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BRIEF INTRODUCTION TO PAINLEVÉ VI

by

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Abstract. — We will give a quick introduction to isomonodromy and the sixth Painlevé differential equation, leading to some questions regarding algebraic solutions.

***Résumé* (Une brève introduction à Painlevé VI).** — Nous donnons une brève introduction à l'isomonodromie et à la sixième équation différentielle de Painlevé, ce qui conduit à des questions sur les solutions algébriques.

Contents

1. Introduction

The sixth Painlevé equation (P_{VI}) is a second order nonlinear differential equation for a complex function $y(t)$:

$$y'' = R(y, y', t)$$

where R is a certain rational function (see below) depending on four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. (Thus we need to fix these parameters to get a particular P_{VI} equation.) The main thing one needs to know about P_{VI} is the following:

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Fact. — Suppose we have a local solution y of $\mathrm{P_{VI}}$ on some disk $D \subset \mathbb{P}^1 \setminus \{0,1,\infty\}$ in the three-punctured sphere. Then y extends, as a solution of $\mathrm{P_{VI}}$, to a meromorphic function on the universal cover of $\mathbb{P}^1 \setminus \{0,1,\infty\}$.

Thus solutions only branch at $0,1,\infty$ and all other singularities are just poles; this is the so-called 'Painlevé property' of the equation.

Thus $\mathrm{P_{VI}}$ shares many of the properties of the Gauss hypergeometric equation, which is a *linear* second order equation whose solutions branch only at $0,1,\infty$.

Another well-known fact about $\mathrm{P_{VI}}$ is that generic solutions $y(t)$ of $\mathrm{P_{VI}}$ are "new" transcendental functions (i.e., they are not expressible in terms of classical special functions). Thus it is very difficult to find explicit solutions to $\mathrm{P_{VI}}$ in general.

However, for special values of the parameters it turns out that there are explicit solutions, and even solutions $y(t)$ which are algebraic, i.e., defined implicitly by polynomial equations

$$F(y,t) = 0. \tag{1}$$

Our aim is to describe some of the geometry behind $\mathrm{P_{VI}}$ leading up to a description of how some of these algebraic solutions may be constructed.

Note immediately that by definition such plane algebraic curves

$$\{(y,t) \mid F(y,t) = 0\} \subset \mathbb{C}^2$$

are covers of the t-line, branched only at $0,1,\infty$ and so are Belyi curves. Also, in all examples so far, the polynomial F turns out to have integer coefficients.

To give a brief taste of the geometry let us mention that, as is often the case, the three-punctured sphere above arises as the moduli space of (ordered) four-tuples of points on another $\mathbb{P}^1$. Explicitly, to each $t \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ we will associate the four-tuple $(0,t,1,\infty)$ of points and in turn the four-punctured sphere

$$\mathbb{P}_t := \mathbb{P}^1 \setminus \{0,t,1,\infty\}.$$

As we will explain, $\mathrm{P_{VI}}$ arises by considering (isomonodromic) deformations of certain non-rigid linear differential equations on theses four-punctured spheres. In particular solving $\mathrm{P_{VI}}$ leads to explicit linear differential equations on the four-punctured sphere with known, non-rigid, monodromy representations.

Acknowledgments. — The reader should note that the literature on $\mathrm{P_{VI}}$ is huge and we will not attempt a survey. (A good bibliography and historical survey may be found in [**DM00**].) This note is written to explain some introductory facts about the method of [**Boa05**], which extends that of Dubrovin and Mazzocco [**DM00**]. I would like to thank Daniel Bertrand and Pierre Dèbes for the invitation to speak at this conference.

2. Monodromy and actions of the fundamental group of the base

Suppose we have a complete flat connection on a fibre bundle $\pi : M \to B$. Choose a basepoint $t \in B$ and let $M_t = \pi^{-1}(t)$ be the fibre of M over t. (See appendix B.)

Then given any loop γ in B based at t, we may integrate the connection on M around γ, yielding an automorphism

$$a_\gamma : M_t \xrightarrow{\cong} M_t$$

of the fibre over t. This automorphism only depends on the homotopy class of the loop γ (since the connection is flat), and in this way one obtains an action of the fundamental group of the base on the fibre, i.e., a homomorphism

$$\pi_1(B) \longrightarrow \mathrm{Aut}(M_t),$$

the *monodromy action*.

This should be compared with the cases of a) linear connections (where the fibre is a vector space V and so one obtains a *representation* $\pi_1(B) \to \mathrm{GL}(V)$), and b) coverings (where the fibre is a finite set and so $\mathrm{Aut}(M_t) = \mathrm{Sym}_n$).

We will be interested in horizontal sections of such flat connections which are finite covers of the base — i.e., sections which only have a finite number of branches. The point to be made here is that, in terms of the monodromy action, such sections correspond precisely to the *finite orbits* of the monodromy action. Given a point of $m \in M_t$ which is in a finite orbit, the horizontal section of the connection through m will extend, by definition, to a section with a finite number of branches.

3. Main example: the $\mathbf{P_{VI}}$ fibrations

The main example of fibre bundle with complete flat connection we are interested in here comes from geometry. It is the simplest *isomonodromy* or *non-abelian Gauss-Manin* connection.

Take the base B to be the three-punctured sphere

$$B := \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

For each point $t \in B$ there is a corresponding four-punctured sphere, namely

$$\mathbb{P}_t := \mathbb{P}^1 \setminus \{0, t, 1, \infty\}.$$

Thus we can think of B as parameterising a (universal) family of four-punctured spheres, with labelled punctures. Write a_1, a_2, a_3, a_4 for these punctures positions:

$$(a_1, a_2, a_3, a_4) := (0, t, 1, \infty).$$

For each $t \in B$ we consider the space of conjugacy classes of $\mathrm{SL}_2(\mathbb{C})$ representations of the fundamental group of $\mathbb{P}_t$

$$\mathrm{Hom}(\pi_1(\mathbb{P}_t), G)/G \tag{2}$$

where $G := \mathrm{SL}_2(\mathbb{C})$, and we have not specified the basepoint used in $\pi_1(\mathbb{P}_t)$, since changing basepoints yields conjugate representations (which are identified in the quotient (2)).

Now suppose we choose four generic conjugacy classes of $G = \mathrm{SL}_2(\mathbb{C})$

$$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 \subset G.$$

Then we can consider the subset of (2),

$$\boldsymbol{\mathcal{C}}_t := \mathrm{Hom}_{\mathcal{C}}(\pi_1(\mathbb{P}_t), G)/G \subset \mathrm{Hom}(\pi_1(\mathbb{P}_t), G)/G$$

of representations which take simple positive loops around a_i into $\mathcal{C}_i$ for $i = 1, 2, 3, 4$.

Explicitly if we choose loops γ_i generating $\pi_1(\mathbb{P}_t)$ such that $\gamma_4 \cdot \gamma_3 \cdot \gamma_2 \cdot \gamma_1$ is contractible and that γ_i is a simple positive loop around a_i. Then each $\rho \in \mathrm{Hom}(\pi_1(\mathbb{P}_t), G)$ determines matrices $M_i = \rho(\gamma_i) \in G$ and we obtain the explicit description:

$$\boldsymbol{\mathcal{C}}_t \cong \{(M_1, M_2, M_3, M_4) \bigm| M_i \in \mathcal{C}_i, M_4 \cdots M_1 = 1\}/G \tag{3}$$

where G acts by overall conjugation. A simple dimension count shows that in general these spaces are of complex dimension two and taking the invariant functions identifies $\boldsymbol{\mathcal{C}}_t$ with an affine cubic surface, (cut out by the so-called "Fricke relation" between the invariants) which is smooth in general (see e.g. [**Iwa02, Boa05**]).

Remark. — One might ask why, in the simplest case, one cannot have dimension one instead, but that is because these spaces of "conjugacy classes of fundamental group representations with fixed local conjugacy classes", have natural holomorphic symplectic structures on them, so are even-dimensional.

Lemma. — *The surfaces $\boldsymbol{\mathcal{C}}_t$ fit together as the fibres of a (nonlinear) fibre bundle*

$$M \longrightarrow B$$

over B and this fibration has a natural complete flat connection defined by identifying representation with the "same" monodromy.

Proof. — Choose $t \in B$ arbitrarily and choose loops generating $\pi_1(\mathbb{P}_t)$ to obtain an explicit description of $\boldsymbol{\mathcal{C}}_t$ as in (3). Then there is a small neighbourhood U of t in B for which we can use the *same loops* to generate $\pi_1(\mathbb{P}_s)$ for any $s \in U$. Thus we have isomorphisms between $\boldsymbol{\mathcal{C}}_s$ and the right-hand side of (3) for any $s \in U$. This gives a preferred trivialisation of M over U (and one obtains the same trivialisation if different loops were initially chosen). Since t was arbitrary we may cover B with such patches U with a preferred trivialisation over each. This is equivalent to giving a complete flat connection. □

Thus we are now in the situation of the previous section, with a complete flat connection on a fibre bundle.

The Painlevé VI equation amounts to an explicit description of this connection. Very briefly one defines two specific functions y, x on a dense open subset of M, which restrict to local coordinates on each fibre. (See appendix A for a better approximation.) Writing out the connection in these coordinates yields a pair of coupled first order non-linear differential equations for $y(t), x(t)$. Eliminating x then yields a second order equation, the $\mathrm{P_{VI}}$ equation, for $y(t)$:

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right).$$

Thus the time t in $\mathrm{P_{VI}}$ is essentially the cross-ratio of the four pole positions (and is the coordinate t on the three-punctured sphere B). Also the four parameters $\alpha, \beta, \gamma, \delta$ in $\mathrm{P_{VI}}$ correspond to the choice of four conjugacy classes $\mathcal{C}_i \subset \mathrm{SL}_2(\mathbb{C})$.

The main point is that from this geometrical viewpoint we see that that branching of solutions $y(t)$ to $\mathrm{P_{VI}}$ corresponds to the monodromy of the connection on $M \to B$. Since this connection is complete, its monodromy amounts to an action of the fundamental group of B on a fibre $\boldsymbol{\mathcal{C}}_t$.

In particular finite-branching solutions of $\mathrm{P_{VI}}$ will be defined on finite covers of B (i.e covers of $\mathbb{P}^1$ branched only over $0, 1, \infty$) and will correspond to finite orbits of the monodromy action.

Explicitly this monodromy action can be described as follows in terms of the standard Hurwitz action.

The three-string braid group B_3 acts on $G^3 = G \times G \times G$ as follows

$$\begin{aligned} \beta_1(M_3, M_2, M_1) &= (M_2, M_2^{-1}M_3M_2, M_1) \\ \beta_2(M_3, M_2, M_1) &= (M_3, M_1, M_1^{-1}M_2M_1) \end{aligned} \tag{4}$$

where $M_i \in G$. The fundamental group of the base B is the free group on two letters $\pi_1(B) = F_2$ and this appears as the subgroup $< \beta_1^2, \beta_2^2 >$ of B_3. This F_2 action on G^3 restricts and descends to an action on $\boldsymbol{\mathcal{C}}_t$ (where the M_i arise as in (3)). Explicitly, with our conventions, the generator β_1^2 corresponds to the monodromy of y around 1 and β_2^2 to the monodromy of y around 0. An equivalent way of thinking of this is to observe this F_2 also arises as the pure mapping class group of the four-punctured sphere, which acts on the conjugacy classes of representations in the natural way, by pullback [**Boa06**].

4. Algebraic solutions

The problem of finding algebraic solutions to $\mathrm{P_{VI}}$ can be broken into two parts:

1) Find all the finite orbits of the explicit braid group action (4) on triples of elements of $\mathrm{SL}_2(\mathbb{C})$. (Since all algebraic solutions will be finite branching these orbits will a priori contain the branches of all algebraic solutions.)

2) For each finite B_3 (or F_2) orbit prove the corresponding $\mathrm{P_{VI}}$ solution is algebraic and, if so, construct it explicitly.

The answer to problem 1) appears to be open in general (even though it is an easily stated algebraic problem about triples of 2×2 matrices). However there is an easy set of solutions; namely take the triple to generate a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. (Of course step 2) is still hard in these cases.) For example all the algebraic solutions of Hitchin [**Hit95, Hit03**] and Dubrovin and Mazzocco [**DM00**] are equivalent to solutions of this form (thus there are dihedral, tetrahedral, octahedral and icosahedral solutions).

However it turns out not to be true that all algebraic solutions of $\mathrm{P_{VI}}$ are related to finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. The main purpose of [**Boa05**] was to construct a solution related to Klein's simple group $\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$, the next simplest simple group after the icosahedral group.

For problem 1) we used a different representation of $\mathrm{P_{VI}}$ as a non-abelian Gauss-Manin connection, by taking representations of the fundamental group of the four-punctured sphere into $\mathrm{GL}_3(\mathbb{C})$ such that three of the local monodromies are pseudo-reflections (i.e., automorphisms of the form "one plus rank one"). One again finds the spaces $\boldsymbol{\mathcal{C}}_t$ are two dimensional and a similar relation to $\mathrm{P_{VI}}$. In fact one can go further (and this is necessary for step 2) and show explicitly how to go between this $\mathrm{GL}_3(\mathbb{C})$ picture and the original (well-known) $\mathrm{SL}_2(\mathbb{C})$ picture sketched above. The key technical step is in a paper [**BJL81**] of Balser–Jurkat–Lutz from 1981 and involves the Fourier–Laplace transformation for certain irregular singular connections. One can deduce from earlier papers of Dubrovin that the approach of Dubrovin and Mazzocco (who did in fact use three-dimensional orthogonal reflections) is along the same lines, although restricting to orthogonal reflections amounts to restricting to a one-parameter subspace of the full four-parameter family of $\mathrm{P_{VI}}$ equations, something we have now managed to avoid.

In the situation of [**Boa05**] problem 2) was solved by adapting (and correcting) a result of Jimbo giving a precise formula for the leading term in the asymptotic expansion of a $\mathrm{P_{VI}}$ solution at zero. This enabled us to pass from a finite braid group orbit of $\mathrm{SL}_2(\mathbb{C})$ triples to the explicit solution: an algebraic curve $F(y,t) = 0$ with seven branches over the t-line and monodromy group A_7, such that the function $y(t)$ solves $\mathrm{P_{VI}}$.

This asymptotic formula of Jimbo is incredibly useful. For example the article [**Boa06**] shows that Jimbo's formula may be used to compute the asymptotics at zero of most of the icosahedral solutions to Painlevé VI. Moreover by inspecting the list of such solutions one sees there is a solution to Painlevé VI whose parameters lie on none of the reflecting hyperplanes of Okamoto's affine F_4 symmetry group; Jimbo's formula facilitates the explicit computation of this "generic" solution.

Since this was a Galois theory conference let us end with a related question. Recall that an algebraic solution to Painlevé VI amounts to giving a Belyi map $t : \Pi \to \mathbb{P}^1$ and a rational function y on Π such that the (local) function $y(t)$ solves the Painlevé VI equation.

Question. — Are all such "Painlevé curves" Π defined over $\mathbb{Q}$?

Appendix A: Riemann–Hilbert map

We wish to describe (dense open subsets) of the spaces of (linear) connections corresponding to the monodromy representations $\rho \in \boldsymbol{\mathcal{C}}_t$. Note that one needs to take care not to confuse the monodromy of P_{VI} (the monodromy of a nonlinear connection on a bundle over the three-punctured sphere) with the monodromy representations $\rho \in \boldsymbol{\mathcal{C}}_t$ which will be the monodromy of the linear connections below (on vector bundles over four-punctured spheres).

Recall we have chosen generic conjugacy classes $\mathcal{C}_i \subset \mathrm{SL}_2(\mathbb{C})$ for $i = 1, 2, 3, 4$. Now choose adjoint orbits $O_i \subset \mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ such that

$$\mathcal{C}_i = \exp(2\pi\sqrt{-1}O_i).$$

Now consider linear meromorphic connections on the trivial rank two vector bundle over $\mathbb{P}^1$ of the form:

$$\nabla := d - A(z)dz; \qquad A(z) = \sum_{i=1}^{3} \frac{A_i}{z - a_i}$$

or, what amounts to the same thing, systems of linear differential equations of the form

$$\frac{d\Phi}{dz} = A(z)\Phi.$$

Here z is a coordinate on $\mathbb{C} \subset \mathbb{P}^1$ and, given $t \neq 0, 1$, we have $(a_1, a_2, a_3, a_4) := (0, t, 1, \infty)$ as before. These connections have four singularities on $\mathbb{P}^1$; simple poles at $0, t, 1, \infty$. Thus on restriction to the four-punctured sphere $\mathbb{P}_t$ they are holomorphic (and therefore flat) connections. Taking their monodromy gives a representation $\rho \in \mathrm{Hom}(\pi_1(\mathbb{P}_t), G)$.

Now fixing the residue A_i to be in the orbit O_i (for $i = 1, 2, 3, 4$ where $A_4 := -A_1 - A_2 - A_3$ is the residue at infinity) implies that the monodromy around a_i is in $\mathcal{C}_i$, so in fact $\rho \in \mathrm{Hom}_{\mathcal{C}}(\pi_1(\mathbb{P}_t), G)$. The conjugacy class of ρ in $\mathcal{C}_t$ is independent of the choice of base point/initial basis chosen to take the monodromy.

The moduli space of such connections thus looks like the space of four-tuples of such matrices:

$$\boldsymbol{\mathcal{O}} := \Big\{ \ (A_1, A_2, A_3, A_4) \ \big| \ A_i \in O_i, \ \sum A_i = 0 \ \Big\} / G$$

where we quotient by diagonal conjugation by G, which corresponds to quotienting by bundle automorphisms (automorphisms of the trivial bundle over $\mathbb{P}^1$ are just these constant automorphisms).

The same dimension count as before gives that $\dim_{\mathbb{C}}(\boldsymbol{\mathcal{O}}) = 2$ so $\boldsymbol{\mathcal{O}}$ is again a complex surface, which we think of as the "additive analogue" of $\mathcal{C}_t$.

Now let $\mathcal{M}^* := B \times \boldsymbol{\mathcal{O}}$ be the product of the base B and the surface $\boldsymbol{\mathcal{O}}$, which we view as a trivial bundle over B with fibre $\boldsymbol{\mathcal{O}}$. A point of $\mathcal{M}^*$ is thus a choice of t and a point of $\boldsymbol{\mathcal{O}}$, and we view these as specifying a connection ∇ as above. The procedure of taking the monodromy representations then gives a holomorphic bundle map, the *Riemann-Hilbert map*:

$$\mathcal{M}^* \xrightarrow{\nu} M$$

from $\mathcal{M}^*$ to the bundle M of monodromy representations. This map is actually injective (as set-up here) and typically its image is the complement of an analytic divisor (the points of this divisor correspond to linear meromorphic connections on *nontrivial* degree zero bundles over $\mathbb{P}^1$).

Now we pull-back (restrict) the connection on M along the map ν to get a connection on the bundle $\mathcal{M}^* \to B$.

It is better to say that P_{VI} is what one gets by writing down this connection on $\mathcal{M}^*$ explicitly — since the coordinates x, y appearing in the definition of P_{VI} are certain *algebraic* functions on $\mathcal{M}^*$ (restricting to local coordinates on each fibre), whereas they are transcendental when viewed from M. The explicit expressions for x, y are well-known and are repeated for example in [**Boa05**].

Appendix B: connections on fibre bundles

A fibre bundle $\pi : M \to B$ is a surjective map π from a manifold M (the total space) to a manifold B (the base). This should satisfy various conditions, e.g. that all fibres are isomorphic: there is some manifold F (the standard fibre) such that each fibre $M_t := \pi^{-1}(t)$ is isomorphic to F and also that the bundle should be "locally trivialisable", meaning each point $t \in B$ of the base has a small neighbourhood $t \in U \subset B$ over which M decomposes as a product $M|_U := \pi^{-1}(U) \cong U \times F$.

By definition a connection on a fibre bundle is a "field of horizontal subspaces of the tangent bundle of M". Namely suppose $m \in M$ lies over $t \in B$ then the tangent space T_mM to M at m is a $\dim(M)$ dimensional vector space with a distinguished $\dim(F)$ dimensional subspace (the vertical directions or tangents to the fibres):

$$V_m := T_m(M_t) \subset T_mM.$$

A connection on M is a (smoothly varying) choice of complementary subspace to V_m in T_mM: i.e., a choice of "horizontal subspaces" $H_m \subset T_mM$ such that $H_m \oplus V_m = T_mM$.

Now if we have a (sufficiently short) smooth path in the base starting at the point t lying under m (i.e., a map $\gamma : [0, 1] \to B$ with $\gamma(0) = t$) then we can use the connection to lift γ to a *unique* path $\widetilde{\gamma}$ in M starting at m. (In brief the projection π sets up an isomorphism between H_m and the tangent space to B at t so the connection enables us to lift any tangent vector to t to a tangent vector to M at m — by requiring the lift to be in H_m. The lifted path $\widetilde{\gamma}$ is got by following these lifted tangent vectors as γ is traversed.)

The connection is said to be "complete" if any (not necessarily short) path in the base can be lifted in this way. (This would be automatic if the fibres were compact, but that will not be the case for our examples.) Given any path in the base, a complete connection thus gives an *isomorphism* between the fibres over the endpoints of the path: namely each point $m \in M_t$ maps to the other end of the lifted path starting at m).

The connection is "flat" if, for any two homotopic paths in the base (with the same end points), the corresponding lifts have the same endpoints. In particular two homotopic loops will lift to paths with the same endpoints. (Infinitesimally this can be restated as follows: the above procedure of using a connection to lift tangent vectors yields a map from the (sheaf of) vector fields on the base to the vector fields on M. The flatness of the connection means precisely that this is a Lie algebra map.)

Complete flat connections thus give preferred "horizontal" local trivialisations: namely if we have a contractible subset $U \subset B$ of the base then any two paths between any two points of U are homotopic. Thus the isomorphism given by the connection between any two fibres over two points of U is independent of the path chosen (in U). Thus we get a preferred trivialisation $M|_U \cong M_t \times U$ for any basepoint $t \in U$.

References

[BJL81] W. Balser, W. Jurkat & D. Lutz – On the reduction of connection problems for differential equations with an irregular singularity to ones with only regular singularities, I., *SIAM J. Math. Anal.* **12** (1981), no. 5, p. 691–721.

[Boa05] P. P. Boalch – From Klein to Painlevé via Fourier, Laplace and Jimbo, *Proc. London Math. Soc.* **90** (2005), no. 3, p. 167–208.

[Boa06] ———, The fifty-two icosahedral solutions to Painlevé VI, 2006.

[DM00] B. Dubrovin & M. Mazzocco – Monodromy of certain Painlevé-VI transcendents and reflection groups, *Invent. Math.* **141** (2000), no. 1, p. 55–147.

[Hit95] N. J. Hitchin – Poncelet polygons and the Painlevé equations, in *Geometry and analysis (Bombay, 1992)*, Tata Inst. Fund. Res., Bombay, 1995, p. 151–185.

[Hit03] ———, A lecture on the octahedron, *Bull. London Math. Soc.* **35** (2003), p. 577–600.

[Iwa02] K. Iwasaki – A modular group action on cubic surfaces and the monodromy of the Painlevé VI equation, *Proc. Japan Acad., Ser. A* **78** (2002), p. 131–135.

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