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LÊ'S CONJECTURE FOR CYCLIC COVERS

by

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Abstract. — We describe the link of the cyclic cover over a singularity of complex surface (S,p) totally branched over the zero locus of a germ of analytic function $(S,p) \to (\mathbf{C},0)$. As an application, we prove Lê's conjecture for this family of singularities *i.e.* that if the link is homeomorphic to the 3-sphere then the singularity is an equisingular family of unibranch curves.

***Résumé* (Conjecture de Lê pour les revêtements cycliques).** — Nous décrivons le « link » du revêtement cyclique sur une singularité de surface complexe (S,p) totalement ramifiée sur le lieu des zéros d'un germe de fonction analytique $(S,p) \to (\mathbf{C},0)$. A titre d'application, nous prouvons la conjecture de Lê pour cette famille de singularités, *i.e.* si le « link » est homéomorphe à la sphère de dimension 3, alors la singularité est une famille équisingulière de courbes unibranches.

1. Introduction

The topology of singularities of complex surfaces has been studied thoroughly in the case of isolated singularities (link, Milnor fibration, monodromy, etc.). For non isolated singularities the situation is less known and more mysterious.

By this work, we start a serie of papers devoted to the study of the link of a non isolated singularity (S,p) and its relations with the geometry of (S,p) through the resolution and with the analytic properties of (S,p).

If (S,p) is a singularity of surface, one denotes by $\mathcal{L}(S,p)$ its link. One of the first questions is to give a topological characterization of a non singular germ. When the singularity (S,p) is isolated, Mumford's theorem gives such a characterization in term of $\mathcal{L}(S,p)$, namely (S,p) is not singular if and only if the link $\mathcal{L}(S,p)$ is homeomorphic to the 3-sphere. If (S,p) is not isolated, this is not true. For instance if $(S,p) \subset (\mathbf{C}^3,0)$ is given by the equation $z^2 - x^3 = 0$, or more generally if (S,p) is an equisingular family of unibranch curves, then $\mathcal{L}(S,p)$ is also homeomorphic to $\mathbf{S}^3$.

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It has been conjectured by Lê D.T. (see for instance [**19**]) that the equisingular families of unibranch curves are the only cases in which $\mathcal{L}(S,p)$ is homeomorphic to $\mathbf{S}^3$. In this paper, we prove Lê's conjecture for the singularities obtained as the cyclic cover over a singularity of complex surface (S,p) totally branched over a curve (Theorem 5.1). The proof is based on the explicit description of the link of such a singularity by means of a plumbing graph which is the aim of Sections 2 to 4.

In Section 2, we study the topological action of the normalization morphism on the links of the singularity. Namely, if (S,p) is a singularity of surface, then the normalization morphism $n : \overline{S} \to S$ restricts to the links, providing a map $n_| : \mathcal{L}(\overline{S}) \to \mathcal{L}(S)$ which is an homeomorphism over the complementary of the singular locus L_{Σ_S} of S, and which is a cyclic cover over each connected component of L_{Σ_S}.

In Section 3, we present some definitions and results about Waldhausen multilinks and their fibrations over the circle which will be applied in the next sections to the Milnor fibrations of some germs of analytic functions $(S,p) \to (\mathbf{C},0)$ defined on a surface singularity (S,p).

In Section 4, we describe the link of any singularity of complex surface obtained as the cyclic cover over some germ of surface (S,p) totally ramified over a germ of curve. These singularities include for instance the germs of hypersurfaces in $(\mathbf{C}^3,0)$ with equations $f(x,y) - z^k = 0$ or $f(x,y) - z^k g(x,y) = 0$. Our method generalizes that developed in [**16**] for the singularities $f(x,y) - z^k = 0$ when f is reduced, using the theory of fibred Waldhausen multilinks developped in Section 3. Similar results have been obtained independently by A. Némethi and A. Szilárd ([**14**]) when (S,p) is normal by performing direct calculus on plumbing graphs.

The method is sumarized in algorithms 4.5 and 4.7. We give several examples to illustrate it, specially of singularities whose links are topological 3-manifolds. We also show through some examples how that the computations presented in these algorithms enable one to describe the link of any singularity $(S,p) \subset (\mathbf{C}^3,0)$ given by an equation $f_d(x,y,z)+f_{d+k}(x,y,z) = 0$ where f_d and f_{d+k} denote two homogeneous polynomials in $\mathbf{C}[X,Y,Z]$ with degrees d and $d+k$. As an application, we prove that the singularity with equation $(y^2-x^2)^2 + y^4x = 0$ gives a negative answer to a question of McEwans and Némethi ([**12**])

In section 5, we prove Lê's conjecture for the singularities $\mathcal{C}(F,k)$ obtained by taking the cyclic cover $\rho : \mathcal{C}(F,k) \to (S,p)$ of a normal surface (S,p) totally branched over the zero locus of a germ of analytic function $F : (S,p) \to (\mathbf{C},0)$ (Theorem 5.1). The link $\mathcal{L}(\mathcal{C}(F,k))$ of $\mathcal{C}(F,k)$ can be defined as the inverse image of $\mathcal{L}(S,p)$ by ρ. Let $L_F \subset \mathcal{L}(S,p)$ be the link of the curve $F^{-1}(0)$. The main argument of the proof of 5.1 is the following surprising fact (Proposition 5.3): when L_F is connected, the minimal Waldhausen decomposition of $\mathcal{L}(\mathcal{C}(F,k))$ such that the link $\rho^{-1}(L_F)$ is a Seifert fibres is also the minimal Waldhausen decomposition of $\mathcal{L}(\mathcal{C}(F,k))$.

2. Topological action of the normalization

Let (S, p) be a reduced germ of complex surface; in particular, the singularity at p is allowed to be non-isolated. One denotes by Σ_S the singular locus of S. Let us identify (S,p) with its image by an embedding $(S,p) \to (\mathbf{C}^N, 0)$. The *link* $\mathcal{L}(S,p)$ of (S,p) (resp. $L(\Sigma_S, p)$ of (Σ_S, p)) is the intersection in $\mathbf{C}^N$ between S (resp. Σ_S) and a sufficiently small sphere $\mathbf{S}^{2N-1}_\varepsilon$ of radius ε centered at the origin of $\mathbf{C}^N$.

According to the cone structure theorem ([**13**]), the homeomorphism class of the pair $(\mathcal{L}(S,p), L(\Sigma_S,p))$ does not depend on N, nor on the embedding of (S,p) in $(\mathbf{C}^n, 0)$, nor on ε when ε is sufficiently small.

If the singularity (S,p) is isolated, then $L(\Sigma_S,p)$ is empty. Otherwise $L(\Sigma_S,p)$ is a 1-dimensional manifold diffeomorphic to a finite disjoint union of circles. $\mathcal{L}(S,p) \setminus L(\Sigma_S,p)$ is a differentiable 3-manifold and the topological singular locus of $\mathcal{L}(S,p)$ is included in $L(\Sigma_S,p)$. Note that $\mathcal{L}(S,p)$ may be a topological manifold even if the singularity (S,p) is not isolated. For example, the link of $(\{(x,y,z) \in \mathbf{C}^3 \mid x^2+y^3=0\}, 0)$ is homeomorphic to the sphere $\mathbf{S}^3$ whereas the singular locus is the z-axis.

In order to lighten the notations when dealing with some germ of analytic space (X,p), we often remove p from the notations when no confusion on the point p is possible, writing for example S, Σ_S, $\mathcal{L}(S)$ and $L(\Sigma_S)$ instead of (S,p), (Σ_S,p), $\mathcal{L}(S,p)$ and $L(\Sigma_S,p)$. Furthermore, we also denote by (X,p) or simply X a sufficiently small neighbourhood of p in X.

Let $(S_1,p),\ldots,(S_r,p)$ be the irreducible components of (S,p). For each $i = 1,\ldots,r$, let $n_i : (\overline{S}_i, p_i) \to (S_i,p)$ be the normalisation of (S_i,p), *i.e.* the morphism, unique up to composition with an analytic isomorphism, such that n_i is proper with finite fibres, the germ $(\overline{S}_i,p_i)$ is normal, $\overline{S}_i \setminus n_i^{-1}(\Sigma_{S_i})$ is dense in $\overline{S}_i$, and the restriction of n_i to $\overline{S}_i \setminus n_i^{-1}(\Sigma_{S_i})$ is biholomorphic. The *normalisation* of (S,p) is the map $n : \coprod_{i=1}^r (\overline{S}_i, p_i) \to (S,p)$ defined by: $\forall\, i = 1,\ldots,r$, $n_{|\overline{S}_i} = n_i$.

We call a *circle* an oriented topological space diffeomorphic to $\mathbf{S}^1 = \{z \in \mathbf{C} \mid |z| = 1\}$.

Definition. — Let T be a topological space, let $C \subset T$ be a circle and let $n \geqslant 1$ be an integer. Let us choose an orientation-preserving diffeomorphic $\gamma : C \to \mathbf{S}^1$. One defines an equivalence relation $\sim$ on T by setting:

$$(x \sim y) \Longleftrightarrow \big((x = y) \text{ or } (x \in C, y \in C, \exists\, k \in \mathbf{Z} \text{ such that } \gamma(x) = e^{2ik\pi/n}\gamma(y)\,)\big)$$

One calls *n-curling on C* the projection $T \to T/{\sim}$.

Note that the homeomorphism class of the quotient space $T/{\sim}$ does not depend on the choice of γ. One denotes by $C/(n)$ the subspace $C/{\sim}$ of $T/{\sim}$.

Definition. — Let T be a topological space and let C and C' be two disjoint circles in T. Let us choose an orientation-preserving diffeomorphism $\delta : C \to C'$ and let us consider the equivalence relation $\sim'$ defined on T by:

$$(x \sim' y) \iff \big((x = y) \text{ or } (x \in C,\, y \in C',\, \delta(x) = y\big)$$

One calls *identification of the two circles C and C'* the projection $T \to T/{\sim'}$.

Note that the homeomorphism class of $T/{\sim'}$ does not depend on the choice of δ. When s is an integer $\geqslant 3$, the identification of s circles in T is defined from this by induction.

Let (S, p) be a singularity of complex surface and let $n : \coprod_{i=1}^{r}(\overline{S}_i, p_i) \to (S, p)$ be its normalisation. According to the theory of semialgebraic or subanalytic neighbourhoods (see [**3**] and [**7**]), there exists a subanalytic rug function $\phi : S \to \mathbf{R}$ for $\{p\}$ in S such that for $\varepsilon > 0$ sufficiently small, $\mathcal{L}(S, p) = \phi^{-1}(\varepsilon)$. As n is analytic, $\phi \circ n$ is a subanalytic rug function for $\coprod_{i=1}^{r}\{p_i\}$ in $\coprod_{i=1}^{r}(S_i, p_i)$. Therefore, if $\varepsilon > 0$ is sufficiently small, then $(\phi \circ n)^{-1}(\varepsilon)$ can be taken as the link of $\coprod_{i=1}^{r}(S_i, p_i)$. In particular, we have that $n^{-1}(\mathcal{L}(S, p)) = \coprod_{i=1}^{r} \mathcal{L}(S_i, p_i)$

Proposition 2.1

(1) *n is an homeomorphism over the complementary of a tubular neighbouhood N of $L(\Sigma_S)$ in $\mathcal{L}(S, p)$.*

(2) *Let $\Sigma_S = \cup_{k=1}^{s}\Gamma_k$, with Γ_k irreducible, and for each k, let $n^{-1}(\Gamma_k) = \cup_{j=1}^{l_k}\Delta_j^k$ with Δ_j^k irreducible. Let a_j^k be the degree of n on Δ_j^k. Then the restriction of n to N is the composition of the a_j^k-curlings on the circles $L(\Delta_j^k)$ for $k = 1, \ldots, s$ and $j = 1, \ldots, l_k$ and of the identifications of the l_k circles $L(\Delta_j^k)/(a_j^k)$ for $k = 1, \ldots, s$.*

Proof. — This follows from the fact that, topologically, the normalisation just separates the branches of the surface at each of its points. □

Remark. — $\mathcal{L}(S, p)$ is a topological manifold if and only if for each irreducible component Γ_k of Σ_S, $l_k = 1$ and $a_1^k = 1$.

Let (S, p) be a normal singularity of complex surface, and let $\pi : Z \to S$ be a resolution of (S, p) whose exceptional divisor $\pi^{-1}(p)$ has normal crossings. The dual graph G_π of the exceptional divisor $\pi^{-1}(p)$ with vertices weighted by the self-intersections and the genus of the irreducible components of $\pi^{-1}(p)$ completely determines the homeomorphism class of $\mathcal{L}(S, p)$; namely, $\mathcal{L}(S, p)$ is homeomorphic to the boundary of the 4-dimensional manifold obtained from G_π by a plumbing process, as described in [**15**].

Let $C \subset S$ be a germ of curve on $(\overline{S}, p)$. One calls *embedded resolution* of C any resolution $\pi : Z \to S$ of (S, p) such that the total transform of C by π has normal crossings. Such a π is obtained by composing any resolution of (S, p) with a suitable finite sequence of blowing-up of points. A *resolution graph* of C is a resolution graph G_π of such a π to which one adds a stalk (see figure 1) for each component of the strict transform of C by π at the corresponding vertex. (usually one uses arrows instead of stalks, but arrows will be used later to represent the components of a multilink associated with a germ of function).

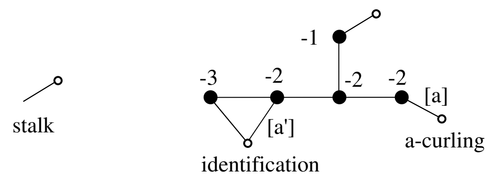

FIGURE 1

Let now (S,p) be an arbitrary singularity of complex surface and let $n : \coprod_{i=1}^{r}(\overline{S}_i,p_i) \to (S,p)$ be its normalization. For each $i \in \{1,\ldots,r\}$, let us choose an embedded resolution π_i of the germ of curve $(n^{-1}(\Sigma_S),p_i) \subset (\overline{S}_i,p_i)$ and let G_{π_i} be the corresponding resolution graph of $(n^{-1}(\Sigma_S),p_i)$. Then, according to Proposition 2.1, the homeomorphism class of the link $\mathcal{L}(S,p)$ is encoded in the *generalised plumbing graph* of $\mathcal{L}(S,p)$ obtained from the disjoint union of the graphs G_{π_i} by performing the following operation for each irreducible component Γ_k of Σ_S. Using again the notations of Proposition 2.1, if $a_j^k \neq 1$, the stalk corresponding to Δ_j^k is weighted by $[a_j^k]$ in order to symbolize the quotient circle $L(\Delta_j^k)/(a_j^k)$. Then the extremities of these l_k stalks are joined in a single extremity which symbolizes the identification of the l_k circles links $L(\Delta_j^k)$. if $l_k = 1$ and $a_1 = 1$, one simply remove from $\coprod_{i=1}^{r} G_i$ the stalk representing $L(\Delta_1^i)$.

Example. — Let $(S,0)$ be the germ of hypersurface at the origin of $\mathbf{C}^3$ with equation $f(x,y) + zg(x,y) = 0$, where $f : (\mathbf{C}^2,0) \to (\mathbf{C},0)$ and $g : (\mathbf{C}^2,0) \to (\mathbf{C},0)$ are two analytic germs which have no irreducible components in common. Let $f : \mathcal{U} \to \mathbf{C}$ and $g : \mathcal{U} \to \mathbf{C}$ be some representatives of the germs f and g. We will describe a generalized resolution graph of the link $\mathcal{L}(S,0)$ from a a resolution of the meromorphic function $h = (f:g) : \mathcal{U} \to \mathbf{P}^1$, *i.e.* a finite sequence $\rho : \widehat{\mathcal{U}} \to \mathcal{U}$ of blowing-up of point such that the map $\widehat{h} : \widehat{\mathcal{U}} \to \mathbf{P}^1$ given by $\widehat{h} = h \circ \rho$ is well defined (see for instance [**9**]).

Let Z_0 (resp. Z_∞) be the union of the irreducible components of the exceptional divisor $\rho^{-1}(0)$ such that $\widehat{h}(Z_0) = (0:1)$ (resp. $\widehat{h}(Z_\infty) = (1:0)$). A component E of $\rho^{-1}(0)$ is *dicritical* if the restriction of $\widehat{h}$ to E is not constant. One denotes by D the union of the dicritical components.

If necessary, one composes ρ with a finite sequence of blowings-up in such a way that the new morphism, again denoted by ρ, verifies that the strict transform of $f^{-1}(0)$ by ρ does not intersect D.

Let $Z_1,\ldots,Z_m$ be the connected components of Z_0. For each $i = 1,\ldots,m$, one denotes by $\widehat{U}_i$ a small regular neighbourhood of Z_i in $\widehat{U}$. As the intersection form restricted to Z_i is negative definite, one obtains a germ of normal surface $(\overline{S}_i,p_i)$ by contracting Z_i to a point p_i ([**5**]). Then the projection $c_i : \widehat{\mathcal{U}}_i \to \overline{S}_i$ is a resolution of $(\overline{S}_i,p_i)$.

Proposition 2.2

(1) *There exists a morphism* $n : \coprod_{i=1}^m (\overline{S_i}, p_i) \to (S, 0)$ *which is the normalisation of* $(S, 0)$.

(2) *Let* L *be the* $z-axis$ *in* $\mathbf{C}^3$. L *contains the singular locus of* $(S, 0)$, *and for each* $i = 1, \ldots, m$, $c_i : \widehat{\mathcal{U}}_i \to \overline{S}_i$ *is an embedded resolution of the germ of curve* (L, p_i), *and the strict transform by* $n \circ c_i$ *of* $(L, 0)$ *is* $D \cap \widehat{U}_i$. *Moreover, for each point* $p \in Z_i \cap D$, *the degree* a_p *of* n *on the germ* (D, p) *equals the multiplicity of* $\widehat{h}$ *along the irreducible component* E *of* Z_i *which intersects* D *at* p, *i.e.* $a_p = m_E(f \circ \rho) - m_E(g \circ \rho)$.

Proof

(1) Let $P : S \to \mathcal{U}$ be the projection $P(x, y, z) = (x, y)$. According to ([**9**], 4.4 and 4.5), there exists a morphism $n : \coprod_{i=1}^m (\overline{S_i}, p_i) \to (S, 0)$ such that $\forall\, i$, $\rho_{|\widehat{U}_i} = P \circ n \circ c_i$ and n is the normalisation of (S, p).

(2) The total transform of L by $n \circ c_i$ is $\rho^{-1} \cap \widehat{U}_i$ which has normal crossings, and its strict transform is $D \cap \widehat{U}_i$. Let $p \in Z_i \cap D$ and let (u, v) be local coordinates at $(\widehat{U}_i, p)$ such that $u = 0$ (resp. $v = 0$) is an equation of E (resp. D). Then, $\widehat{h}$ is locally given by $\widehat{h}(u, v) = (u^{m_1} w_1 : u^{m_2} w_2) = (u^{m_1 - m_2} w : 1)$ where $m_1 = m_E(f \circ \rho)$, $m_2 = m_E(g \circ \rho)$ and w_1, w_2 and w are a unities. This implies that locally, $(n \circ c_i)(u, 0) = (0, 0, u^{m_1 - m_2} w)$. Therefore, $a_p = \deg(n_{|c_i(D)}) = m_1 - m_2$. $\square$

Propositions 2.1 and 2.2 enable one to explicitly compute a generalized plumbing graph of the link $\mathcal{L}(S, 0)$ from any resolution graph of the meromorphic function $h = (f : g)$ weighted by the multiplicities of h along the irreducible components of the exceptional divisor $\rho^{-1}(0)$. In particular, the dual graph of the divisor Z_0 is a plumbing graph of the link $\coprod_{i=1}^r \mathcal{L}(\overline{S}_i, p_i)$.

For each example below, the figure represents the exceptional divisor of a resolution of the meromorphic function $(f : g)$ and a generalized plumbing graph of the link $\mathcal{L}(S, 0)$. The numbers between parenthesis are the multiplicities of $\widehat{h}$ along the irreducible components of the exceptional divisor, and the numbers without parenthesis are their self-intersections in $\widehat{U}$.

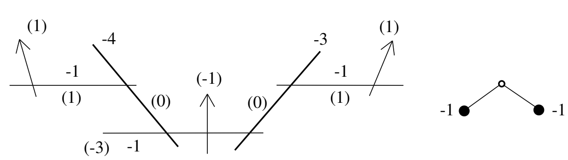

FIGURE 2. $f(x, y) = xy$; $g(x, y) = x^2 + y^3$.

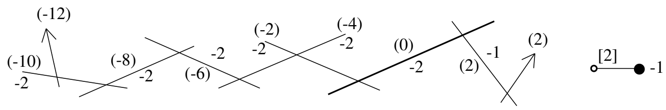

FIGURE 3. $f(x,y) = x^2$; $g(x,y) = y^{12}$.

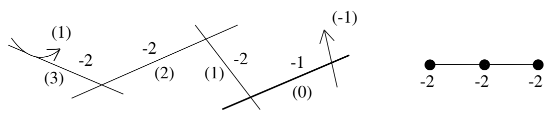

FIGURE 4. $f(x,y) = x^4 + y^5$; $g(x,y) = y$. Note that in this case, $(S,0)$ is normal.

3. Waldhausen multilinks and horizontal fibrations

In this section, we present classical definitions and some results on Waldhausen multilinks which fibre over the circle. This section does not contain any proof as it is an easy generalization to multilinks of definitions and results already presented in [**16**] and [**17**], which concern Waldhausen links (called "marked Waldhausen manifolds" in [**16**]). This will be applied in the next sections to the Milnor fibration $F/|F| : \mathcal{L}(S,p) \setminus L_F \to \mathbf{S}^1$, of a germ $F : (S,p) \to \mathbf{C}, 0$ of analytic function defined on a normal singularity of surface (S,p).

A *Waldhausen manifold* is a compact oriented 3-manifold M such that there exists a finite family $\mathcal{T}$ of tori embedded in M, called *separating family*, which has the following property: if $\mathcal{U}(\mathcal{T})$ is a sufficiently small regular neighbourhood of $\mathcal{T}$ in M, then each connected component of $M \setminus \mathcal{U}(\mathcal{T})$ is a Seifertic manifold. The manifold M is equiped with a *Waldhausen decomposition* if a separating family $\mathcal{T}$ and a Seifert fibration on each connected component V_ν of $M \setminus \mathcal{U}(\mathcal{T}) = \coprod_{\nu=1}^{m} V_\nu$ are fixed.

In this paper, we only consider Seifert fibrations whose base are orientable, as this is the case for each Seifert fibration appearing in singularity theory.

A *multilink* is a 1-dimensional link L in a 3-manifold whose components are weighted by some integers which are called the *multiplicities* of the components of L.

A *Waldhausen link* is a pair (M, L) where M is a Waldhausen manifold without boundary and where L is a finite union of Seifert fibres in a Waldhausen decomposition of M. Such a decomposition is called a *Waldhausen decomposition* of (M, L). A Waldhausen graph $G(M, L)$ of (M, L) is a graph of M associated to such a Waldhausen decomposition decorated with arrows corresponding to the components of L. For more details, see [**17**]. When L is a multilink, one says that (M, L) is a *Waldhausen multilink*.

Let (M, L) be a Waldhausen multilink equipped with a Waldhausen decomposition $M \setminus \mathcal{U}(\mathcal{T}) = \coprod_{\nu=1}^{m} V_\nu$. Let us fix an orientation of the Seifert fibres on each V_ν. One defines the *Waldhausen graph* of (M, L) associated with this decomposition and this choice of orientation as follows. If $L = \varnothing$, then one defines by the same way the Waldhausen graph of M.

1) The vertices (resp. the edges) of $G(M)$ are in bijection with the Seifert manifolds (resp. with the torii of $\mathcal{T}$) in such a way that for each $T \in \mathcal{T}$ and for $\nu, \nu' \in \{1, \dots, m\}$, the edge corresponding to T joins the vertices ν and ν' if and only if $\partial\overline{\mathcal{U}(T)} = \overline{\mathcal{U}(T)} \cap (V_\nu \cup V_{\nu'})$, where $\mathcal{U}(T)$ denotes the connected component of $\mathcal{U}(\mathcal{T})$ which contains T.

2) each edge is is endowed with an arbitrary orientation, and then, is weighted by the normalized triple $(\alpha, \beta, \varepsilon)$ defined as in [**18**] (see also [**15**] p. 322) as follows: Let T be a separating torus between the Seifert components V_ν and V_{ν_i}, let $T_i \subset V_\nu$ and $T'_i \subset V_{\nu_i}$ be the two connected components of the boundary of $\mathcal{U}(T)$. Let us orient $T_i \cup T'_i$ as the boundary of $\mathcal{U}(T)$. Let b_i (resp. b'_i) be a Seifert fibre of V_ν on T_i (resp. of V_{ν_i} on T'_i) and let a_i (resp. a'_i) be an oriented closed curve on T_i (resp. T'_i) such that $a_i \cdot b_i = +1$ in $H_1(T_i, \mathbf{Z})$ (resp. $a'_i \cdot b'_i = +1$ in $H_1(T'_i, \mathbf{Z})$). Let $h : T_i \to T'_i$ be an reversing orientation homeomorphism induced by the product structure of $\overline{\mathcal{U}(T)}$. There exist some unique integers $\varepsilon_i \in \{1, -1\}$, $\alpha_i > 0$ and $\beta_i, \beta'_i \in \mathbf{Z}$ such that $\varepsilon_i h^{-1}(b'_i) = \alpha_i a_i + \beta_i b_i$ in $H_1(T_i, \mathbf{Z})$ and $\varepsilon_i h(b_i) = \alpha_i a'_i + \beta'_i b'_i$ in $H_1(T'_i, \mathbf{Z})$.

Moreover, there exists up to homology a unique choice of the curves a_i and a'_i such that the pair (α_i, β_i) is normalized, *i.e.* $0 \leqslant \beta_i < \alpha_i$ et $0 \leqslant \beta'_i < \alpha_i$. If $\alpha_i > 1$, the integers β_i and β'_i are related by $\beta_i \beta'_i \equiv 1 \bmod \alpha_i$.

If the edge joining ν_i and ν in $G(M)$ is oriented from ν_i and ν, it is weighted by the normalized triple $(\alpha_i, \beta_i, \varepsilon_i)$ as on figure 5. Otherwise, it is weighted by $(\alpha_i, \beta'_i, \varepsilon_i)$.

3) For each Seifert fibre of $L \cap V_\nu$ (resp. for each exceptional fibre of V_ν which is not a component of L) one attaches to the vertex ν an arrow (resp. a stalk) whose extremity is weighted, as on figure 5, by the normalized Seifert invariants (α_i, β_i) defined as follows: let $\mathcal{N}_i$ be a saturated small tubular neighbourhood of the Seifert fibre of V_ν indexed by i ($i \in \{1, \dots, d'\}$), and let b_i be a Seifert fibre on $\partial\mathcal{N}_i$. The torus $\partial\mathcal{N}_i$ being oriented as the boundary of $\mathcal{N}_i$, one choose on it an oriented closed curve a_i such that $a_i \cdot b_i = +1$ in $H_1(\partial\mathcal{N}_i, \mathbf{Z})$. There then exists a unique pair (α_i, β_i) such that $\alpha_i a_i + \beta_i b_i = 0$ in $H_1(\partial\mathcal{N}_i, \mathbf{Z})$. Moreover, there exists up to homology a unique choice of the curve a_i such that (α_i, β_i) is normalized, *i.e.* $0 \leqslant \beta_i < \alpha_i$.

Moreover, the extremities of the arrows are weighted by the multiplicities μ_i of the corresponding components of L.

4) Each vertex ν is weighted by the genus g_ν of the base of the Seifert fibration of V_ν and by the Euler class e_ν defined in the following classical way: Let $\mathcal{N}$ be a saturated solid torus in $V_\nu \setminus \coprod_{i=1}^{d'} \mathcal{N}_i$ and let b be a Seifert fibre on $\partial\mathcal{N}$. Let F be a surface in $V_\nu \setminus \coprod_{i=1}^{d'} \mathcal{N}_i$ which is horizontal in the sense of Waldhausen ([**18**]), *i.e.* transversal to each Seifert fibre and whose boundary is the union of the d curves a_i defined in 3) and of $a = F \cap \partial\mathcal{N}$. Let us endow F with the orientation compatible

with that of the a_i's, and then, let us orient a as a component of ∂F. The Euler class e_ν is defined by

$$a - e_\nu b = 0 \text{ in } H_1(V, \mathbf{Z})$$

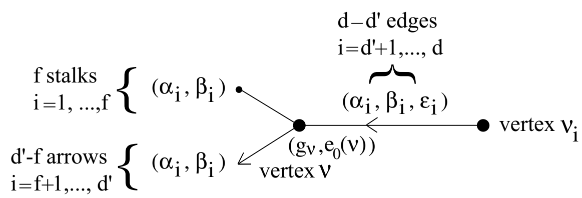

Figure 5

According to [**15**], the set of Waldhausen manifolds coincides up to homeomorphism with the set of the boundaries of the 4-manifolds obtained by plumbing processes, and there is a dictionary between the Waldhausen graphs and the plumbing graphs which constitutes an important part of the so-called plumbing calculus. For more details, see [**15**].

If (S,p) is a normal singularity of surface, then its link $\mathcal{L}(S,p)$ is a Waldhausen manifold. A Waldhausen graph of $\mathcal{L}(S,p)$ can be computed from any resolution graph of (S,p) by using plumbing calculus. Moreover, if $F : (S,p) \to (\mathbf{C}, 0)$ is an analytic germ, one denotes by L_F the multilink associated with F, that is, for $\varepsilon > 0$ sufficiently small, the link $\mathbf{S}^{2N-1} \cap F^{-1}(0)$ $((S,p) \subset (\mathbf{C}^N, 0))$ whose components are weighted by the multiplicities of the corresponding branches of F. Then the pair $(\mathcal{L}(S), L_F)$ is a Waldhausen multilink. A Waldhausen graph of $(\mathcal{L}(S), L_F)$ can be computed from any resolution graph of F by plumbing calculus.

Now, let (M,L) be a multilink which is fibred in the sense of [**4**]. Let $\Phi : M \setminus L \to \mathbf{S}^1$ be a fibration. Assume that we are not in the following special situation, which can be treated by hand: M is a lens space and L is included in the union of the cores of the two torii whose union is M. Then, if (M,L) is Waldhausen, the fibration Φ is *horizontal*, *i.e.* each fiber of Φ is, up to isotopy, transversal to the separating family $\mathcal{T}$ and to each Seifert fibre of $M \setminus L$.

Let $\mathcal{F}$ be a compact oriented surface with strictly negative Euler class. An orientation-preserving diffeomorphism $h : \mathcal{F} \to \mathcal{F}$ is *quasi-periodic* if there exists a finite family $\mathcal{C}$ of simple disjoint closed curves on $\mathcal{F}$ such that the restriction of h to the complementary of a small regular neighbourhood $\mathcal{U}(\mathcal{C})$ of $\mathcal{C}$ in $\mathcal{F}$ is periodic. One calls such a $\mathcal{C}$ is a *reduction system* of h.

Let (M,L) is a fibred multilink, let $\Phi : M \setminus L \to \mathbf{S}^1$ be a fibration and let $\mathcal{F}$ be a fibre of Φ. The *monodromy* of Φ is the conjugation class in the group of difféotopies of $\mathcal{F}$ of a diffeomorphism $h : \mathcal{F} \to \mathcal{F}$ defined as the first return on $\mathcal{F}$ of a flow transversal to the fibres of Φ. This class is independant from the choice of the transversal flow. Such a diffeomorphism h is called a *representant of the monodromy*.

If (M,L) is a fibred Waldhausen multilink and if $\Phi : M \setminus L \to \mathbf{S}^1$ is an horizontal fibration, then the monodromy of Φ admits some quasi-periodic representant. Indeed, if $\mathcal{F}$ is a fibre of Φ transversal to the Waldhausen structure, then the diffeomorphism h of first return on $\mathcal{F}$ of the Seifert fibres of $M \setminus \mathcal{U}(\mathcal{T})$ extends to a quasi-periodic representant of the monodromy of ϕ whose reduction system is $\mathcal{C} = \mathcal{T} \cap \mathcal{F}$. Let us consider the neighbourhood $\mathcal{U}(\mathcal{C}) = \mathcal{F} \cap \mathcal{U}(\mathcal{T})$ of $\mathcal{C}$ and let N be the smaller positive interger such that $h^N_{|\mathcal{F}\setminus\mathcal{U}(\mathcal{C})} = \mathrm{Id}_{|\mathcal{F}\setminus\mathcal{U}(\mathcal{C})}$. If c denotes a curve of $\mathcal{C}$, then the restriction of h^N to $\overline{\mathcal{U}(c)}$ de $\mathcal{U}(c)$ is a Dehn twist which is characterized by the rational number t, called *rational twist*, defined as follows: let $\mu : \mathbf{S}^1 \times [0,1] \to \overline{\mathcal{U}(c)}$ be a trivialization of the annulus $\overline{\mathcal{U}(c)}$ such that $\mu(\mathbf{S}^1 \times \{\frac{1}{2}\}) = c$. Let δ be the oriented path in $\overline{\mathcal{U}(c)}$ defined by $\delta(s) := \mu(x,s)$, where x is fixed on $\mathbf{S}^1$ and where $s \in [0,1]$. Let us orient c in such a way that $\delta \cdot c = +1$ in $\overline{\mathcal{U}(c)}$. There then exists a unique rational number t, such that the cycles Ntc and $h^N\delta - \delta$ are holomogous in $\overline{\mathcal{U}(c)}$.

Example. — If (S,p) is a normal singularity of surface and if $F : (S,p) \to (\mathbf{C},0)$ is an analytic germ, then the pair $(\mathcal{L}(S), L_F)$ is a fibred multilink by considering the Milnor fibration $\Phi_F : \mathcal{L}(S) \setminus L_F \to \mathbf{S}^1$ defined by

$$\forall\, \sigma \in \mathcal{L}(S) \setminus L_F, \quad \Phi_F(\sigma) = \frac{F(\sigma)}{|F(\sigma)|}$$

Let $h : \mathcal{F} \to \mathcal{F}$ be a quasi-periodic diffeomorphism of surface. A graph $G(h)$, called *Nielsen graph* of h, is defined in both [**16**] and [**17**] from the works of J. Nielsen. Let us recall this definition.

Let F be a compact connected oriented surface and let $\tau : F \to F$ be an orientation-preserving periodic diffeomorphism with order n. The projection $\pi : F \to O$ on the orbits space of τ is a n-sheeted cyclic cover branched over a finite number of *exceptional* orbits. Let $D_1, \ldots, D_f$ be some open disjoint disks, neighbourhoods of the exceptionnal orbits and let $\check{O} = O \setminus \coprod_{i=1}^{f} D_i$. One associates to each oriented simple closed curve Γ on O a triple (m,λ,σ) called *valence* of Γ which is defined as follows: m is the number of connected components of $\pi^{-1}(\Gamma)$ and $\lambda = n/m$. Let $\rho : H_1(\check{O},\mathbf{Z}) \to \mathbf{Z}/n\mathbf{Z}$ be the homomorphism associated to the cover π over $\check{O}$, σ the integer defined modulo λ by $\rho([\Gamma]) = m \cdot \sigma$.

Let us orient O as F via π. For each $i \in \{1,\ldots,f\}$, the valency of the exceptional orbit indexed by i is by definition the valency of the curve ∂D_i oriented as a component of the boundary of $\check{O}$.

The *Nielsen graph* of τ is the graph $\mathcal{G}(\tau)$ represented on figure 6. It has a single vertex which carry some "stalks" and "boundary-stalks" which represent respectively the exceptional orbits and the components of the boundary of O. This graph is weighted by the following numerical data:

- n, the order of τ,
- g, the genus of O,
- $(m_i, \lambda_i, \sigma_i)$, $i = 1, \ldots, f$, the valencies of the f exceptional orbits of O,
- $(m_i, \lambda_i, \sigma_i)$, $i = f+1, \ldots, d$, the valencies of the $f - d$ boundary components of O oriented as the boundary of O.

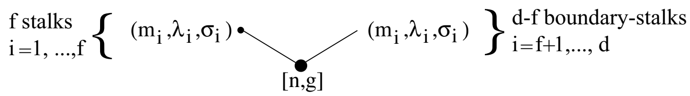

FIGURE 6

Now let $h : \mathcal{F} \to \mathcal{F}$ be a quasi-periodic diffeomorphism and let $\mathcal{C}$ be a reduction system of h. Let $\mathcal{G}_h$ be the graph defined as follows: the vertices (resp. the edges) of $\mathcal{G}_h$ are in bijection with the connected components of $\mathcal{F} \setminus \mathcal{C}$ (resp. with the curves of $\mathcal{C}$) in such a way that if F et F' are some connected components of $\mathcal{F} \setminus \mathcal{C}$ and if c is a curve of $\mathcal{C}$ such that $c \subset \overline{F} \cap \overline{F'}$, then the edge $A(c)$ joins the vertices $S(F)$ and $S(F')$

Let $\overline{\mathcal{G}_h}$ be the quotient graph of the action induced by h on the graph $\mathcal{G}_h$. The Nielsen graph $\mathcal{G}(h)$ of h is constructed from $\overline{\mathcal{G}_h}$ as follows.

Let ν be a vertex of $\overline{\mathcal{G}_h}$ and let r_ν be the number of connected components of $\mathcal{F} \setminus \mathcal{U}(\mathcal{C})$ represented by ν. The diffeomorphism h cyclically permutes these r_ν connected components, and if F_ν is one of them, the diffeomorphism $h_\nu = h^{r_\nu}_{|F_\nu}$ is a periodic diffeomorphism of F_ν. For each vertex ν of $\overline{\mathcal{G}}_h$, one endow the vertex ν of the graph $\mathcal{G}(h_\nu)$ with the weight r_ν.

For each edge A of $\overline{\mathcal{G}_h}$ with extremities ν and ν' ($\nu = \nu'$ is allowed), one performs the following operation: let c be a curve of $\mathcal{C}$ represented by A, let t be the twist of h around c, and let $\mathcal{U}(c)$ be the connected component of $\mathcal{U}(\mathcal{C})$ which contains c. The boundary components of the annulus $\mathcal{U}(c)$ are represented by two distinct boundary stalks T et T' belonging respectively to the graphs $\mathcal{G}(h_\nu)$ and $\mathcal{G}(h_{\nu'})$. One constructs an edge joining the vertices $\mathcal{G}(h_\nu)$ and $\mathcal{G}(h_{\nu'})$ by attaching T and T' by their extremities, and then, the middle of this edge is weighted by the corresponding rational twist t (figure 7).

At last, let us call circuit of a graph G any subgraph of G isomophic to the graph whose set of vertices is $\{1, \ldots, n\}$ and whose set of edges is $\{(1,2), (2,3), \ldots, (n-1, n), (n, 1)\}$. Each oriented circuit $\overline{c}$ of the obtained graph $G(h)$ is weighted by the class $\omega_{\overline{c}}$ modulo $\gcd(r_\nu,\ \nu$ vertex of $\overline{c})$ defined in [**1**] as follows. The circuit $\overline{c}$ being also a circuit $\overline{c}$ of $\overline{\mathcal{G}_h}$, let c be an oriented circuit of $\mathcal{G}_h$ such that $p(c) = \overline{c}$, where $p : \mathcal{G}_h \to \overline{\mathcal{G}_h}$ denotes the projection. Let ν be a vextex on $\overline{c}$ and let s be a vertex on c such that $p(s) = \nu$. Then the vertices $p^{-1}(\nu) \cap c$ appear in the following order on the oriented

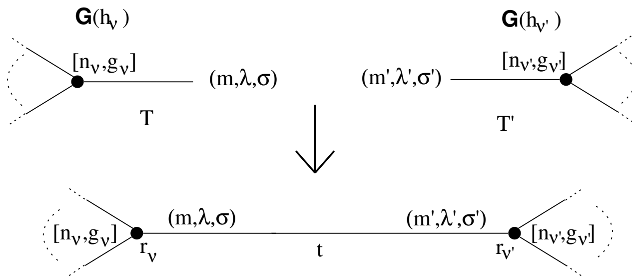

FIGURE 7

cycle c:

$$s, h^{\omega_{\overline{c}}}(s), h^{2\omega_{\overline{c}}}(s)\dots$$

This achieves the definition of the Nielsen graph $G(h)$.

The following result gives a dictionnary beetween the Waldhausen graph of a fibred Waldhausen multilink and the Nielsen graph of a quasi-periodic representant of its monodromy.

Lemma 3.1. — *Let (M, L) be a Waldhausen multilink admitting an horizontal fibration $\Phi : M \smallsetminus L \to \mathbf{S}^1$. Let $G(M, L)$ be a Waldhausen graph of (M, L) and let $G(h)$ be the Nielsen graph of the corresponding quasi-periodic representative $h : \mathcal{F} \to \mathcal{F}$ of the monodromy of Φ. There exists an isomorphism between the graphs $G(M, L)$ and $G(h)$ which sends:*

- *the vertices of $G(M, L)$ on the vertices of $G(h)$,*
- *the edges of $G(M, L)$ on the edges of $G(h)$,*
- *the stalks of $G(M, L)$ on the stalks of $G(h)$,*
- *the arrows of $G(M, L)$ on the boundary-stalks of $G(h)$.*

Moreover, let ν be a vertex of $G(M, L)$ as on figure 8. The corresponding vertex ν of $g(h)$ is also represented on figure 8. Let us set $N_\nu = n_\nu r_\nu$. For each valency (m, λ, σ), there exists a representative σ in his class modulo λ in such a way that the following equalities hold:

$$\forall\, i \in \{1, \dots, f\}, \quad (\alpha_i, \beta_i) = (\lambda_i, \sigma_i) \tag{1}$$

$$\forall\, i \in \{f+1, \dots, d'\}, \quad m_i = 1,\ \lambda_i = N_\nu,\ \textit{and}\ \alpha_i\sigma_i - N_\nu\beta_i = -\frac{\mu_i}{m_i} \tag{2}$$

$$\forall\, i \in \{d'+1, \dots, d\}, \quad \varepsilon_i = \frac{-N_{\nu_i} t_i \lambda_i}{|-N_{\nu_i} t_i \lambda_i|},\ \alpha_i = |-N_{\nu_i} t_i \lambda_i|, \quad \textit{and}\ \beta_i = \varepsilon_i \frac{N_{\nu_i} - N_\nu N_{\nu_i} t_i \sigma_i}{N_\nu} \tag{3}$$

Let $e_0(\nu) = e_\nu - \Sigma_{i=1}^d \beta_i/\alpha_i$. *Then*

$$e_0(\nu) = \sum_{i=1}^{d} \left(\frac{\sigma_i}{\lambda_i} - \frac{\beta_i}{\alpha_i} \right) \tag{4}$$

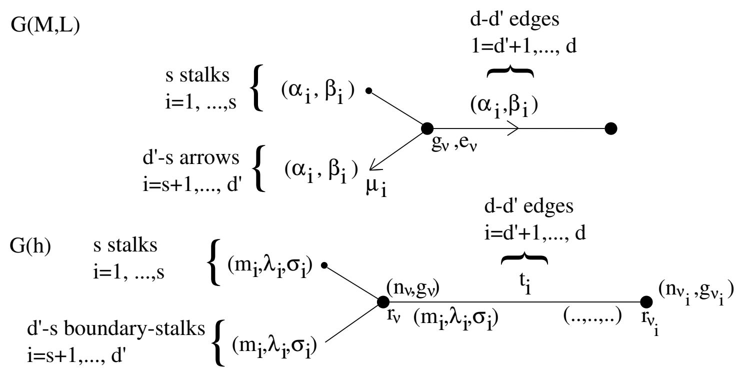

FIGURE 8

Proof. — This result is a generalization of ([P2], Lemma 2.2) which treats the case of a Waldhausen link. In fact, the dictionary is identical, except the second formula which takes into account the multiplicity μ_i of the corresponding component K_i of the multilink and the number m_i of boundary-components of the horizontal fibre in the neighbourhood of K_i. □

According to Lemma 3.1, the graph $G(M, L)$ is entirely computed from the graph $G(h)$ and the multiplicities of the components of L as Seifert fibres and as components of a multilink. Conversely, as mentioned in [**17**], if M is a rational homology sphere (*i.e.* if $G(M, L)$ is a tree and if all its vertices carry genus zero), then the graph $G(h)$ is completely determined by the graph $G(M, L)$. In particular, if (S, p) is a normal singularity of surface whose boundary is a rational homology sphere and if $F : (S, p) \to (\mathcal{C}, 0)$ is any analytic germ, then the Nielsen graph of the quasi-periodic monodromy of its Milnor fibration is completely determined from any resolution graph of F. The explicit calculus can be performed by using the formulae of Du Bois - Michel ([**2**], Proposition 1.6 and Theorem 2.21, see also [**16**] for the formulation in terms of Nielsen graphs). In these papers, these formulae concern the case S smooth and F reduced, but their proof in [**2**] also hold without any change in the general case.

4. Branched cyclic cover over a singularity of surface

In this section, we describe the link of any singularity of complex surface obtained as the cyclic cover over some germ of surface (S, p) totally ramified over a germ of curve. This singularities include for instance the germs of hypersurfaces in $(\mathbf{C}^3,0)$ with equations $f(x,y)-z^k=0$ or $f(x,y)-z^kg(x,y)=0$.

Our method generalizes that developed in [**16**] for the singularities $f(x,y)-z^k=0$ when f is reduced, using the theory of fibred Waldhausen multilinks presented in Section 3.

Let (S,p) be a germ of complex surface, let $F:(S,p)\to(\mathbf{C},0)$ be an analytic function such that $F^{-1}(0)$ is a curve, and let $k\geqslant 1$ be an integer. One denotes by $\mathcal{C}(F,k)$ ($\mathcal{C}$ for "cover") the germ of hypersurface at $(p,0)$ in $S\times\mathbf{C}$ with equation $F-z^k=0$. In other words, $\mathcal{C}(F,k)$ is the fibre-product of F and of the branched cover $\rho_k:\mathbf{C}\to\mathbf{C}$ defined by $\rho_k(z)=z^k$. In particular, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C}(F,k) & \xrightarrow{\rho} & S \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{C} & \xrightarrow{\rho_k} & \mathbf{C} \end{array}$$

where ρ and F' are the restrictions of the natural projections $S\times\mathbf{C}\to S$ and $S\times\mathbf{C}\to\mathbf{C}$ respectively. The map ρ is nothing but the k-sheeted cyclic over (S,p) totally branched over the germ of curve with equation $F=0$.

Examples

(1) $(S,p)=(\mathbf{C}^2,0)$. Then, for any germ $F:(\mathbf{C}^2,0)\to(\mathbf{C},0)$, $\mathcal{C}(F,k)$ is the germ of hypersurface at the origin of $\mathbf{C}^3$ with equation $F(x,y)-z^k=0$.

(2) Let $f:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ and $g:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ be two analytic germs, let $(S,0)$ be the germ of hypersurface in $\mathbf{C}^3$ with equation $f(x,y)+zg(x,y)=0$, and let $F:(S,p)\to(\mathbf{C},0)$ defined by $F(x,y,z)=z$. Then $\mathcal{C}(F,k)$ is analytically isomorphic to the germ of hypersurface at the origin of $\mathbf{C}^3$ with equation $f(x,y)-z^kg(x,y)=0$.

Let $I:(S,p)\to(\mathbf{C}^N,0)$ be an embedding. Let us identify $\mathcal{C}(F,k)$ with the image of its embedding in $\mathbf{C}^{N+1}$ obtained by restricting the map $I\times\mathrm{Id}_{\mathbf{C}}:S\times\mathbf{C}\to\mathbf{C}^N\times\mathbf{C}$. According to [**3**], when ε and $\varepsilon'>0$ are sufficiently small, the link of $\mathcal{C}(F,k)$ can be defined as the intersection in $\mathbf{C}^{N+1}$ between the complex surface $\mathcal{C}(F,k)$ and the boundary of the "ball with corners"

$$\mathbf{B}^{2N+2}=\{(x,z)\in\mathbf{C}^N\times\mathbf{C}\mid \|x\|\leqslant\varepsilon,\ |z|\leqslant\varepsilon'\}$$

Let us now choose ε' so that $\forall\, x\in\mathbf{C}^N$ such that $\|x\|\leqslant\varepsilon,\ |F(x)|^k<\varepsilon'$. Then the link $\mathcal{L}(\mathcal{C}(F,k))$ is contained in $\mathbf{S}^{2N-1}_{\varepsilon}\times\{z\in\mathbf{C}\mid |z|<\varepsilon'\}$.

Let $\eta:\overline{\mathcal{C}}(F,k)\to\mathcal{C}(F,k)$ be the normalisation of $\mathcal{C}(F,k)$. As mentioned before Proposition 1.3, one can define $\mathcal{L}(\overline{\mathcal{C}}(F,k)$ as $\eta^{-1}(\mathcal{L}(\mathcal{C}(F,k)))$. Let us denote again by $\eta:\mathcal{L}(\overline{\mathcal{C}}(F,k))\to\mathcal{L}(\mathcal{C}(F,k))$ the restriction of η.

Proposition 4.1

a) *The restriction $\rho : \mathcal{L}(\mathcal{C}(F,k)) \to \mathcal{L}(S)$ is a k-sheeted cyclic cover totally branched over L_F.*

b) *The map $\overline{\rho} = \rho \circ \eta : \mathcal{L}(\overline{\mathcal{C}}(F,k)) \to \mathcal{L}(S)$ is a k-sheeted cyclic cover with branching locus included in the link $L_F \cup L(\Sigma_S)$.*

Proof. — a) follows from the definition of ρ. The singular locus of $\mathcal{C}(F,k)$ is included in the strict tranform by ρ of the germ of curve $\Sigma_S \cup F^{-1}(0)$. This leads to b). ☐

Our aim is to describe a generalized plumbing graph of the multilink $(\mathcal{L}(\mathcal{C}(F,k)), L_{F'})$ from a resolution graph of the germ F by using the properties of the covers ρ and $\overline{\rho}$. Our study consists of two parts, the first one dealing with the particular case when the singularity (S,p) is normal.

I - Description of the multilink $(\mathcal{L}(\mathcal{C}(F,k)), L_{F'})$ when (S,p) is normal. — This first step is a generalization of the method developed in [**16**] in the smooth case and for $F : \mathbf{C}^2, 0 \to \mathbf{C}, 0$ reduced.

Let us denote by $\overline{F} : \overline{\mathcal{C}}(F,k) \to (\mathbf{C},0)$ the analytic germ defined by $\overline{F} = F' \circ \eta$. Then $L_{\overline{F}} = \overline{\rho}^{-1}(L_F)$, and a Waldhausen decomposition of the multilink $(\mathcal{L}(\overline{\mathcal{C}}(F,k)), L_{\overline{F}})$ can be defined via the branched cyclic cover $\overline{\rho}$ from any Waldhausen decomposition of the multilink $(\mathcal{L}(S), L_F)$ — with separating family say $\mathcal{T}$ — as follows: the separating family is $\mathcal{T}' := \overline{\rho}^{-1}(\mathcal{T})$, $\mathcal{U}(\mathcal{T}') := \overline{\rho}^{-1}(\mathcal{U}(\mathcal{T}))$, and the Seifert fibres of $\mathcal{L}(S) \setminus \mathcal{U}(\mathcal{T})$ are the images by $\overline{\rho}$ of the Seifert fibres of $\mathcal{L}(\overline{\mathcal{C}}(F,k)) \setminus \mathcal{U}(\mathcal{T}')$.

Let us now fix on $(\mathcal{L}(S), L_F)$ a Waldhausen decomposition with separating family $\mathcal{T}$. We will describe the Waldhausen decomposition of the multilink $(\mathcal{L}(\overline{\mathcal{C}}(F,k), L_{\overline{F}})$ induced by this way via the cyclic cover $\overline{\rho}$.

Lemma 4.2. — *Let $\mathcal{F}$ be a fibre of the Milnor fibration $\Phi_F = F/|F| : \mathcal{L}(S,p) \setminus L_F \to \mathbf{S}^1$. Then $\overline{\rho}^{-1}(\mathcal{F})$ is the disjoint union of k fibres of the Milnor fibration $\Phi_{\overline{F}}$ of $\overline{F}$, and if $\mathcal{F}'$ denotes one of them, the restriction $\overline{\rho} : \mathcal{F}' \to \mathcal{F}$ is a diffeomorphism. Furthermore, if $h : \mathcal{F} \to \mathcal{F}$ denotes a quasi-periodic representative of the monodromy of Φ_F, then a quasi-periodic representative of the monodromy of $\Phi_{\overline{F}}$ is $\overline{\rho}^{-1} \circ h^k \circ \overline{\rho}$.*

Proof. — The proof is analogous to that of ([**16**] 1.5) by using the commutativity of the diagram

$$(d1) \qquad \begin{array}{ccc} \mathcal{L}(\overline{\mathcal{C}}(F,k)) \setminus L_{\overline{F}} & \xrightarrow{\ \rho\ } & \mathcal{L}(S) \setminus L_F \\ \Phi_{\overline{F}} \big\downarrow & & \big\downarrow \Phi_F \\ \mathbf{S}^1 & \xrightarrow{\ \rho_k\ } & \mathbf{S}^1 \end{array}$$

where $\rho_k(z) = z^k$. ☐

Lemma 4.3

(1) *Let K be a component of L_F and let m be the number of connected components of the intersection of $\mathcal{F}$ with a small tubular neighbourhood of K in $\mathcal{L}(S)$. Then $\overline{\rho}^{-1}(K)$ is a disjoint union of* $\gcd(m,k)$ *components of $L_{\overline{F}}$.*

(2) *Let K' be one of them and let μ be the multiplicity of K as a component of the multilink L_F. Then the multiplicity μ' of K' as a component of the multilink $L_{\overline{F}}$ is given by*

$$\mu' = \frac{\mu}{\gcd(\mu,k)}$$

(3) *Let V' be the Seifert component of $\mathcal{L}(\overline{\mathcal{C}}(F,k)) \smallsetminus \mathcal{U}(\mathcal{T}')$ which contains K'. If α' (resp. α) is the multiplicity of K' (resp. K) as a Seifert fibre of V' (resp. $\overline{\rho}(V')$), and if N is the order of the periodic diffeomorphism $h_{|\mathcal{F}\cap\overline{\rho}(V')}$, then*

$$\alpha' = \frac{k\gcd(m,k)\alpha}{\gcd(N,k)\gcd(\mu,k)}$$

(4) *η performs a $\frac{\gcd(\mu,k)}{\gcd(m,k)}$-curling on each of the* $\gcd(m,k)$ *components of $\overline{\rho}^{-1}(K)$, and then, identifies the quotients in a single circle.*

The proof of Lemma 4.3 uses the following topological result which will be used again later on.

Lemma 4.4. — *Let $\mathbf{S}^1_0$, $\mathbf{S}^1_1$ and $\mathbf{S}^1_2$ be three copies of $\mathbf{S}^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ and let $r_1 : \mathbf{S}^1_2 \to \mathbf{S}^1_0$ and $r_2 : \mathbf{S}^1_1 \to \mathbf{S}^1_0$ be two cyclic covers with degrees respectively d_1 and d_2. Let Y be the fibre-product of r_1 and r_2 and let $r'_j : Y \to \mathbf{S}^1_j$, $j = 1,2$ be the natural projections. Then Y is the disjoint union of* $\gcd(d_1,d_2)$ *copies of $\mathbf{S}^1$ and r'_j is a d_j-sheeted cyclic cover.*

$$\begin{array}{ccc} Y & \xrightarrow{r'_1} & \mathbf{S}^1_1 \\ {\scriptstyle r'_2}\downarrow & & \downarrow{\scriptstyle r_2} \\ \mathbf{S}^1_2 & \xrightarrow{r_1} & \mathbf{S}^1_0 \end{array}$$

Proof. — Let us identify $\mathbf{S}^1_j$ with the standard circle $\mathbf{S}^1$. Then, up to conjugacy, r_j is the map defined by $r_j(e^{2i\pi t}) = e^{2i\pi d_j t}$. Therefore $Y = \{(x,y) \in \mathbf{S}^1_1 \times \mathbf{S}^1_2 \mid y^{d_1} = x^{d_2}\}$, that is Y is the disjoint union of $\gcd(d_1,d_2)$ parallel copies of the torus knot $(\frac{d_2}{\gcd(d_1,d_2)}, \frac{d_1}{\gcd(d_1,d_2)})$ on the torus $\mathbf{S}^1_1 \times \mathbf{S}^1_2$. □

Proof of Lemma 4.3

1) Let $\mathcal{U}(K)$ be a small tubular neighbourhood of K saturated with Seifert fibres and let T be its boundary. As h cyclically permutes the m connected components of $\mathcal{F} \cap T$, then these m curves split among $\gcd(m,k)$ orbits of the action of h^k on $\mathcal{F}$. Then, according to Lemma 4.2, $\overline{\rho}^{-1}(\mathcal{U}(K))$ is the disjoint union of $\gcd(m,k)$ solid torii, and $\overline{\rho}^{-1}(K)$ is the disjoint union of their cores.

2) and 4) Let γ be a meridian of $\mathcal{U}(K)$ that is a simple closed curve on T which borders a disk of compression in $\mathcal{U}(K)$. Restricting (d1), one obtains the following diagram

$$\begin{tikzcd} \overline{\rho}^{-1}(\gamma) \arrow[r, "\overline{\rho}"] \arrow[d, "\Phi_{\overline{F}}"'] & \gamma \arrow[d, "\Phi_F"] \\ \mathbf{S}^1 \arrow[r, "\rho_k"] & \mathbf{S}^1 \end{tikzcd}$$

which expresses that $\overline{\rho}^{-1}(\gamma)$ is the fibre-product of the cyclic covers ρ_k and Φ_F : $\gamma \to \mathbf{S}^1$. Applying Lemma 4.4 to these covers, whose degrees are respectively k and μ, one obtains that $\overline{\rho}^{-1}(\gamma)$ is the disjoint union of $\gcd(\mu, k)$ curves which split uniformically as meridians of the $\gcd(m, k)$ components of $\overline{\rho}^{-1}(K)$. Therefore, the restriction $\overline{\rho} : K' \to \overline{\rho}^{-1}(K)$ is a $\frac{\gcd(\mu,k)}{\gcd(m,k)}$-sheeted cyclic cover.

Furthermore, let T' be the boundary of the component of $\overline{\rho}^{-1}(\mathcal{U}(K))$ which contains K'; according to Lemma 4.4, the restriction $\Phi_{\overline{F}} : \overline{\rho}^{-1}(\gamma) \cap T' \to \mathbf{S}^1$ is a $\frac{\mu}{\gcd(\mu,k)}$-sheeted cyclic cover. By definition, this number of sheets is equal to the multiplicity μ'.

3) Let b be a Seifert fibre of $\overline{\rho}(V')$ on T. Let us orientate T as the boundary of $\mathcal{U}(K)$ and γ in such a way that $\gamma \cdot b > 0$ in $H_1(T, \mathbf{Z})$. Let us orientate $\overline{\rho}^{-1}(T)$ (resp. $\overline{\rho}^{-1}(\gamma)$, resp. $\overline{\rho}^{-1}(b)$) as T (resp. γ, resp. b) via $\overline{\rho}$. Then

$$\big(\overline{\rho}^{-1}(\gamma) \cap T'\big) \cdot \big(\overline{\rho}^{-1}(b) \cap T'\big) = \frac{k}{\gcd(m,k)} \gamma \cdot b$$

as the restriction $\overline{\rho}_| : T' \to T$ is a $\frac{k}{\gcd(m,k)}$-sheeted cyclic cover.

According to Lemma 4.2, $\rho^{-1}(b) \cap T'$ is the disjoint union of $\frac{\gcd(N,k)}{\gcd(m,k)}$ Seifert fibres. If b' is one of them and if γ' is one of the $\frac{\gcd(\mu,k)}{\gcd(m,k)}$ components of $\overline{\rho}^{-1}(\gamma) \cap T'$, one therefore obtains

$$\frac{\gcd(\mu,k)}{\gcd(m,k)} \times \frac{\gcd(N,k)}{\gcd(m,k)} \gamma' \cdot b' = \frac{k}{\gcd(m,k)} \gamma \cdot b$$

This leads to 3) as $\gamma \cdot b = \alpha$ and $\gamma' \cdot b' = \alpha'$. □

Let (S,p) be a normal singularity of surface, let $F : (S,p) \to (\mathbf{C}, 0)$ be any analytic germ, and let $k \geqslant 2$ be an integer. Lemma 3.1, Proposition 4.1, Lemma 4.2 and Lemma 4.3 lead to Algorithm 4.5, which computes a generalized plumbing graph of the multilink $(\mathcal{L}(\mathcal{C}(F,k)), L_{\overline{F}})$ from any resolution graph of F and from the Nielsen graph of a quasi-periodic representative of the monodromy $h : \mathcal{F} \to \mathcal{F}$ of the Milnor fibration Φ_F.

Each step of the algorithm is illustrated on the example $F : (\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ defined by $F(x,y) = (x^2 + y^3)^2$ and $k = 3$. $\mathcal{C}(F,3)$ is the germ of hypersurface in $(\mathbf{C}^3, 0)$ with equation $(x^2+y^3)^2 + z^3 = 0$ and $\overline{F} : (\mathcal{C}(F,3), 0) \to \mathbf{C}$ is the analytic germ defined by $\overline{F}(x,y,z) = z$. Figure 9 represents a resolution graph G of F and a

Nielsen graph of the quasi-periodic representative of the monodromy of Φ_F performed from G by using Du Bois-Michel's formulae ([**2**]).

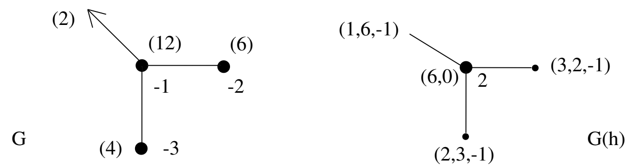

FIGURE 9

Algorithm 4.5
Step 1. Using Lemma 4.3, one computes the multiplicities of the components of the multilink $L_{\overline{F}}$ as Seifert fibres and as components of a multilink. In the example, the single component of $L_{\overline{F}}$ has multiplicity $\alpha' = 1$ as Seifert fibre and $\mu' = 2$ as component of the multilink.
Step 2. One computes the Nielsen graph $G(h^k)$ from $G(h)$ (using for instance ([**16**], 2.2 and 2.3; in particular, the classes ω of the circuits of $G(h^k)$ are obtained by analogous formulas as the valencies of the curves).

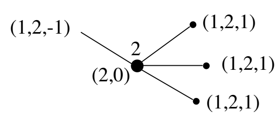

FIGURE 10. $G(h^3)$

Step 3. According to Lemma 4.2, h^k is a quasi-periodic representative of the monodromy of the horizontal fibration $\Phi_{\overline{F}}$ of the multilink $(\mathcal{L}(\overline{\mathcal{C}}(F,k)), L_{\overline{F}})$. Using Lemma 3.1, one computes the Waldhausen graph of this multilink from $G(h^k)$ and from the multiplicities of the components of $L_{\overline{F}}$ as Seifert fibres and as components of the multilink.

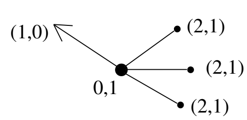

FIGURE 11

Step 4. Using plumbing calculus again, one computes from this Waldhausen graph a plumbing graph of the link $(\mathcal{L}(\overline{\mathcal{C}}(F,k)), L_{\overline{F}})$.

Step 5. Using 4) of Lemma 4.3, one computes from this a generalized plumbing graph of $(\mathcal{L}(\mathcal{C}(F,k)), L_{F'})$. In our example, the restriction $\overline{\rho} : L_{\overline{F}} \to \rho^{-1}(L_F)$ is a homeomorphism. Figure 12 represents the plumbing graph of $(\mathcal{L}(\mathcal{C}(F,k)), L_{F'})$.

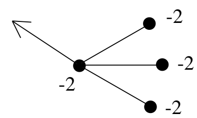

FIGURE 12

II - Description of $(\mathcal{L}(\mathcal{C}(F,k)), L_{F'})$ *in the general case.* — Let (S,p) be any singularity of surface and let $F : (S,p) \to (\mathbf{C},0)$ be an analytic germ such that $F^{-1}(0)$ is a curve. Let $n : \coprod_{i=1}^r (\overline{S}_i, p_i) \to (S,p)$ be the normalisation of S, and for each $i \in \{1,\ldots,r\}$, let $F_i : (\overline{S}_i, p_i) :\to (S,p)$ be the germ defined by $F_i = F \circ n$. Then $\coprod_{i=1}^r \mathcal{C}(F_i,k)$ is the fibre-product of n and ρ, as is expressed in the commutative diagram below, where n' and ρ' denote the natural projections. Again by argument of [**3**], this diagram restricts to the links.

$$\begin{tikzcd}
\coprod_{i=1}^r \mathcal{C}(F_i,k) \arrow[r, "\rho'"] \arrow[d, "n'"'] & \coprod_{i=1}^r (\overline{S}_i, p_i) \arrow[d, "n"] \\
\mathcal{C}(F,k) \arrow[r, "\rho"] \arrow[d, "F'"'] & (S,p) \arrow[d, "F"] \\
\mathbf{C} \arrow[r, "\rho_k"] & \mathbf{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod_{i=1}^r \mathcal{L}(\mathcal{C}(F_i,k)) \arrow[r, "\rho'"] \arrow[d, "p"'] & \coprod_{i=1}^r \mathcal{L}(\overline{S}_i) \arrow[d, "n"] \\
\mathcal{L}(\mathcal{C}(F,k)) \arrow[r, "\rho"] & \mathcal{L}(S)
\end{tikzcd}$$

As in Section 1, one denotes by Σ_S the singular locus of S and by $L(\Sigma_S) \subset \mathcal{L}(S)$ its link. The following result describes the map $n : \coprod_{i=1}^r \mathcal{L}(\mathcal{C}(F_i,k)) \to \mathcal{L}(\mathcal{C}(F,k))$.

Lemma 4.6. — *The map* $n' : \coprod_{i=1}^r \mathcal{L}(\mathcal{C}(F_i,k)) \to \mathcal{L}(\mathcal{C}(F,k))$ *is a homeomorphism over* $\rho^{-1}(L(\Sigma_S))$. *Moreover, let* K *be a component of* $L(\Sigma_S)$ *and let* $K_j,\ j = 1,\ldots,l_K$ *be the components of* $n^{-1}(K)$. *For each* $j = 1,\ldots,l_K$, *one denotes by* a_j *the degree of the restriction* $n : K_j \to K$.

a) *If* $K \subset L_F$, *then* $\rho^{-1}(K)$ *is a single Seifert fibre in* $\mathcal{L}(\mathcal{C}(F,k))$, *and for each* $j = 1,\ldots,l_K$, $\rho'^{-1}(K_j)$ *is a single Seifert fibre in* $\coprod_{i=1}^r \mathcal{L}(\mathcal{C}(F_i,k))$. *The restriction* $n' : \rho'^{-1}(K_j) \to \rho^{-1}(K)$ *is an* a_j*-sheeted cyclic cover.*

b) *If* $K \not\subset L_F$, *let* m *be the degree of* $F_{|K}$ *and let* m_j *be that of* $(F \circ n)_{|K_j}$ *(note that* $m_j = ma_j$. *Then* $\rho^{-1}(K)$ *is the disjoint union of* $\gcd(m,k)$ *Seifert fibres in* $\mathcal{L}(\mathcal{C}(F,k))$, $\rho'^{-1}(K_j)$ *is the disjoint union of* $\gcd(m_j,k)$ *Seifert fibres in* $\coprod_{i=1}^r \mathcal{L}(\mathcal{C}(F_i,k))$, *and the restriction* $n' : \rho'^{-1}(K_j) \to \rho^{-1}(K)$ *is an* a_j*-sheeted cyclic cover.*

Proof

a) The cover $\rho : \mathcal{L}(\mathcal{C}(F,k)) \to \mathcal{L}(S)$ is a homeomorphism over K as K is contained in its ramification locus. Then, a) follows by applying Lemma 3.4 to the covers $\rho : \rho^{-1}(K) \to K$ and $n : K_j \to K$.

b) Applying Lemma 3.4 to ρ_k and $F : K \to \mathbf{C}^*$, one obtains that $\rho^{-1}(K)$ has $\gcd(m,k)$ connected components. Applying Lemma 3.4 to ρ_k and $F \circ n : K_j \to \mathbf{C}^*$, one obtains that $\rho'^{-1}(K_j)$ has $\gcd(m_j,k)$ connected components. The degree of the cyclic cover $n' : \rho'^{-1}(K_j) \to \rho^{-1}(K)$ is a_j as the following diagram expresses that $\rho'^{-1}(K_j)$ is the fiber product of $\rho : \rho^{-1}(K) \to K$ and $n : K_j \to K$

$$\begin{tikzcd}
\rho'^{-1}(K_j) \arrow[r, "\rho'"] \arrow[d, "n'"'] & K_j \arrow[d, "n"] \\
\rho^{-1}(K) \arrow[r, "\rho"] & K
\end{tikzcd}$$

□

The following algorithm computes a generalized plumbing graph of the multilink $(\mathcal{L}(\mathcal{C}(F,k)), L_{F'})$ from any resolution graphs of F_i, from the Nielsen graphs of the quasi-periodic monodromies of the Milnor fibrations Φ_{F_i}, and from the curlings and identifications performed by the normalization n over the link of the singular locus Σ_S.

Algorithm 4.7

Step 1. By using Algorithm 4.5, one computes a generalized plumbing graph of the multilinks $(\mathcal{C}(F_i,k), L_{F_i})$ for each $i \in \{1,\dots,r\}$.

Step 2. Using Lemma 4.6, one indicates on the disjoint union of these graphs the curlings and identifications which have to be performed on the link $(n \circ \overline{\rho})^{-1}(L(\Sigma_S))$ to obtain $\mathcal{L}(\mathcal{C}(F,k))$ from $\coprod_{i=1}^{r} \mathcal{L}(\mathcal{C}(F_i,k))$.

Examples

(1) In the case $(S,0) = (\mathbf{C}^2,0)$ and $F : (\mathbf{C}^2,0) \to (\mathbf{C},0)$ reduced, a lot of examples are computed by this method in [**16**]. See also [Nem].

(2) $f : (\mathbf{C}^2,0) \to (\mathbf{C},0)$ and $g : (\mathbf{C}^2,0) \to (\mathbf{C},0)$ are the two analytic germs defined by $f(x,y) = x^2$ and $g(x,y) = y^{12}$, $(S,0)$ is the germ of hypersurface in $\mathbf{C}^3$ with equation $x^2 + zy^{12} = 0$, and $F : (S,p) \to (\mathbf{C},0)$ defined by $F(x,y,z) = z$. Then $\mathcal{C}(F,k)$ is the germ of hypersurface at the origin of $\mathbf{C}^3$ with equation $x^2 - z^k y^{12} = 0$ and $\overline{F} : (\mathcal{C}(F,k),0) \to \mathbf{C}$ is the analytic germ defined by $\overline{F}(x,y,z) = z$. Figure 13 represents a generalized plumbing graph G of the multilink L_F, computed from Figure 3, the Nielsen graph $G(h)$ of the monodromy of L_F, the Nielsen graph $G(h^k)$, a generalized plumbing graph of the multilink $(\mathcal{L}(\mathcal{C}(F,k)), L_{F'})$ obtained by using Algorithm 4.7, and the underlying minimal plumbing graph obtained from the latter by some suitable blowing-down.

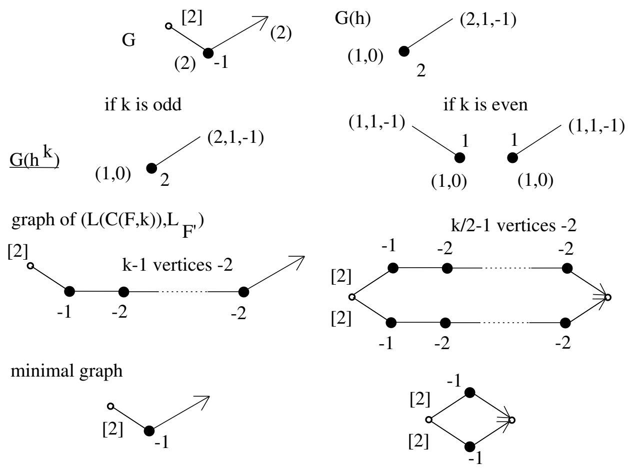

FIGURE 13

Let $(S, 0)$ be the germ of hypersurface in $\mathbf{C}^3$ with equation

$$f_d(x, y, z) + f_{d+k}(x, y, z) = 0,$$

where f_i is homogeneous of degree i. Let $\phi : (\widetilde{S}, D) \to (S, 0)$ be the blowing-up of S with center 0. If $\widetilde{p} \in D$ is a point in the exceptional divisor corresponding to a tangent direction (x_0, y_0, z_0) with $z_0 \neq 0$ then a local equation of $(\widetilde{S}, \widetilde{p})$ is $f_d(x, y, 1) + z^k f_{d+k}(x, y, 1) = 0$, and one can obtain a resolution graph of (S, O) using the previous algorithms. Moreover one can give precise geometric conditions on f_d and f_{d+k} in order the link $\mathcal{L}(S, p)$ to be a topological manifold (*cf.* [**11**].)

As an example let us consider the surface germ $(S, 0)$ with equation

$$(zy - x^2) + y^4 x = 0.$$

We can see that the resolution graph of $(S, 0)$ is $\circ - 2 \quad \circ -2 \quad \circ -2$. This is the resolution graph of the Hirzebruch-Jung singularity with equation $xy + z^4 = 0$ (A_3 in Arnold's notation). This example gives a negative answer to a question of McEwans and Némethi (see [**12**] for other related questions). In order to state the question, let us remind the concept of a quasi-ordinary surface singularity. A germ of surface $(S, 0) \subset (\mathbf{C}^3, 0)$ is called quasi-ordinary (QO) if there exists a linear projection $p : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ such that $p_{|S}$ is finite and its discriminant has normal crossings at 0. This means that one can take local coordinates (x, y, z) such that the projection is given by $p(x, y, z) = (x, y)$, the local equation f of is $(S, 0)$ is a Weierstrass polynomial in z and its discriminant $Disc_z(f) = x^a y^b \varepsilon(x, y)$ with $\varepsilon(0, 0) \neq 0$. McEwans and Némethi asked for some intrinsic characterization of quasi-ordinary surface germ, for instance in terms of its link. In our case $(S, 0)$ is not QO and has the same (abstract) link than A_3, namely the lens space $L(2, 1)$, but as embedded links in S^5 which may be different. Question 2.9(a) of [**12**] asked if the fact that the normalization of $(S,0)$ is is quasi-ordinary characterize the quasi-ordinary singularities in $(\mathbf{C}^3,0)$. The above example gives a negative answer to this question, because $(S,0)$ is not QO, but the above computations gives that its normalization is A_3. One can check directly that the map $n(x_1,y_1,z_1)=(-x_1+y_1z_1,y_1,-z_1^2)$ from the germ $(S_1,0)$ defined by $x_1y_1-z_4^4$ to $(S,0)$ is the normalization of $(S,0)$. In order to prove that $(S,0)$ is not QO in any system of coordinates we can use the next property from [**10**]: Let $(S,0)$ be QO in the coordinates (x,y,z) and let us denote by $PN(f)\subset\mathbf{N}^3$ its Newton polygon ($O_0=(0,0,k)\in PN(f)$) and by $\pi_0:\mathbf{N}^3\to\mathbf{N}^2\times\{0\}$ the projection from the point O_0. If the coefficient of f in z^{k-1} is 0, then there exists a point $O_1=(a,b,0)$ such that $\pi_0(PN(f))=O_1+(\mathbf{N}^2\times\{0\})$ Let us denote by l the segment $[O_0,O_1]$ then $PN(f)\subset l+\mathbf{N}^3$. As the tangent cone is a double conic then one can see that $O_0=(0,0,4)$ and $O_1=(2,2,0)$). For instance if $a+b\geqslant 0$ then $k=4$ and the tangent cone is a plane (the other cases are similar). Using again that $PN(f)\subset l+\mathbf{N}^3$ one can see that the two axis E_x and E_y are contained in the singular locus Σ_S and this gives a contradiction because Σ_S is a line.

5. Lê's conjecture for the cyclic covers

Definition. — A germ of surface (S,p) is *equisingular* if either it is smooth or (Σ_S,p) is a smooth germ and the topological type of the germ of curve obtained by hyperplane transversal section of S at a point $p'\in\Sigma_S$ close from p does not depend of p'.

The link of an equisingular germ of surface with a single branch is homeomorphic to the sphere $\mathbf{S}^3$. The converse has been conjectured by Lê D.T. in [**19**]:

Lê's Conjecture. — *If the link of a singularity of complex surface is homeomorphic to the sphere* $\mathbf{S}^3$ *then the singularity is equisingular with a single branch.*

The aim of this Section is to prove Lê's Conjecture when the singularity is a cyclic cover of a germ of normal surface totally branched over a germ of curve:

Theorem 5.1. — *Lê's Conjecture is true for the singularity* $\mathcal{C}(F,k)$*, where* $F:(S,p)\to(\mathbf{C},0)$ *is an analytic function on a germ* (S,p) *of normal surface such that* $F^{-1}(0)$ *is a curve.*

In the sequel, (S,p) is a singularity of normal surface and $F:(S,p)\to(\mathbf{C},0)$ is an analytic germ. Using again the notations of Section 3, one denotes by $\overline{\rho}:(\mathcal{L}(\overline{\mathcal{C}}(F,k)),L_{\overline{F}})\to(\mathcal{L}(S,p),L_F)$ the k-sheeted cyclic cover associated to F and k. In order to prove Theorem 5.1, we will describe the minimal Waldhausen decomposition of $\mathcal{L}(\overline{\mathcal{C}}(F,k))$. The following is obtained as ([**8**],1.2.3) by using arguments of [**6**].

Proposition 5.2. — *Let M be a Waldhausen manifold without boundary which is not homeomorphic to a lens space, or to a torus bundle over the circle. There then exists, up to isotopy, a unique Waldhausen decomposition of M in which the following conditions hold:*

(i) if $\mathcal{T}$ is the separating family, then the Seifert fibres on both sides of any torus $T \subset \mathcal{T}$ are not homologous on T.

(ii) No connected component of $M \setminus \mathcal{U}(\mathcal{T})$ is a solid torus or a product torus $\times$ interval.

This unique decomposition is called the minimal Waldhausen decomposition of M.

Let $\pi : Z \to S$ be a resolution of F and let G_π the dual graph of the resolution. Recall that G_π is a plumbing graph of the multilink $(\mathcal{L}(S,p), L_F)$. One denotes by $\mathcal{R}$ the set of *rupture* vertices of G_π, that is the set of vertices carrying a non zero genus or admitting at least 3 neighbouring vertices, the extremities of the arrows being considered as true vertices. A *long edge* of G_π is the adherence of a connected component of $G_\pi \setminus \mathcal{R}$ whose two tips are rupture vertices.

If $\mathcal{R} \neq \varnothing$, then it follows from [**15**] that $\mathcal{L}(S,p)$ admits a Waldhausen decomposition whose Seifert components are in bijection with the rupture vertices of G_π and whose separating family $\mathcal{T}$ is in bijection with the set of long edges of G_π. Actually, this Waldhausen decomposition, which does not depend on G_π, is the minimal one such that L_F is a union of Seifert fibres. Indeed, if $\mathcal{U}(L_F)$ denotes a small open tubular neighbourhood of L_F saturated with Seifert fibres, it coincides on the complementary of $\mathcal{U}(L_F)$ with the minimal Waldhausen decomposition of $\mathcal{L}(S,p) \setminus \mathcal{U}(L_F)$.

The following result generalizes ([**16**], 6.4).

Proposition 5.3. — *Let us assume that the graph G_π admits at least one rupture vertex and let us equip $\mathcal{L}(S,p)$ with the minimal Waldhausen decomposition such that L_F is a union of Seifert fibres. Then the link $\mathcal{L}(\overline{\mathcal{C}}(F,k))$ is neither a lens space nor a torus bundle over the circle. Moreover, if $k \geqslant 3$ or if $k=2$ and L_F has a single component, then the Waldhausen decomposition of $\mathcal{L}(\overline{\mathcal{C}}(F,k))$ induced via $\overline{\rho}$ by that of $\mathcal{L}(S,p)$ is the minimal one.*

Proof. — Let $\mathcal{T}$ be the separating family of $\mathcal{L}(S,p)$ and let $\mathcal{T}' := \overline{\rho}^{-1}(\mathcal{T})$ be that of $\mathcal{L}(\overline{\mathcal{C}}(F,k))$.

At first, let us prove that the Waldhausen decomposition of $\mathcal{L}(\overline{\mathcal{C}}(F,k))$ has the properties (i) and (ii) of 5.3.

(i) holds by the same arguments as in the proof of ([**16**], 6.4).

(ii) Let V' be a Seifert component of $\mathcal{L}(\overline{\mathcal{C}}(F,k)) \setminus \mathcal{U}(\mathcal{T}')$ and let V be the Seifert component of $\mathcal{L}(S,p) \setminus \mathcal{U}(\mathcal{T})$ defined by $V = \overline{\rho}(V')$. Let us assume that V' is either a solid torus or a product *torus* $\times$ *interval*.

Let $\mathcal{F}'$ be a fibre of $\Phi_{\overline{F}}$ and let $\mathcal{F}$ be the fibre of Φ_F defined by $\mathcal{F} = \overline{\rho}(\mathcal{F}')$.

Assume that $V' \cap L_{\overline{F}} = \varnothing$. Then $V' \cap \mathcal{F}'$ is either a disjoint union of disks or a disjoint union of rings. As the restriction $\overline{\rho} : \mathcal{F}' \to \mathcal{F}$ is a diffeomorphism, $\mathcal{F} \cap V$ also is a disjoint union of disks or a disjoint union of rings. Therefore V is is either a solid torus or a product *torus* $\times$ *interval* as $V \cap L_F = \varnothing$. This contradicts the minimality of the Waldhausen decomposition of $\mathcal{L}(S,p) \smallsetminus \mathcal{U}(L_F)$. Therefore $V' \cap L_{F'} \neq \varnothing$

If K is a component with multiplicity μ of the multilink L_F, it follows from Lemma 3.3 that $\gcd(\mu, k) = 1$ as $\mathcal{L}(S,p)$ is a manifold. If r denotes the number of connected components of $\mathcal{F} \cap V$, one then obtains that $\gcd(r,k) = 1$ as r divides μ. This means that V' is the single connected component of $\overline{\rho}^{-1}(V)$.

Under these conditions, we may use the following generalization of ([**17**], 6.2) which follows from the computations of Section 3. $\square$

Lemma 5.4. — *Let $p : V \to B$ be the Seifert fibration of V and let $p' : V' \to B'$ be that of V'. The map $c : B' \to B$ defined by the commutativity of the diagram*

$$\begin{array}{ccc} V' & \xrightarrow{\rho} & V \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{c} & B \end{array}$$

is a cyclic finite branched cover such that:

1) the number k' of sheets of c is equal to $\gcd(n,k)$ where n is the order of the restriction of h^r to one of the connected components of $\mathcal{F} \cap V$.

2) Let K' be a Seifert fibre of V' and let α' (resp. α) be the multiplicity of K' (resp. $\overline{\rho}(K')$) as Seifert fibre.

a) If K' is an exceptional fibre of V' and is not a component of $L_{F'}$ then $(p \circ \overline{\rho})(K')$ is a branching point of c of order $\frac{\alpha}{\alpha'}$.

b) If K' is component of L_F (not necessarily exceptional as Seifert fibre of V') then $(p \circ \overline{\rho})(K')$ is a branching point of c of order k', and $\alpha' = \frac{k \times \alpha}{k'}$.

c) Otherwise $(p \circ \overline{\rho})(K')$ is a regular point for c.

Let us now apply the Hurwitz formula to the to the k'-sheeted cover $c : B' \to B$: let g be the genus of B, let u (resp. u') be the number of boundary components of B (resp. B'), and let v be the number of connected components of $V \cap L_F$.

Let $\alpha_1, \ldots, \alpha_l$ be the multiplicities of the l exceptional fibres of V which are not components of L_F, and for all $i \in \{1, \ldots, l\}$, let α'_i be the multiplicity of the Seifert fibres of V' which project onto the exceptional fibre of V indexed by i. One then obtains:

$$(*) \qquad 2 - \sum_{i=1}^{l} k' \frac{\alpha'_i}{\alpha_i} - u' - v = k'(2 - 2g - l - u - v)$$

If V' is a solid torus, then $u = u' = 1$, V' has a maximum of one exceptional fibre, and $(*)$ becomes

$$(k'-1)(v-1) + 2k'g + \sum_{i=1}^{l} k'\big(1 - \frac{\alpha'_i}{\alpha_i}\big) = 0$$

Hence $g = 0$, ($k' = 1$ or $v = 1$) and $\alpha'_i = \alpha_i$ for each i. This last condition implies that the inverse image by $\overline{\rho}$ of any exceptional component of $V \setminus L_F$ is an exceptional component of V'. Moreover, according to Lemma 5.4 (2b), the inverse image by $\overline{\rho}$ of any exceptional component of V which is a component of L_F is also exceptional in V'. Therefore, V is a solid torus.

But by minimality of the Waldhausen decomposition of $\mathcal{L}(S,p)\setminus\mathcal{U}(L_F)$, $V\setminus\mathcal{U}(L_F)$ is not a product *torus* $\times$ *interval*. There must then exists a regular fibre K of V which is a component of L_F and whose inverse image by $\overline{\rho}$ is regular. This leads to $k' = k$ (Lemma 5.4 (2b)) and $v = 1$. Hence $l \geqslant 1$, as $V \setminus \mathcal{U}(L_F)$ is not a product *torus* $\times$ *interval*, and the inverse image by $\overline{\rho}$ of the Seifert fibre with multiplicity α_1 consists of k exceptional fibres of V' (Lemma 3.1). Then V' is not a solid torus as it has more than one exceptional fibre. Contradiction.

If V' is a product torus $\times$ interval, then $u' = 2$, and V' has no exceptional fibres. In particular $\alpha'_i = 1$ for each i, and, according to Lemma 5.4, $k' = k$. Furthermore $u \in \{1, 2\}$.

If $u = 2$, $(*)$ becomes

$$v(k-1) + 2gk' + \sum_{i=1}^{l} k\big(1 - \frac{1}{\alpha_i}\big) = 0$$

This contradicts $k \geqslant 2$.

If $u = 1$, $(*)$ becomes

$$v(k-1) - k + 2gk' + \sum_{i=1}^{l} k\big(1 - \frac{1}{\alpha_i}\big) = 0$$

Hence $l \in \{0, 1\}$, as $\alpha_i \geqslant 2$ for each i. If $l = 0$ (resp. if $l = 1$), then $v = \frac{k}{k-1}$ (resp. $v = \frac{k}{(k-1)\alpha_1}$). As v is an integer, this implies that $k = 2$ and $v = 2$, *i.e.* L_F has more than one component. This is a contradiction as we have assumed that L_F has a single component when $k = 2$.

Then condition (ii) holds.

Let us now prove that $\mathcal{L}(\overline{\mathcal{C}}(F,k)$ is neither a lens space nor a torus bundle over the circle.

Let us assume that $\mathcal{L}(\overline{\mathcal{C}}(F,k)$ is a lens space. Then, the condition (ii) implies that $\mathcal{T}'$ and $\mathcal{T}$ are empty. Using the previous notations again with $V = \mathcal{L}(S,p)$ and

$V' = \mathcal{L}(\overline{\mathcal{C}}(F,k)$, one obtains

$$(v-2)(k'-1) + \sum_{i=1}^{l} k'\big(1 - \frac{1}{\alpha_i}\big) = 0$$

Therefore $g = 0$, $l = 0$ and $v = 2$. Then the graph G_π has no rupture vertex. This contradicts our hypothesis.

Let us now assume that $\mathcal{L}(\overline{\mathcal{C}}(F,k)$ is a torus bundle over the circle. Then $\mathcal{T}'$ is empty (condition (ii) again), and $\mathcal{L}(\overline{\mathcal{C}}(F,k)$ is homeomorphic to the product $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$. This is not possible as Grauert's condition implies that the link of a normal surface does not fibre over the circle. □

Proof of Theorem 5.1. — Let us assume that the link of $\mathcal{C}(F,k)$ is homeomorphic to the sphere $\mathbf{S}^3$. Then it follows from P.A. Smith's theorem that the link $L_{\overline{F}}$ have a single component. Therefore, the link L_F have also a single component, and, according to Proposition 5.3, the graph G_π has no rupture vertex. Then the graph G_π is as on figure 14, the multiplicities being writen between parenthesis.

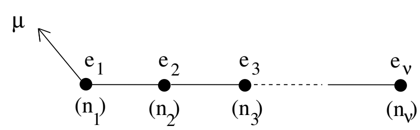

Figure 14

Let us equip the pair $(\mathcal{L}(S), L_F)$ with the Seifert fibration obtained by extending that of the manifold represented by the vertex of G_π weighted by n_1 and let us equip $\mathcal{C}(F,k)$ with the Seifert fibration obtained by lifting that of $(\mathcal{L}(S), L_F)$ via $\overline{\rho}$. As $\mathcal{C}(F,k)$ is homeomorphic to the 3-sphere, *i.e.* to a lens space $L(p,q)$ such that $p = 1$, the corresponding Waldhausen graph of the link $L_{F'}$ has the following form:

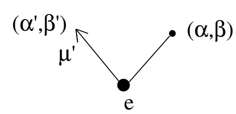

Figure 15

with

(1) $$1 = p = \beta\alpha' + \beta'\alpha - e\alpha\alpha'$$

Moreover, the order of the corresponding periodic monodromy of $\Phi_{F'}$ is equal to $n_1/\gcd(n_1,k)$. Then the monodromical system (see [**17**],) of the multilink $L_{F'}$ consists of the single equality

$$\Big(\frac{\beta}{\alpha}+\frac{\beta'}{\alpha'}-e\Big)\cdot\frac{n_1}{\gcd(n_1,k)}=\frac{\mu'}{\alpha'}$$

Then (1) implies

$$\frac{n_1}{\gcd(n_1,k)}=\mu'\alpha$$

Therefore μ' divides n_1. As $\mathcal{C}(F,k)$ is a manifold, $\mu=\mu'$, then μ divides n_1. If m denotes the greatest common divisor of the multiplicities carried by two consecutive vertices of G_π (it does not depend of the choice of the two vertices), one therefore obtains: $m=\gcd(n_1,\mu)=\mu$.

Let us choose G_π minimal, that is either ($\nu=1$ and $e_1=-1$) or $e_i\leqslant -2$ for each i. Let us assume that $\nu\geqslant 2$. Then, the equalities

$$e_\nu n_\nu+n_{\nu-1}=0 \text{ and for } i\in\{2,\dots,\nu-1\},\ e_in_i+n_{i-1}+n_{i+1}=0$$

lead to $n_\nu=\mu<n_{\nu-1}<\cdots<n_2<n_1$. This contradicts the equality $e_1n_1+n_2+\mu=0$ as μ divides n_1 and n_2 and as $e_1<-1$.

Then, in fact, $\nu=1$, and the equality $e_1n_1+\mu=0$ leads to $e_1=-1$ as μ divides n_1. This means that the link of (S,p) is homeomorphic to the sphere $\mathbf{S}^3$. As (S,p) is normal, it implies that (S,p) is smooth according to [Mu]. Hence the germ F is analytically equivalent to the germ $G:(\mathbf{C}^2,0)\to(\mathbf{C},0)$ defined by $G(x,y)=x^\mu$ and then, $\mathcal{C}(F,k)$ is equisingular with a single branch as it is equivalent to the germ of hypersurface at the origin of $\mathbf{C}^3$ with equation $x^\mu+z^k=0$.

This achieves the proof of Theorem 4.1. □

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