t+2^{-1}-n)\psi_{0,l}(t)\ \mathrm{d}t \int_{\mathbb{R}} \psi(s+2^{-1}-n)\overline{\psi_{0,l}(s)}\ \mathrm{d}s \bigg) \\
&= \frac{1}{2},
\end{aligned}$$

for every integer $l$. Hence the projection of the function $\rho_X(\cdot,\cdot) - \rho_X(\cdot+2^{-1}, \cdot+2^{-1})$ on $\mathbf{W}_0$ is not zero. This implies that $\rho_X(\cdot,\cdot) - \rho_X(\cdot+2^{-1}, \cdot+2^{-1}) \notin \big(\mathcal{W} \setminus \{0\}\big)$ is not true.

EXAMPLE 3. The Ornstein–Uhlenbeck process.

Let $\{B_t\}_{t\in\mathbb{R}}$ be a Brownian motion process and let us define the process $\{Z_t\}_{t\in\mathbb{R}}$ as

$$Z_t = e^{-t} B_{e^{2t}},$$

for every real $t$; $\{Z_t\}_{t\in\mathbb{R}}$ is called the *Ornstein–Uhlenbeck process* and it is an important stochastic process in many areas, including statistical mechanics and mathematical finance (see for example [3]).

By definition, $Z_t$ has normal $(0,1)$ distribution, for each $t$. Moreover, $\{Z_t\}$ is stationary, and its correlation function is given by $\rho_Z(t,s) = e^{|t-s|}$, for every $t, s \in \mathbb{R}$. Clearly $\rho_Z(\cdot,\cdot) - \rho_Z(\cdot+2^{-j}h, \cdot+2^{-j}h) = 0$ for every $j, h \in \mathbb{Z}$; hence the discrete wavelet transform of $\{Z_t\}$ is weakly stationary, at every level $j$.

We remark that in Examples 1–3 the correlation function of the process is not twice continuously differentiable. The same is true for the *continuous time fractional Gaussian noise* (see [10]).

In order to give a charaterization and for the sake of completeness, we give the following well-known result.

LEMMA 2. *Let $\psi$ be a wavelet that satisfies (3). Let $\{X_t\}_{t\in\mathbb{R}}$ be a second order process with a correlation function $\rho_X$ and assume that (4) holds.*
*If $\{X_t\}_{t\in\mathbb{R}}$ has weakly stationary increments, then $\{W(j,k)\}_{k\in\mathbb{Z}}$ is weakly stationary for all $j \in \mathbb{Z}$.*

*Proof.* Since, by (3), we have

$$\begin{aligned}
&E\big(W(j,k)\overline{W(j,k')}\big)\\
&= \int_{\mathbb{R}^2} \rho_X(t,s)\psi_{j,k}(t)\,\overline{\psi_{j,k'}(s)}\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_{\mathbb{R}^2} \big(\rho_X(t,s) - \rho_X(t',s) - \rho_X(t,s') + \rho_X(t',s')\big)\psi_{j,k}(t)\,\overline{\psi_{j,k'}(s)}\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}$$

the thesis is obvious. □

Let $\mathcal{L}$ be the space of the functions $f : \mathbb{R}^2 \to \mathbb{R}$, such that there exist two functions $h_1$ and $h_2$, with $h_1, h_2 : \mathbb{R} \to \mathbb{R}$, such that

$$f(x,y) = h_1(x) + h_2(y), \qquad \forall (x,y) \in \mathbb{R}^2.$$

Note that $\mathcal{W} \cap \mathcal{L} = \{0\}$. The following result is a consequence of the previous lemmas.

THEOREM 2. *Let $\psi$ be a wavelet that satisfies (3). Let $\{X_t\}_{t\in\mathbb{R}}$ be a second order process with a correlation function $\rho_X$ and assume that (4) holds. Suppose that there exist two functions $\rho_{\mathcal{W}}$ and $\rho_{\mathcal{L}}$ that satisfy the following properties:*

(i) $\rho_X = \rho_{\mathcal{W}} + \rho_{\mathcal{L}}$;

(ii) *for every $u \in \mathbb{R}$, $\rho_{\mathcal{L}}(\cdot,\cdot) - \rho_{\mathcal{L}}(\cdot + u, \cdot + u) \in \mathcal{L}$*

(iii) $\rho_{\mathcal{W}}(t+u,s+u) - \rho_{\mathcal{W}}(t+u,s'+u) - \rho_{\mathcal{W}}(t'+u,s+u) + \rho_{\mathcal{W}}(t'+u,s'+u) = \rho_{\mathcal{W}}(t,s) - \rho_{\mathcal{W}}(t',s) - \rho_{\mathcal{W}}(t,s') + \rho_{\mathcal{W}}(t',s')$, $\forall t,s,t',s',u \in \mathbb{R}$.

*The following properties are equivalent:*

(j) $\{X_t\}_{t\in\mathbb{R}}$ *has weakly stationary increments;*

(jj) *for all $j \in \mathbb{Z}$, $\{W(j,k)\}_{k\in\mathbb{Z}}$ is weakly stationary;*

(jjj) $\displaystyle\int_{\mathbb{R}^2} \big(\rho_{\mathcal{W}}(t,s) - \rho_{\mathcal{W}}(t + 2^{-j}h, s + 2^{-j}h)\big)\psi_{j,l}(t)\overline{\psi_{j,l'}(s)}\,\mathrm{d}t\,\mathrm{d}s = 0$, $\forall j,h,l,l' \in \mathbb{Z}$.

*Proof.* As in Lemma 2 (jj) $\Rightarrow$ (j). In order to prove the converse implication, let $\{W(j,k)\}_{k\in\mathbb{Z}}$ be weakly stationary, for every $j \in \mathbb{Z}$. By (ii), for every $u \in \mathbb{R}$ there exist two function $h_1^u$ and $h_2^u$ such that $\rho_{\mathcal{L}}(t+u,s+u) = \rho_{\mathcal{L}}(t,s)+h_1^u(t)+h_2^u(s),\ \forall(t,s) \in \mathbb{R}^2$. Hence, (i)–(iii) give us that

$$\begin{aligned}
&\big((X_{t+u} - X_{t'+u})(\overline{X_{s+u} - X_{s'+u}})\big) \\
&\quad = \rho_{\mathcal{W}}(t+u,s+u) - \rho_{\mathcal{W}}(t+u,s'+u) - \rho_{\mathcal{W}}(t'+u,s+u) + \\
&\qquad +\rho_{\mathcal{W}}(t'+u,s'+u) + \rho_{\mathcal{L}}(t+u,s+u) - \rho_{\mathcal{L}}(t+u,s'+u) - \\
&\qquad +\rho_{\mathcal{L}}(t'+u,s+u) + \rho_{\mathcal{L}}(t'+u,s'+u) \\
&\quad = \rho_{\mathcal{W}}(t,s) - \rho_{\mathcal{W}}(t,s') - \rho_{\mathcal{W}}(t',s) + \rho_{\mathcal{W}}(t',s') + \rho_{\mathcal{L}}(t,s) + \\
&\qquad +h_1^u(t) + h_2^u(s) - \rho_{\mathcal{L}}(t,s') - h_1^u(t) - h_2^u(s') - \rho_{\mathcal{L}}(t',s) \\
&\qquad -h_1^u(t') + h_2^u(s) + \rho_{\mathcal{L}}(t',s') + h_1^u(t') + h_2^u(s') + \\
&\quad = E\big((X_t - X_{t'})(\overline{X_s - X_{s'}})\big).
\end{aligned}$$

Therefore (j) $\Rightarrow$ (jj).
Since, by (3) and (i)–(ii), for every $j,k,k',h \in \mathbb{Z}$ we have that

$$\begin{aligned}
&E\big(W(j,k)\overline{W(j,k')}\big) - E\big(W(j,k+h)\overline{W(j,k'+h)}\big) \\
&\quad = \int_{\mathbb{R}^2}\big[h_1^{2^{-j}h}(t) + h_2^{2^{-j}h}(s) + \rho_{\mathcal{W}}(t,s) - \rho_{\mathcal{W}}(t+2^{-j}h, s+2^{-j}h)\big] \\
&\qquad\quad \psi_{j,k}(t)\,\overline{\psi_{j,k'}(s)}\ \mathrm{d}t\ \mathrm{d}s \\
&\quad = \int_{\mathbb{R}^2}\big[\rho_{\mathcal{W}}(t,s) - \rho_{\mathcal{W}}(t+2^{-j}h, s+2^{-j}h)\big]\psi_{j,k}(t)\,\overline{\psi_{j,k'}(s)}\ \mathrm{d}t\ \mathrm{d}s
\end{aligned}$$

the implication (jj) $\Leftrightarrow$ (jjj) is obvious. □

We remark that the decomposition (i) of the correlation function depends on the choice of the wavelet $\psi$. Moreover, we remark that if $\rho_X \in \mathcal{C}^2(\mathbb{R}^2)$, then conditions (i) and (iii) of the Theorem 2 imply that

$$\frac{\partial^2}{\partial x \partial y}\big(\rho_X(x,y) - \rho_X(x+s,y+s)\big) = 0 \qquad \forall x,y,s \in \mathbb{R}.$$

This relation plays a crucial role in the proof of the characterization in Theorem 1 and follows from the weak stationarity of $\{W(j,k)\}_{k\in\mathbb{Z}}$.

EXAMPLE 4. The DWT of a fBm: $2^{nd}$ part.
Let us go back to the case of the discrete wavelet transform of a fBm studied in Example 1. Let $\psi$ be the Haar wavelet and let $\{B_t^H\}_{t\geq 0}$ be a fBm. It is clear that in this case the decomposition (i) of Theorem 2 of the function $\rho_{B^H}$ is given by

$$\rho_{\mathcal{L}}(t,s) = \frac{\sigma_H^2}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big) \qquad \text{and} \qquad \rho_{\mathcal{W}} = 0.$$

Since,

$$\rho_{\mathcal{L}}(t,s) - \rho_{\mathcal{L}}(t+u,s+u) = \frac{\sigma_H^2}{2}\big(|t|^{2H} - |t+u|^{2H}\big) + \frac{\sigma_H^2}{2}\big(|s|^{2H} - |s+u|^{2H}\big), \quad \forall u \in \mathbb{R}$$

condition (ii) and (iii) are clearly satisfied. Hence, for all $j \in \mathbb{Z}$, $\{W(j,k)\}_{k\in\mathbb{Z}}$ is weakly stationary.

Stronger conditions on $\psi$ give us further results.

REMARK 2. Let $r$ be positive integer and let $\psi_r$ be a wavelet with $r+1$ vanishing moments, i.e. such that

$$\int_{\mathbb{R}} x^s \psi_r(x)\,\mathrm{d}x = 0, \qquad \forall s \in [0,r] \cap \mathbb{N}.$$

Let $\mathcal{L}_r$ be defined as

$$\mathcal{L}_r = \Big\{ f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = \sum_{i=0}^{r} y^i h_{1,i}(x) + x^i h_{2,i}(y),\ \forall (x,y) \in \mathbb{R}^2 \Big\}.$$

If in the statement of Theorem 2 we replace $\psi$ with $\psi_r$ and $\mathcal{L}$ with $\mathcal{L}_r$, then the new theorem is true.

We note that in the definition of the space $\mathcal{L}_r$ on the function we need no assumptions on the functions $h_{d,i}$, with $d = 1,2,\ 1 \le i \le r,\ h_{d,i} : \mathbb{R} \to \mathbb{R}$. Hence, they can be very irregular. We leave the easy proof to the interested reader.

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