DYNAMICS OF THE SIXTH PAINLEVÉ EQUATION

by

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Abstract. — The sixth Painlevé equation is hiding extremely rich geometric structures behind its outward appearance. In this article, we give a complete picture of its dynamical nature based on the Riemann-Hilbert approach recently developed by the authors and using various techniques from algebraic geometry.

A large part of the contents can be extended to Garnier systems, while this article is restricted to the original sixth Painlevé equation.

***Résumé* (Dynamique de la sixième équation de Painlevé)**. — Malgré une apparente simplicité, l'équation de Painlevé VI cache des structures géométriques très riches. Nous en décrivons les aspects dynamiques en nous appuyant sur l'approche de type Riemann-Hilbert récemment développée par les auteurs et en utilisant différentes techniques issues de la géométrie algébrique.

Une grande partie de ces résultats peut être étendue aux systèmes de Garnier. Toutefois, dans cet article, nous nous limitons au cas de l'équation de Painlevé VI.

1. Introduction

The sixth Painlevé equation $P_{VI} = P_{VI}(\kappa)$ is among the six kinds of differential equations that were discovered by Painlevé [**65**] and his student Gambier [**18**] around the turn of the twentieth century. It is a second-order nonlinear ordinary differential

equation with an independent variable $x \in \mathbb{P}^1 - \{0, 1, \infty\}$ and an unknown function $q = q(x)$,

$$
\begin{aligned}
q_{xx} \quad &= \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-x}\right)q_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{q-x}\right)q_x \\
&\quad + \frac{q(q-1)(q-x)}{2x^2(x-1)^2}\left\{\kappa_4^2 - \kappa_1^2\frac{x}{q^2} + \kappa_2^2\frac{x-1}{(q-1)^2} + (1-\kappa_3^2)\frac{x(x-1)}{(q-x)^2}\right\},
\end{aligned}
\tag{1}
$$

depending on parameters $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ in a 4-dimensional affine space

$$
\mathcal{K} = \{\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 \ : \ 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1\}.
\tag{2}
$$

This highly nonlinear and seemingly rather ugly equation is only a small visible part of a more substantial entity. The large invisible part has extremely rich geometric structures that are related to symplectic geometry, moduli spaces of stable parabolic connections, moduli spaces of representations of a fundamental group, Riemann-Hilbert correspondence, geometry of cubic surfaces, braid and modular groups, simple isolated singularities and their resolutions of singularities, affine Weyl groups, discrete dynamical systems, and so on. The aim of this survey article is to discuss various aspects of these illuminating structures, giving a complete picture of the sixth Painlevé equation.

Among other features, Painlevé equation is primarily a dynamical system and a dynamical system in general is characterized by two aspects: *laws* and *phenomena*. Mathematically, laws refer to equations that govern the dynamics, symmetries of the system, etc., while phenomena refer to solutions of the equations, (global) behaviors of trajectories, etc. These two aspects often show a sharp contrast. For example, in classical mechanics, the simple laws of Newton create extremely rich and complicated phenomena. The Painlevé dynamics is also in this case, being algebraic in its laws and transcendental in its phenomena (see Table 1).

aspect	contents	nature
laws	equations, symmetry, ...	algebraic
phenomena	solutions, trajectories, ...	transcendental

TABLE 1. Two aspects of Painlevé equation

For comparison, we should remark that there exists an interesting dynamics whose laws are already transcendental, like a dynamics on a K3 surface recently explored by McMullen [**49**], who showed that the existence of Siegel disks implies the transcendence of the K3 surface.

Generally speaking, the two principal approaches to dynamical systems are perhaps:

$$(\mathrm{L}) \ \text{Lyapunov's methods}, \qquad (\mathrm{C}) \ \text{conjugacy methods}.$$

In Lyapunov's methods (L), we examine, control, or confine the behaviors of trajectories by estimating suitable functions called "Lyapunov functions". Main tools of the methods are estimations by inequalities. On the other hand, in the conjugacy methods (C), we try to find a "conjugacy map" that converts the difficult dynamical system we want to study to a more tractable one, to extract informations from the latter, and to send feedback to the former (see §2.2 for more details). Our approach to the Painlevé equation, which we call the *Riemann-Hilbert approach*, falls into this category (C), making use of Riemann-Hilbert correspondence as a conjugacy map between Painlevé flow and isomonodromic flow.

Of course, the Riemann-Hilbert approach is closely related to the isomonodromic approach represented by the classical works of Fuchs [**17**], Schlesinger [**71**], Garnier [**20**], Jimbo, Miwa and Ueno [**37, 38**] and others, but differs from the latter in its definitive employment of the method of conjugacy maps and in its extensive use of a complete solution to the Riemann-Hilbert problem. The Riemann-Hilbert approach *a priori* has a global nature once Riemann-Hilbert correspondence is formulated appropriately, while the isomonodromic approach mostly stands on the infinitesimal point of view and pays little attention to the target space of Riemann-Hilbert correspondence, namely, moduli space of monodromy representations. In the Riemann-Hilbert appraoch, we consciously distinguish the Painlevé flow on the moduli space of stable parabolic connections and the isomonodromic flow on the moduli space of monodromy representations, and build a bridge between them via the Riemann-Hilbert correspondence.

This approach has been explored by Iwasaki [**31, 32, 33, 34**], Hitchin [**24**], Kawai [**40, 41**], Boalch [**5, 6, 7**], Dubrovin and Mazzocco [**14**] and others. Recently it was thoroughly developed by Inaba, Iwasaki and Saito [**29, 28**]. The exposition of this article is largely based on the contents of the last papers. We focus our attention on the original case of P_{VI} with the aim of presenting, for the most basic model, those materials which can be expected to be universal throughout various generalizations. A large part of the contents is extended to Garnier systems, a several-variable version of P_{VI}; see [**29**].

In addtion to the general methods represented by approaches (L) and (C), which are conceivable in general situations, there are also various particular methods applicable to various particular dynamical systems. For example, for the class of dynamical systems that are called completely integrable systems, there exist

<center>(CI) methods for complete integration,</center>

which are characterized by such keywords as τ-functions, bilnear equations, Lax pairs, separations of variables, combinatorics and representation theory, etc. Painlevé equations are usually thought of as a member of this class and many works have been done from this point of view. See Conte [**10**], Noumi [**56**] and the references therein. But we shall not touch on this aspect in this article. Among other things, we wish to lay

a sound foundation on the sixth Painlevé equation to such an extent that it can be a basis for the investigations into the *transcendental* nature of P_{VI}. To do so, many things should be done, even within the general framework of dynamical systems, before entering into those subjects which are particular to integrable systems. Therefore the integrable aspects should be discussed later and elsewhere.

Lyapunov-type approaches to Painlevé equations will not at all be discussed in this article. There have been long traditions as well as recent developments of establishing Painlevé property by these methods. We refer to Painlevé [**65**], Hukuhara [**25**] (see Okamoto and Takano [**64**] for a part of these unpublished lectures), Joshi and Kruskal [**39**], Steinmetz [**76**], Shimomura [**72**], Iwasaki, Kimura, Shimomura and Yoshida [**35**], Gromak, Laine and Shimomura [**22**] and the references therein.

The organization of this article is as follows: In Section 2 we introduce a general formalism of dynamical systems and cast P_{VI} into this framework. We present the Guiding Diagram that encodes major dynamical natures of P_{VI}. Section 3 is devoted to the construction of moduli spaces of stable parabolic connections, which, in the dynamical context, means the construction of phase spaces of P_{VI}. In Section 4, after setting up moduli spaces of monodromy representations, we formulate Riemann-Hilbert correspondence, RH, and settle Riemann-Hilbert problems in suitable ways. In the dynamical context, theses parts correspond to the construction of conjugacy maps. In Section 5 we formulate isomonodromic flows $\mathcal{F}_{IMF}$ and Painlevé flows $\mathcal{F}_{P_{VI}}$ in such a manner that RH yields analytic conjugacy from $\mathcal{F}_{P_{VI}}$ to $\mathcal{F}_{IMF}$. Section 6 is devoted to the construction of a family of affine cubic surfaces, which enables us to describe all the previous constructions more explicitly. In Section 7 we give a characterization of Bäcklund transformations, namely, the symmetries of P_{VI}, in terms of Riemann-Hilbert correspondence. In Section 8 we describe the nonlinear monodromy or the Poincaré return map of P_{VI} that extracts the global nature of trajectories of P_{VI}. In Section 9 we characterize the classical components of P_{VI}, called the Riccati flows, in terms of singularities on cubic surfaces and their resolutions of singularities. In Section 10 we construct canonical coordinate systems of moduli spaces (phase spaces) which make it possible to write down the Painlevé dynamics explicitly. This article is closed with a brief summary, especially with the Closing Diagram, in Section 11.

2. Painlevé Equation as a Dynamical System

A complete picture of the sixth Painlevé equation is most clearly described in the framework of dynamical systems, or, more specifically as a time-dependent Hamiltonian system with Painlevé property. So we begin by establishing a general formalism of dynamical systems, based on which we shall develop our whole story.
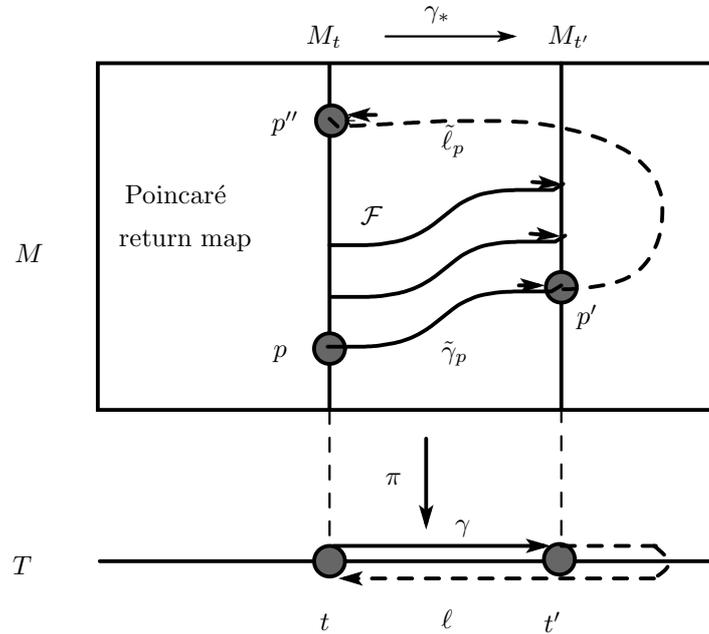
FIGURE 1. Dynamical system with Painlevé property

2.1. General Formalism of Dynamical Systems

Definition 2.1 **(Time-Dependent Dynamical System)**. — A *time-dependent dynamical system* $(M, \mathcal{F})$ is a fibration $\pi : M \to T$ of complex manifolds together with a complex foliation $\mathcal{F}$ on M that is transverse to each fiber $M_t = \pi^{-1}(t)$, $t \in T$. The total space M is referred to as the *phase space*, while the base space T is called the *space of time-variables*. Moreover, the fiber M_t is called the *space of initial conditions* at time t.

The space of initial conditions becomes a meaningful concept if the dynamical system enjoys Painlevé property. It is this property that makes it possible to think of Poincaré return maps or the nonlinear monodromy, which is the discrete dynamical system on a space of initial conditions that represents the global nature of a continuous dynamical system.

Definition 2.2 **(Geometric Painlevé Property)**. — We say that a dynamical system $(M, \mathcal{F})$ has *geometric Painlevé property* (GPP) if for any path γ in T and any point $p \in M_t$, where t is the initial point of γ, there exists a unique $\mathcal{F}$-horizontal lift $\tilde{\gamma}_p$ of γ with initial point at p (see Figure 1). Here a curve in M is said to be $\mathcal{F}$-horizontal if it lies in a leaf of $\mathcal{F}$.

Remark 2.3 **(Uniqueness)**. — Under the *transversality* assumption, the lifting problem is reduced to solving a Cauchy problem for a regular ODE along the curve γ. Hence the local existence and uniqueness of the lift $\tilde{\gamma}_p$ always hold, due to the classical Cauchy theorem on ODE's. The question in Definition 2.2 is the existence of the *global* lift $\tilde{\gamma}_p$ for any path γ in T.

Definition 2.4 **(Poincaré Return Map)**. — If $(M, \mathcal{F})$ has geometric Painlevé property, then any path γ in T with initial point t and terminal point t' induces an isomorphism

$$\gamma_* : M_t \to M_{t'}, \qquad p \mapsto p', \tag{3}$$

where p' is the terminal point of the lift $\tilde{\gamma}_p$. When γ is a loop in T with base point at t, we have an automorphism γ_* of M_t, which is called the *Poincare return map* along the loop γ. Since γ_* depends only on the homotopy class of γ, we have the group homomorphism

$$\pi_1(T, t) \to \operatorname{Aut} M_t, \qquad \gamma \mapsto \gamma_*,$$

which we call the *nonlinear monodromy* of the dynamical system $(M, \mathcal{F})$.

Definition 2.5 **(Hamiltonian System)**. — A dynamical system $(M, \mathcal{F})$ with Painlevé property is said to be *Hamiltonian* if $\pi : M \to T$ is a fibration of symplectic manifolds and the map (3) is a symplectic isomorphism for any path γ in T. If $(M, \mathcal{F})$ is Hamiltonian, then there exists a unique closed 2-form Ω on M, called the *fundamental 2-form* for $(M, \mathcal{F})$, such that

(1) the form Ω restricted to each fiber M_t yields the symplectic structure Ω_t on M_t,

(2) the form Ω vanishes along the foliation $\mathcal{F}$, that is,

$$\iota_v \Omega = 0, \tag{4}$$

for any $\mathcal{F}$-horizontal vector field v, where $\iota_v \Omega = \Omega(v, \cdot)$ stands for the interior product of Ω by v.

Remark 2.6 **(Differential Equations)**. — The condition (4), when expressed in terms of canonical local coordinates on M, leads to a Hamiltonian system of differential equations.

There are two definitions of Painlevé property; one is the geometric definition given in Definition 2.2, which is addressed to a foliation, and the other is the analytic one addressed to a *nonlinear differential equation*. As for the latter, a differential equation is said to have Painlevé property if any solution has no movable singularities other than movable poles. This traditional but rather ambiguous definition can be made rigorous by the following definition.
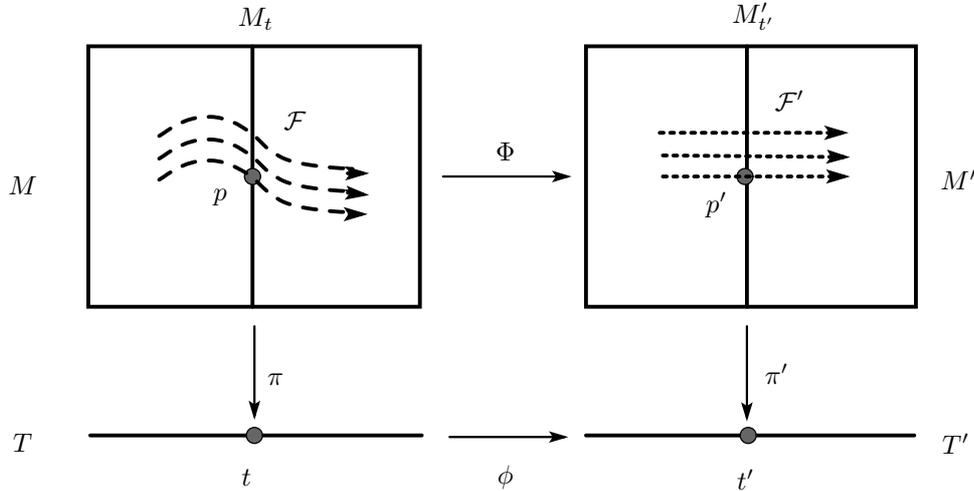
FIGURE 2. Conjugacy map

Definition 2.7 (**Analytic Painlevé Property**). — A nonlinear differential equation in a domain X is said to have *analytic Painlevé property* (APP) if any meromorphic solution germ at any point $x \in X$ has an analytic continuation as a meromorphic function, along any path γ in X emanating from x.

Here is a relation between geometric and analytic Painlevé properties.

Remark 2.8 (**GPP Versus APP**). — Given a dynamical system (foliation), assume that its phase space is an *algebraic variety*. Then, in terms of affine algebraic coordinates, the foliation is expressed as a differential equation and the geometric Painlevé property for the foliation implies the analytic Painlevé property for the differential equation.

In this sense the algebraicity of phase space is an important issue. Remark 2.8 will be applied to the Hamiltonian system of differential equaitons in Remark 2.6 and in particular to the Painlevé equation (see Theorem 10.12).

2.2. Conjugacy Maps. — As is mentioned in the Introduction, one of the major approaches in dynamical system theory is to find out a conjugacy map that converts a "difficult" dynamical system to an "easy" one; to extract as much information as possible from the latter; and to send feedback to obtain nontrivial (hopefully striking) results on the former.

Definition 2.9 (**Conjugacy**). — A *conjugacy map* Φ between two dynamical systems $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ is a commutative diagram as in Figure 2 such that

   (1) the map Φ is an isomorphism that preserves geometric structures under consideration, e.g., measure, topology, analytic structure, Hamiltonian structure, etc.,

(2) the foliation $\mathcal{F}$ is the pull-back of $\mathcal{F}'$ by Φ, that is, $\Phi^*\mathcal{F}' = \mathcal{F}$.

It is expected that a good conjugacy map should be highly transcendental, reflecting the difficulty of the "difficult" dynamical system. This transcendental nature would make the problem not so tractable but at the same time so attractive. A few examples of conjugacy maps are presented in Table 2, with some explanations below.

***Example 2.10* (Examples of Conjugacy)**

(1) The KdV flow is conjugated to an isospectral flow by the scattering map, whose inversion is the Gel'fand-Levitan-Marchenko procedure; a seminal discovery by Gardner, Green, Kruskal and Miura [**19**] which opened up the soliton theory.

(2) The quadratic dynamics $P_c(z) = z^2 + c$ on $\mathbb{C} - K_c$ is conjugated to the standard angle-doubling $P_0(z) = z^2$ on $\mathbb{C} - \bar{\mathbb{D}}$ by the Böttcher function (if K_c is connected), where K_c is the filled Julia set of P_c and $\bar{\mathbb{D}}$ is the closed unit disk. Using this fact, Douady and Hubbard [**13**] made deep studies on Julia sets and the Mandelbrot set for the quadratic dynamics on $\mathbb{C}$.

(3) If a dynamical system has a transverse homoclinic point, then it admits a horseshoe subdynamics, which in turn is conjugated to a symbolic dynamics. This is a famous discovery by Smale [**75**] which enables us to easily look at "chaos" generated by a homoclinic tangle.

(4) The Painlevé flow on a moduli space of stable parabolic connections is conjugated to an isomonodromic flow on a moduli space of monodromy representations by a Riemann-Hilbert correspondence.

The fourth example is exactly what is focused on in this article. As is mentioned in the Introduction, our approach is closely related with the isomonodromic deformation theory. But the latter has been mainly concerned with infinitesimal deformations of linear differential equations, without paying serious attentions to the global structure of Riemann-Hilbert correspondence, RH, especially to its target space, moduli space of monodromy representations. Let us repeat to say that our objective is to set up the source and target spaces of RH firmly; to establish a precise correspondence between them via RH; to interrelate two dynamics on both sides; and to understand the dynamics of P_{VI} through all these procedures. A similar situation seems to have occurred with KdV: While the machinery of inverse scattering method had been known since 1967 ([**19**]), it was not so soon that the precise correspondence was established between a class of potentials (of Schrödinger equations) and a class of scattering data, as in Deift and Trubowitz [**11**].

According to Definition 2.9, a conjugacy map Φ must be an isomorphism, namely, a *biholomorphism* in the case of holomorphic dynamics. However we sometimes come across such cases where this condition is too restrictive, that is, where the injectivity of Φ slightly fails to hold. To cover those cases, we make the following definition.

	"difficult" dynamics	"easy" dynamics	conjugacy map
1	KdV flow	isospectral flow	scattering map
2	quadratic dynamics	angle-doubling map	Böttcher function
3	horseshoe dynamics	symbolic dynamics	Smale's trick
4	Painlevé flow	isomonodromic flow	Riemann-Hilbert map

TABLE 2. Examples of conjugacy maps

Definition 2.11 **(Semi-Conjugacy)**. — A *semi-conjugacy map* Φ between two dynamical systems $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ is a commutative diagram as in Figure 2 such that the following conditions are satisfied:

(1) the map Φ is a surjective, proper, holomorphic map,
(2) there exists an $\mathcal{F}'$-invariant, analytic-Zariski closed, proper subset $Z' \subset M'$ such that $\Phi : M - Z \to M' - Z'$ is a biholomophism that preserves geometric structures under consideration, where $Z = \Phi^{-1}(Z')$,
(3) the foliation $\mathcal{F}$ restricted to $M - Z$ is the pull-back of $\mathcal{F}'$ on $M' - Z'$,
(4) Φ maps $\mathcal{F}$-trajectories in Z to $\mathcal{F}'$-trajectories in Z'.

Here the time-correspondence $\phi : T \to T'$ is assumed to be biholomorphic.

The properness condition on the map Φ in Definition 2.11 has a significant meaning for the geometric Painlevé property. Indeed the following lemma follows from the properness of Φ.

Lemma 2.12 **(Properness and GPP)**. — *Let* $\Phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ *be a semi-conjugacy map. Assume that the target dynamics* $(M', \mathcal{F}')$ *has geometric Painlevé property, then so does the source dynamics* $(M, \mathcal{F})$.

Proof. — Let γ be any compact path in T with initial point t, and let p be any point on M_t. By GPP for $(M', \mathcal{F}')$, the path $\phi(\gamma)$ in T' is lifted to $K' = \widetilde{\phi(\gamma)}_{\Phi(p)}$ emanating from $\Phi(p) \in M'_{\phi(t)}$. Since K' is compact, the properness of Φ implies that $K = \Phi^{-1}(K')$ is also compact. By conditions (3) and (4) of Definition 2.11, the Cauchy problem for constructing the lift $\tilde{\gamma}_p$ has a solution within K. Since K is compact, the solution $\tilde{\gamma}_p$ exists globally over γ. Hence GPP for $(M, \mathcal{F})$ follows. $\quad\square$

Lemma 2.12 is a guiding principle in establishing Painlevé property by the conjugacy method: The GPP for a difficult dynamical system follows from that for an easier one.

It will turn out that the Riemann-Hilbert correspondence is a conjugacy map (in the strict sense) for generic values of $\kappa \in \mathcal{K}$, but is only a semi-conjugacy map for nongeneric values of $\kappa \in \mathcal{K}$, due to the presence of what we call the Riccati locus

(see Definition 4.9). This locus carries classical trajectories that can be linearized in terms of Gauss hypergeometric equations (see Theorem 5.15).

2.3. Application to Painlevé Equation. — We apply the general formalism developed in the previous subsections to the Painlevé equation $\mathrm{P_{VI}}(\kappa)$. In Figure 3 we present the Guiding Diagram that will serve as guidelines on what we shall develop in the sequel. The main ingredients of the diagram are stated as follows.

***Ingredients of Guiding Diagram* (Figure 3)**

- T *is the configuration space of mutually distinct ordered four points in* $\mathbb{P}^1$,

$$T = \{\, t = (t_1, t_2, t_3, t_4) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 : t_i \neq t_j \quad for \quad i \neq j \,\},$$

  which serves as the space of time-variables (see also Remark 2.13).
- $\mathcal{M}(\kappa)$ *is the moduli space of rank-two stable parabolic connections on* $\mathbb{P}^1$ *having four regular singular points with fixed local exponents* $\kappa \in \mathcal{K}$ *(see Definitions 3.1 and 3.6). It serves as the phase space of* $\mathrm{P_{VI}}(\kappa)$ *as a dynamical system (the Painlevé flow in Theorem 5.10).*
- $\mathcal{M}_t(\kappa)$ *is the moduli space of stable parabolic connections with singular points fixed at* $t \in T$. *It serves as the space of initial conditions at time* t *for the Painlevé flow.*
- $\mathcal{R}_t(a)$ *is the moduli space of monodromy representations (up to Jordan equivalence),*

$$\rho : \pi_1(\mathbb{P}^1 - \{t_1, t_2, t_3, t_4\}, *) \to SL_2(\mathbb{C}),$$

  with a fixed local monodromy data $a \in A$ *(see Definition 4.2). It serves as the space of initial conditions at time* t *for the isomonodromic flow (see Definition 5.1).*
- $\mathcal{R}(a)$ *is the disjoint union of* $\mathcal{R}_t(a)$ *over* $t \in T$, *that is,*

$$\pi_a : \mathcal{R}(a) = \coprod_{t \in T} \mathcal{R}_t(a) \to T, \tag{5}$$

  which serves as the phase space of the isomonodromic flow (see Definition 5.1).
- RH_κ *is the Riemann-Hilbert correspondence that associates to each stable parabolic connection its monodromy representation (see Definition 4.5). It plays the role of a (semi-)conjugacy map between the Painlevé flow and the isomonodromic flow.*
- $\mathcal{S}(\theta)$ *is an affine cubic surface, which is a concrete realization of the moduli space* $\mathcal{R}_t(a)$ *of monodromy representations (see Theorem 6.5).*

Some further explanations should be added about the objects on the moduli space $\mathcal{M}(\kappa)$. Each point $Q \in \mathcal{M}(\kappa)$ is representing a rank-two stable parabolic connection on $\mathbb{P}^1$ with four singular points, which is a refined notion of a Fuchsian system with four regular singular points on $\mathbb{P}^1$, consisting of a data on an algebraic vector bundle, a logarithmic connection on it, a prescribed determinantal structure, and a parabolic
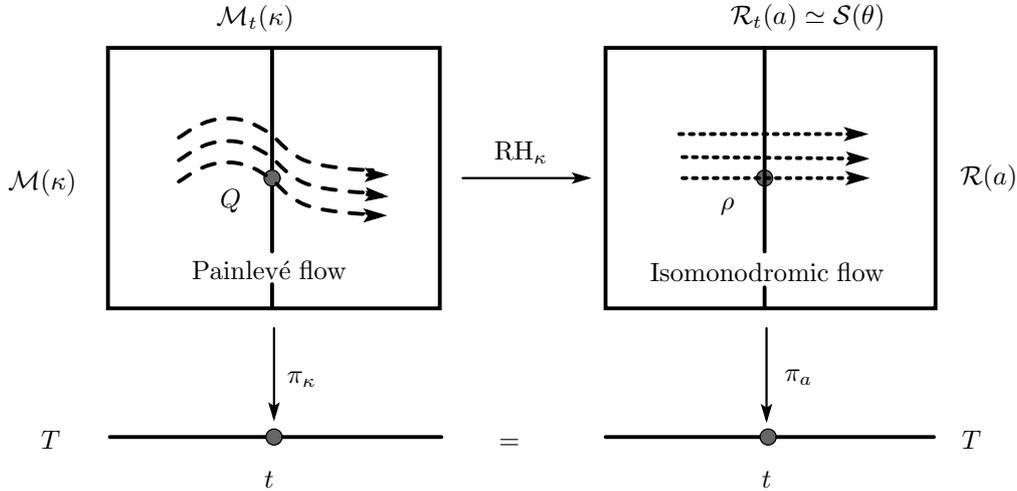
FIGURE 3. Guiding Diagram for P_{VI}

structure at the singular points (see Definition 3.1). The map $\pi_\kappa : \mathcal{M}(\kappa) \to T$ is the canonical projection associaitng to each connection Q its ordered regular singular points $t = (t_1, t_2, t_3, t_4)$.

As for the space of time variables, the following remark should be in order at this stage.

Remark 2.13 (**Reduction of Time Variables**). — The original Painlevé equation (1) has only one time-variable x, while our dynamical system has four time-variables $t = (t_1, t_2, t_3, t_4)$. The transition from t to x is achieved by a symplectic reduction that is explained as follows. The group of Möbius transformations $PSL_2(\mathbb{C})$ acts on T diagonally and this action can be lifted symplectically to the phase space $\mathcal{M}(\kappa)$ in such a manner that the lifted action is commutative with the Painlevé flow. So the space of time-variables T can be reduced to the quotient space

$$T/PSL_2(\mathbb{C}) \cong \mathbb{P}^1 - \{0, 1, \infty\}.$$

It is well known that a natural coordinate of the quotient space is given by the cross ratio

$$x = \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_2)(t_3 - t_4)}, \tag{6}$$

which gives the independent variable of the original Painlevé equation (1). This reduction amounts to just taking the normalization $t_1 = 0$, $t_2 = 1$, $t_3 = x$, $t_4 = \infty$. The transition from t to x brings slightly larger symmetry to the Painlevé equation (see Remark 7.6).

For the most part, we shall work with four time-variables $t = (t_1, t_2, t_3, t_4)$, but occasionally we shall make use of three time-variables $t = (t_1, t_2, t_3)$ upon putting $t_4 = \infty$, when such a convention is more convenient.

3. Moduli Spaces of Parabolic Connections – Phase Spaces

In our dynamical approach to Painlevé equation, first of all, we have to set up an appropriate phase space of P_{VI} as a dynamical system. It is realized as the moduli space of certain stable parabolic connections on $\mathbb{P}^1$. Following Inaba, Iwasaki and Saito [29] we shall briefly sketch its construction.

Before entering into the subject, we should remark that there exist related works by Arinkin and Lysenko [1, 3], who introduced moduli spaces of $SL(2)$-bundles with connections on $\mathbb{P}^1$ in the context of Painlevé equation. Unfortunately, they treated the moduli spaces mostly as stacks and restricted themselves to generic parameters in $\mathcal{K}$ to avoid reducible or resonant connections. For a full understanding of the Painlevé equation, however, the locus of nongeneric parameters often plays a significant part. In order to cover all parameters, we should take parabolic structures into account (see Remark 3.4 for this and for another reason). Moreover, in order to develop a good moduli theory in the framework of geometric invariant theory [51], we need the concept of stability. These demands lead us to consider stable parabolic connections.

3.1. Parabolic Connections. — In what follows, a vector bundle will be identified with the locally free sheaf associated to it. For a vector bundle E on $\mathbb{P}^1$ and a point $x \in \mathbb{P}^1$, we denote by $E|_x$ the fiber of E over x (not the stalk at x), namely we have $E|_x = E/E(-x)$ with $E(-x) = E \otimes \mathcal{O}_{\mathbb{P}^1}(-x)$.

Definition 3.1 (Parabolic Connection). — Given any $(t, \kappa) \in T \times \mathcal{K}$, a (t, κ)-*parabolic connection* is a quadruple $Q = (E, \nabla, \psi, l)$ such that the following conditions are satisfied:

(1) E is a rank-two vector bundle over $\mathbb{P}^1$.
(2) $\nabla : E \to E \otimes \Omega^1_{\mathbb{P}^1}(D_t)$ is a connection, where D_t is the divisor

$$D_t = t_1 + t_2 + t_3 + t_4.$$

(3) $\psi : \det E \to \mathcal{O}_{\mathbb{P}^1}(-t_4)$ is a horizontal isomorphism, where $\mathcal{O}_{\mathbb{P}^1}(-t_4) \subset \mathcal{O}_{\mathbb{P}^1}$ is equipped with the connection d_{t_4} induced from the exterior differentiation $d : \mathcal{O}_{\mathbb{P}^1} \to \Omega^1_{\mathbb{P}^1}$.
(4) $l = (l_1, l_2, l_3, l_4)$, where l_i is a 1-dimensional subspace of the fiber $E|_{t_i}$ over t_i such that

$$(\mathrm{Res}_{t_i}(\nabla) - \lambda_i \, \mathrm{id}_{E|_{t_i}})|_{l_i} = 0,$$

singularity	t_1	t_2	t_3	t_4
first exponent	$-\lambda_1$	$-\lambda_2$	$-\lambda_3$	$-\lambda_4$
second exponent	λ_1	λ_2	λ_3	$\lambda_4 - 1$
difference	κ_1	κ_2	κ_3	κ_4

TABLE 3. Riemann scheme: first exponents correspond to parabolic structures

namely, l_i is an eigenline of $\mathrm{Res}_{t_i}(\nabla)$ with eigenvalue λ_i, where $\mathrm{Res}_{t_i}(\nabla) \in \mathrm{End}(E|_{t_i})$ is the residue of ∇ at t_i and the parameter λ_i is defined so that

$$\kappa_i = \begin{cases} 2\lambda_i & (i = 1, 2, 3), \\ 2\lambda_4 - 1 & (i = 4). \end{cases} \tag{7}$$

The data l is called the *parabolic structure* of the parabolic connection Q.

Remark 3.2 (**Riemann Scheme**). — An eigenvalue of $-\mathrm{Res}_{t_i}(\nabla)$ is called a *local exponent* of ∇ at t_i. By condition (4) of Definition 3.1, one exponent at t_i is $-\lambda_i$ corresponding to the parabolic structure l_i (the first exponent). By condition (3) the eigenvalues of $\mathrm{Res}_{t_i}(\nabla)$ are summed up to the residue $\mathrm{Res}_{t_i}(d_{t_4})$ of the connection d_{t_4} on the line bundle $\mathcal{O}_{\mathbb{P}^1}(-t_4)$. Since

$$\mathrm{Res}_{t_i}(d_{t_4}) = \begin{cases} 0 & (i = 1, 2, 3), \\ 1 & (i = 4), \end{cases}$$

the exponents of ∇ at each singular point are given as in Table 3. Formula (7) means that κ_i is the *difference* of the second exponent from the first exponent at t_i. For brevity κ_i is often referred to as the local exponent at t_i. We remark that the fourth singular point t_4 is somewhat distinguished from the others.

Remark 3.3 (**Determinantal Structure**). — In condition (3) of Definition 3.1, the horizontal isomorphism $\psi : \det E \to \mathcal{O}_{\mathbb{P}^1}(-t_4)$ is referred to as the *determinantal structure* of the parabolic connection Q. Here the choice of $\mathcal{O}_{\mathbb{P}^1}(-t_4)$ as the target line bundle of ψ is just for convenience. More generally, a determinantal structure relative to L is conceivable for any line bundle L with a connection d_L, as a horizontal isomorphism $\psi : \det E \to L$.

There are, at least, two advantages of taking parabolic structures into account.

Remark 3.4 (**Advantages of Parabolic Structures**)

(1) The Riemann-Hilbert approach and the isomonodromic approach become feasible for *all* parameters $\kappa \in \mathcal{K}$. Without parabolic structures, people usually avoid nongeneric parameters for "technical" reasons, but they cannot be ruled out becuase many interesting phenomena occur at nongeneric parameters both moduli-theoretically and special-function-theoretically. Moreover, what is called the technical difficulty is in fact an essential difficulty.

(2) The technique of elementary transformations becomes available. Here elementary transformations are certain kinds of gauge transformations canonically associated to parabolic structures (see Definition 3.5). They, together with the powerful technique of Langton [**42**], play an important part in solving the Riemann-Hilbert problem (see Remark 4.13). They also serve as some portions of the Bäcklund transformations (see Definition 7.2).

Definition 3.5 (**Elementary Transformation**). — Let $Q = (E, \nabla, \psi, l)$ be a parabolic connection with a determinantal structure $\psi : \det E \to L$ and a parabolic structure $l = (l_1, l_2, l_3, l_4)$. The *elementary transform* of Q at t_i is the parabolic connection $\tilde{Q} = (\tilde{E}, \tilde{\nabla}, \tilde{\psi}, \tilde{l})$ defined in the following manner (see also Figure 4).

(1) The bundle $\tilde{E}$ is the subsheaf $\tilde{E} = \mathrm{Ker}[\, E \to E|_{t_i}/l_i\,]$, where $E \to E|_{t_i}/l_i$ is the composite of the canonical projections $E \to E/E(-t_i) = E|_{t_i}$ and $E|_{t_i} \to E|_{t_i}/l_i$. Note that $\tilde{E}$ is locally free, and hence is a vector bundle.

(2) The connection $\tilde{\nabla} = \nabla|_{\tilde{E}} : \tilde{E} \to \tilde{E} \otimes \Omega^1_{\mathbb{P}^1}(D_t)$ is the restriction of ∇ to the subsheaf $\tilde{E} \subset E$. This is well-defined because condition (4) of Definition 3.1 implies that ∇ maps $\tilde{E}$ into $\tilde{E} \otimes \Omega^1_{\mathbb{P}^1}(D_t)$.

(3) The determinantal structure $\tilde{\psi} = \psi|_{\det \tilde{E}} : \det \tilde{E} \to \tilde{L} := L(-t_i)$ is the restriction of ψ to the subsheaf $\det \tilde{E} \subset \det E$, where $L(-t_i) = L \otimes \mathcal{O}_{\mathbb{P}^1}(-t_i)$ is equipped with the connection $d_L \otimes d_{t_i}$ with d_{t_i} being the connection on $\mathcal{O}_{\mathbb{P}^1}(-t_i) \subset \mathcal{O}_{\mathbb{P}^1}$ induced from the exterior differentiation $d : \mathcal{O}_{\mathbb{P}^1} \to \Omega^1_{\mathbb{P}^1}$. This is well-defined because one has $\det \tilde{E} = (\det E)(-t_i)$ and ψ maps $(\det E)(-t_i)$ to $L(-t_i)$ by condition (4) of Definition 3.1.

(4) The parabolic structure $\tilde{l}_j$ at t_j is defined by

$$
\tilde{l}_j = \begin{cases} E(-t_i)/\tilde{E}(-t_i) & (j = i), \\ l_j & (j \neq i), \end{cases}
$$

where $\tilde{l}_i$ is well-defined, since $\tilde{l}_i = E(-t_i)/\tilde{E}(-t_i) \subset \tilde{E}/\tilde{E}(-t_i) = \tilde{E}|_{t_i}$.

There are other types of elementary transformations defined in similar manners; see [**29**]. Elementary transformations were intensively studied by Maruyama [**47**] and others. For parabolic structures appearing in various moduli problems, we refer to Maruyama and Yokogawa [**46**], Nakajima [**52**], Inaba [**27**] and the references therein.

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E(-t_i) & \longrightarrow & \tilde{E} & \longrightarrow & l_i & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E(-t_i) & \longrightarrow & E & \longrightarrow & E|_{t_i} & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & E|_{t_i}/l_i & == & E|_{t_i}/l_i & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

FIGURE 4. Diagram related to an elementary transformation

See also Huybrechts and Lehn [26], Nitsure [55] Simpson [73, 74] for related moduli problems.

3.2. Stability. — To obtain a good moduli space, namely, to avoid non-Hausdorff phenomena, we require a concept of stability for parabolic connections.

Definition 3.6 **(Stability)**. — A *weight* is a sequence of mutually distinct rational numbers

$$
\alpha = (\alpha_1, \alpha_1', \ldots, \alpha_4, \alpha_4') \qquad \text{such that} \qquad 0 < \alpha_i < \alpha_i' < 1.
$$

Given a weight α, a parabolic connection $Q = (E, \nabla, \psi, l)$ is said to be α-*stable* if for any proper subbundle $F \subset E$ such that $\nabla(F) \subset F \otimes \Omega^1_{\mathbb{P}^1}(D_t)$, one has

$$
\frac{\operatorname{pardeg} F}{\operatorname{rank} F} < \frac{\operatorname{pardeg} E}{\operatorname{rank} E}, \tag{8}
$$

where $\operatorname{pardeg} E$ and $\operatorname{pardeg} F$, called the parabolic degrees, are defined by

$$
\operatorname{pardeg} E = \deg E + \sum_{i=1}^{4} \left\{ \alpha_i \dim(E|_{t_i}/l_i) + \alpha_i' \dim l_i \right\} = \deg E + \sum_{i=1}^{4}(\alpha_i + \alpha_i'),
$$

$$
\operatorname{pardeg} F = \deg F + \sum_{i=1}^{4} \left\{ \alpha_i \dim(F|_{t_i}/l_i \cap F|_{t_i}) + \alpha_i' \dim(l_i \cap F|_{t_i}), \right\}
$$

The concept of α-*semistability* is defined in a similar manner by weakening the condition (8) so that it allows equality. A weight α is said to be *generic* if every α-simistable object is α-stable. Hereafter the weight will be assumed to be generic.

3.3. Moduli Space of Stable Parabolic Connections. — Based on arguments from geometric invariant theory, we can establish the following result [**29**].

Theorem 3.7 (Moduli Space). — *Fix a generic weight α.*

(1) *There exists a fine moduli scheme $\mathcal{M}_t(\kappa)$ of stable (t, κ)-parabolic connections.*
(2) *The moduli space $\mathcal{M}_t(\kappa)$ is a smooth, irreducible, quasi-projective surface.*
(3) *As a relative setting, there exists a family of moduli spaces*

$$\pi : \mathcal{M} \to T \times \mathcal{K}, \tag{9}$$

such that π is a smooth morphism whose fiber over $(t, \kappa) \in T \times \mathcal{K}$ is just $\mathcal{M}_t(\kappa)$.
(4) *Fixing an exponent $\kappa \in \mathcal{K}$, one can also speak of the family*

$$\pi_\kappa : \mathcal{M}(\kappa) \to T. \tag{10}$$

We insist that the fibration (10) gives a precise phase space of $\mathrm{P}_{\mathrm{VI}}(\kappa)$ as a time-dependent dynamical system. In this regard the following remark should be in order.

Remark 3.8 (Connections on Trivial Vector Bundle). — In the isomonodromic approach to P_{VI}, people usually work with linear Fuchsian systems of the form

$$\frac{dY}{dz} = A(z)\, Y, \qquad A(z) = \sum_{i=1}^{4} \frac{A_i}{z - t_i}, \tag{11}$$

namely, Fuchsian connections on the *trivial* vector bundle, and derive the Schlesinger system [**71**],

$$\frac{\partial A_i}{\partial t_i} = \sum_{k \neq i} \frac{[A_i, A_k]}{t_k - t_i}, \qquad \frac{\partial A_i}{\partial t_j} = \frac{[A_i, A_j]}{t_j - t_i} \qquad (i \neq j), \tag{12}$$

and then recast it to the Painlevé equation. In that case they are supposing that the totality of the connections in (11) forms a phase space of $\mathrm{P}_{\mathrm{VI}}(\kappa)$. However, it is only isomorphic to a Zariski-open proper subset of the true phase space, that is, our moduli space $\mathcal{M}(\kappa)$, and some trajectories actually escape from this open subset. Thus, with such a naïve setting of phase space as in (11), the geometric Painlevé property is not fulfilled, (although the analytic Painlevé property for the system (12) holds true as was proved[1] by Malgrange [**44**] and Miwa [**50**]). This is why we had to consider connections on nontrivial vector bundles together with the extra data of parabolic structures, in order to build a complete phase space. In our setting, the geometric Painlevé property holds quite naturally and then the analytic Painlevé property follows from this and the algebraicity of the phase space (see Theorem 5.12, Remark 2.8 and Theorem 10.12).

---

[1]under generic conditions on exponents

3.4. Parabolic ϕ-Connection. — As the moduli space $\mathcal{M}_t(\kappa)$ is quasi-projective, it is natural to pose the following problem.

Problem 3.9 (Compactification). — Compactify the moduli space $\mathcal{M}_t(\kappa)$ in a natural manner.

This problem is settled by introducing the notion of parabolic ϕ-connection, which is a generalized object of parabolic connections, allowing some degeneracy in the exterior differential part. This procedure reminds us of semi-classical limits of Schrödinger equations as the Planck constant tends to zero; we compactify the moduli space by adding some "semi-classical" objects.

Definition 3.10 (Parabolic ϕ-Connection). — For a fixed $(t, \kappa) \in T \times \mathcal{K}$, a (t, κ)-*parabolic ϕ-connection* is a sextuple $Q = (E_1, E_2, \phi, \nabla, \psi, l)$ such that the following conditions are satisfied:

(1) E_1 and E_2 are rank-two vector bundles over $\mathbb{P}^1$ of the same degree $\deg E_1 = \deg E_2$.

(2) $\phi : E_1 \to E_2$ is an $\mathcal{O}_{\mathbb{P}^1}$-homomorphism.

(3) $\nabla : E_1 \to E_2 \otimes \Omega^1_{\mathbb{P}^1}(D_t)$ is a $\mathbb{C}$-linear map such that

$$\nabla(fs) = \phi(s) \otimes df + f\nabla(s) \qquad \text{for} \qquad f \in \mathcal{O}_{\mathbb{P}^1},\, s \in E_1.$$

(4) $\psi : \det E_2 \to \mathcal{O}_{\mathbb{P}^1}(-t_4)$ is a horizontal isomorphism in the sense that

$$(\psi \otimes 1)(\phi(s_1) \wedge \nabla(s_2) + \nabla(s_1) \wedge \phi(s_2)) = d_{t_4}(\psi(\phi(s_1) \wedge \phi(s_2))) \qquad \text{for} \quad s_1, s_2 \in E_1.$$

(5) $l = (l_1, l_2, l_3, l_4)$, where l_i is a 1-dimensional subspace of the fiber $E_1|_{t_i}$ over t_i such that

$$(\mathrm{Res}_{t_i}(\nabla) - \lambda_i\, \phi|_{t_i})|_{l_i} = 0,$$

where $\mathrm{Res}_{t_i}(\nabla) \in \mathrm{Hom}(E_1|_{t_i}, E_2|_{t_i})$ is the residue of ∇ at t_i and λ_i is defined by formula (7).

We remark that a parabolic ϕ-connection is isomorphic to a parabolic connection if ϕ is an isomorphism, while it is thought of as a degenerate object if ϕ is not an isomorphism.

3.5. Stability. — Again, to get a good moduli space, we need a concept of stability for parabolic ϕ-connections. The following definition may be intricate at first glance, but works well in practice.

Definition 3.11 (Stability). — A *weight* is a sequence $\alpha = (\alpha_1, \alpha'_1, \ldots, \alpha_4, \alpha'_4)$ of mutually distinct rational numbers, together with positive integers β_1, β_2, γ, such that

$$(\beta_1 + \beta_2)\alpha_i < (\beta_1 + \beta_2)\alpha'_i < \beta_1, \qquad \gamma \gg 0.$$

A (t, κ)-parabolic ϕ-connection $Q = (E_1, E_2, \phi, \nabla, \psi, l)$ is said to be (α, β, γ)-*stable* if for any proper subbundle $(F_1, F_2) \subset (E_1, E_2)$ such that $\phi(F_1) \subset F_2$ and $\nabla(F_1) \subset F_2 \otimes \Omega^1_{\mathbb{P}^1}(D_t)$, one has

$$\frac{\mathrm{pardeg}(F_1, F_2)}{\beta_1 \mathrm{rank}\, F_1 + \beta_2 \mathrm{rank}\, F_2} < \frac{\mathrm{pardeg}(E_1, E_2)}{\beta_1 \mathrm{rank}\, E_1 + \beta_2 \mathrm{rank}\, E_2}, \tag{13}$$

where $\mathrm{pardeg}(E_1, E_2)$ and $\mathrm{pardeg}(F_1, F_2)$ are define by

$$
\begin{aligned}
\mathrm{pardeg}(E_1, E_2) &= \beta_1 \deg E_1(-D_t) + \beta_2 (\deg E_2 - \gamma \mathrm{rank}\, E_2) \\
&\quad + (\beta_1 + \beta_2) \sum_{i=1}^{4} \left\{ \alpha_i \dim(E_1|_{t_i}/l_i) + \alpha'_i \dim l_i \right\}, \\
\mathrm{pardeg}(F_1, F_2) &= \beta_1 \deg F_1(-D_t) + \beta_2 (\deg F_2 - \gamma \mathrm{rank}\, F_2) \\
&\quad + (\beta_1 + \beta_2) \sum_{i=1}^{4} \left\{ \alpha_i \dim(F_1|_{t_i}/l_i \cap F_1|_{t_i}) + \alpha'_i \dim(l_i \cap F_1|_{t_i}) \right\},
\end{aligned}
$$

The concept of (α, β, γ)-*semistability* is defined in a similar manner by weakening the condition (13) so that it allows equality. A weight (α, β, γ) is said to be *generic* if every (α, β, γ)-simistable object is (α, β, γ)-stable. Hereafter the weight will be assumed to be generic.

3.6. Moduli Space of Stable Parabolic ϕ-Connections. — Again, based on arguments from geometric invariant theory, we have the following result [**29**].

Theorem 3.12 (**Moduli Space**). — *Fix a generic weight* (α, β, γ).

(1) *There is a coarse moduli scheme* $\overline{\mathcal{M}}_t(\kappa)$ *of stable* (t, κ)-*parabolic ϕ-connections.*

(2) *The moduli space* $\overline{\mathcal{M}}_t(\kappa)$ *is a smooth, irreducible, projective surface.*

(3) *The moduli space* $\mathcal{M}_t(\kappa)$ *is embedded into the compactified space* $\overline{\mathcal{M}}_t(\kappa)$ *by the natural map*

$$\mathcal{M}_t(\kappa) \hookrightarrow \overline{\mathcal{M}}_t(\kappa), \qquad (E, \nabla, \psi, l) \mapsto (E, E, \mathrm{id}, \nabla, \psi, l),$$

the image of which is the open subscheme of all stable (t, κ)-*parabolic ϕ-connections* $Q = (E_1, E_2, \phi, \nabla, \psi, l)$ *such that* $\phi : E_1 \to E_2$ *is an isomorphism.*

(4) *As a relative setting, there exists a family of moduli spaces*

$$\overline{\pi} : \overline{\mathcal{M}} \to T \times \mathcal{K},$$

such that $\overline{\pi}$ *is a smooth, projective morphism whose fiber over* $(t, \kappa) \in T \times \mathcal{K}$ *is just the compactified moduli space* $\overline{\mathcal{M}}_t(\kappa)$.

(5) *There exists a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathrm{embedding}} & \overline{\mathcal{M}} \\
\pi \downarrow & & \downarrow \overline{\pi} \\
T \times \mathcal{K} & =\!=\!=\!= & T \times \mathcal{K}.
\end{array}
$$

3.7. Realization of Moduli Spaces. — The moduli space $\mathcal{M}_t(\kappa)$ of stable parabolic (t, κ)-connections, together with its compactification $\overline{\mathcal{M}}_t(\kappa)$, admits a concrete realization in terms of the Hirzebruch surface Σ_2 of degree 2. The surface Σ_2 is the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ whose cross section at infinity, denoted by F_0, has self-intersection number -2. Moreover $\Sigma_2 - F_0$ is isomorphic to the line bundle $\Omega^1_{\mathbb{P}^1}(D_t)$ over $\mathbb{P}^1$. Given any $t = (t_1, t_2, t_3, t_4) \in T$ and $i \in \{1, 2, 3, 4\}$, let F_i denote the fiber over t_i of the fibration $\Sigma_2 \to \mathbb{P}^1$. Then we have the following theorem from Inaba, Iwasaki and Saito [**29**].

Theorem 3.13 (Realization of Moduli Spaces). — *Let $(t, \kappa) \in T \times \mathcal{K}$ be fixed.*

(1) *$\overline{\mathcal{M}}_t(\kappa)$ is an 8-point blow-up of the Hirzebruch surface Σ_2 of degree 2, blown up at certain two points on each fiber F_i, $i = 1, 2, 3, 4$. The location of the blowing-up points, possibly infinitely near, is determined by the value of κ.*

(2) *$\overline{\mathcal{M}}_t(\kappa)$ has a unique effective anti-canonical divisor*

$$Y_t(\kappa) = 2E_0 + E_1 + E_2 + E_3 + E_4 \in \left| -K_{\overline{\mathcal{M}}_t(\kappa)} \right|,$$

where E_i is the strict transform of F_i for $i = 0, 1, 2, 3, 4$. Each irreducible component E_i of $Y_t(\kappa)$ satisfies the condition

$$K_{\overline{\mathcal{M}}_t(\kappa)} \cdot E_i = 0 \qquad (i = 0, 1, 2, 3, 4).$$

(3) *$\mathcal{M}_t(\kappa)$ is obtained from $\overline{\mathcal{M}}_t(\kappa)$ by removing $Y_t(\kappa)_{\mathrm{red}}$.*

Note that $\overline{\mathcal{M}}_t(\kappa)$ is an example of generalized Halphen surfaces (see Definition 3.14), which were introduced and classified by Sakai [**70**]; namely, a surface of type $D_4^{(1)}$ in his classification.

Definition 3.14 (Generalized Halphen Surface). — A smooth, projective, rational surface S is called a *generalized Halphen surface* if S has an effective anti-canonical divisor

$$Y \in |-K_S| \qquad \text{such that} \qquad K_S \cdot Y_i = 0 \quad (i = 1, \ldots, r), \tag{14}$$

where $Y_1, \ldots, Y_r$ are the irreducible components of Y.

This notion was introduced to construct discrete Painlevé equations as Cremona transformations of generalized Halphen surfaces and to obtain continuous Painlevé equations as their continuous limits (Cremona approach in Remark 3.17).

Furthermore, the pair $(\overline{\mathcal{M}}_t(\kappa), Y_t(\kappa))$ is an instance of Okamoto-Painlevé pairs (see Definition 3.15), which were introduced and classified by Saito, Takebe and Terajima [**66, 67**]; namely, a pair of type $\tilde{D}_4$ (or of type I_0^* in Kodaira's notation) in their classification.

Definition 3.15 (Okamoto-Painlevé Pair). — A pair (S, Y) is called a *generalized Okamoto-Painlevé pair* if S is a smooth, projective surface and $Y \in |-K_S|$ is an
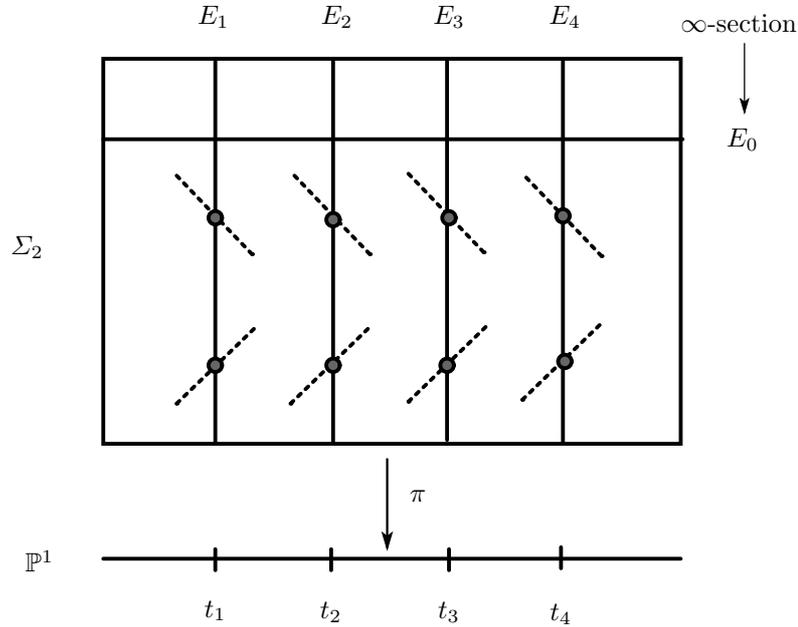
FIGURE 5. 8-point Blow-up of Hirzebruch surface of degree 2

effective anti-canonical divisor satisfying the condition (14). It is called an *Okamoto-Painlevé pair* if moreover $S - Y_{\mathrm{red}}$ contains an affine plane $\mathbb{C}^2$ as a Zariski open subset and $F := S - \mathbb{C}^2$ is a (reduced) divisor with normal crossings.

This notion was introduced to construct continuous Painlevé equations as Kodaira-Spencer deformations of Okamoto-Painlevé pairs (Kodaira-Spencer approach in Remark 3.17).

Definitions 3.14 and 3.15 were invented by speculating on the meanings of the spaces constructed by Okamoto [**59**]. Here is a comparison of our moduli spaces with his spaces.

Remark 3.16 (Comparison with Okamoto's space). — Theorem 3.13 implies that our phase space $\mathcal{M}(\kappa)$ is isomorphic to the space constructed by Okamoto [**59**]. He constructed it by hand, chasing trajectories of differential equation (1)[(2)], blowing up the points where distinct trajectories meet together and removing the vertical leaves. Our construction is more theoretical and intrinsic[(3)]. More importantly, our moduli-theoretical construction immediately allows us to consider the Riemann-Hilbert correspondence from the constructed space (to a moduli space of monodromy representations), since each point of which represents a parabolic connection. This means

---

[(2)]to be more precise, a Hamitoninan system associated to equation (1).

[(3)]Painlevé property follows from our construction, while it was presupposed in his construction.

that we are in a happy situation that the construction of the phase space immediately results in the construction of a natural conjugacy map.

Digressively, we take this opportunity to collect the major approaches to Painlevé equations we have ever encountered. Gathering those mentioned in the Introduction and those remarked after Definitions 3.14 and 3.15, we have (at least) five approaches.

***Remark 3.17* (Approaches to Painlevé Equations)**

(1) Isomonodromic (Fuchs) approach (2) Lyapunov approach

(3) Cremona approach (4) Kodaira-Spencer approach

(5) Riemann-Hilbert approach

As is mentioned in the Introduction, the isomonodromic approach and the Riemann-Hilbert approach are close relatives. In this context, the meaning of our moduli-theoretical construction is that we were able to match Okamoto's spaces with the isomonodromic picture, which had hitherto existed independently, in the framework of Riemann-Hilbert approach. On the other hand, his spaces have *a priori* had their raison d'être in the Cremona and Kodaira-Spencer approaches, since these approaches originate from searches for their intrinsic meanings.

4. Riemann-Hilbert Correspondence — Conjugacy Map

In the Riemann-Hilbert approach, undoubtedly, the Riemann-Hilbert correspondence plays a central part, as a (semi-)conjugacy map between the Painlevé flow and the isomonodromic flow. We start with some basic notions concerning monodromy representations.

4.1. Monodromy Representations. — Given $t \in T$, we consider representations of the fundamental group $\pi_1(\mathbb{P}^1 - D_t, *)$ into $SL_2(\mathbb{C})$, where the divisor D_t is identified with the 4-point set $\{t_1, t_2, t_3, t_4\}$. Recall that two representations ρ_1 and ρ_2 are said to be *isomorphic* if there exists a matrix $P \in SL_2(\mathbb{C})$ such that

$$\rho_2(\gamma) = P\rho_1(\gamma)P^{-1} \qquad \text{for any} \quad \gamma \in \pi_1(\mathbb{P}^1 - D_t, *).$$

For a precise formulation of the Riemann-Hilbert correspondence, we need the concept of Jordan equivalence of representations, which is closely related to the categorical-quotient construction in algebraic geometry. We insist that the usual equivalence up to isomorphisms is not appropriate, because the set of all representations up to isomorphisms is *not* an algebraic variety. A more substantial reason will gradually be clear in the course of discussions: by a categorical-quotient formulation, the Riemann-Hilbert correspondence will become a resolution of singularities.
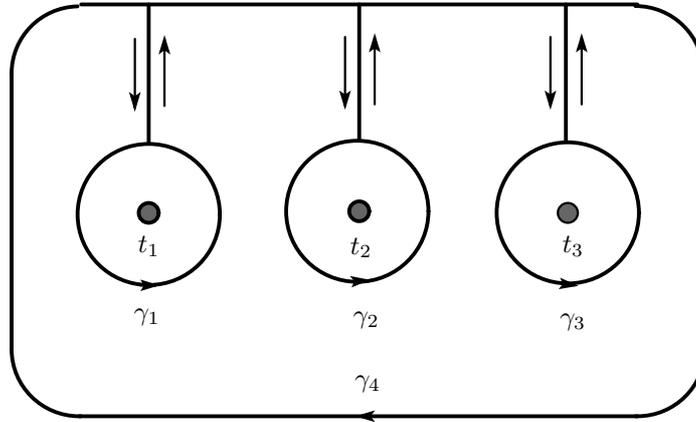
FIGURE 6. The loops γ_i; the fourth point t_4 is outside γ_4, invisible.

Definition 4.1 (**Jordan Equivalence**). — A *semisimplification* of a representation ρ is the associated graded of a composition series of ρ. Two representations ρ_1 and ρ_2 are said to be *Jordan equivalent* if they have isomorphic simisimplifications, that is, if either

(1) they are both irreducible and isomorphic, or

(2) they are both reducible and their semisimplifications $\rho_1' \oplus \rho_1/\rho_1'$ and $\rho_2' \oplus \rho_2/\rho_2'$ are isomorphic, where ρ_1' and ρ_2' are 1-dimensional subrepresentations of ρ_1 and ρ_2, respectively.

If there is no danger of confusion, a representation and its Jordan equivalence class will be denoted by the same symbol. For each $t \in T$ let $\mathcal{R}_t$ denote the set of all Jordan equivalence classes of $SL_2(\mathbb{C})$-representarions of $\pi_1(\mathbb{P}^1 - D_t, *)$. We can also speak of the family

$$\mathcal{R} = \coprod_{t \in T} \mathcal{R}_t. \tag{15}$$

Definition 4.2 (**Local Monodromy Data**). — We put $A := \mathbb{C}^4$ and consider the map

$$\pi_t : \mathcal{R}_t \to A, \quad \rho \mapsto a = (a_1, a_2, a_3, a_4), \qquad a_i = \operatorname{Tr} \rho(\gamma_i). \tag{16}$$

where $\gamma_i \in \pi_1(\mathbb{P}^1 - D_t, *)$ is a loop surrounding the point t_i anti-clockwise, leaving the remaining three points outside, as in Figure 6. Note that a_i is well-defined, that is, it depends only on the Jordan equivalence class of ρ and does not depend on the choice of loop γ_i. We call a the *local monodromy data* of ρ. For each $a \in A$ let $\mathcal{R}_t(a)$ denote the fiber of the map (16) over a. As the relative setting of (16) over T, we have the family

$$\pi : \mathcal{R} \to T \times A,$$

where $\mathcal{R}$ is defined by (15). For a fixed $a \in A$ we also have the family $\pi_a : \mathcal{R}(a) \to T$ as in (5).

4.2. Riemann-Hilbert Correspondence. — To formulate the Riemann-Hilbert correspondence, we first set it up in the parameter level.

Definition 4.3 **(Riemann-Hilbert Correspondence in Parameter Level)**
We consider the correspondence of local exponents to local monodromy data

$$\mathrm{rh} : \mathcal{K} \to A, \qquad \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \mapsto a = (a_1, a_2, a_3, a_4). \tag{17}$$

From Table 3 the monodromy matrix $\rho(\gamma_i)$ along the loop γ_i has eigenvalues $\exp(\pm 2\pi \sqrt{-1} \lambda_i)$ and hence has trace $2 \cos 2\pi \lambda_i$. Then (7) and (16) imply that in terms of exponents $\kappa \in \mathcal{K}$, the local monodromy data of ρ is expressed as

$$a_i = \begin{cases} 2\cos \pi \kappa_i & (i = 1, 2, 3), \\ -2\cos \pi \kappa_4 & (i = 4). \end{cases} \tag{18}$$

The map (17) with (18) is called the *Riemann-Hilbert correspondence in the parameter level.*

Definition 4.4 **(Riemann-Hilbert Correspondence)**. — Given $t \in T$, any stable parabolic connection $Q = (E, \nabla, \psi, l) \in \mathcal{M}_t$, upon restricted to $\mathbb{P}^1 - D_t$, induces a flat connection

$$\nabla|_{\mathbb{P}^1 - D_t} : E|_{\mathbb{P}^1 - D_t} \to E|_{\mathbb{P}^1 - D_t} \otimes \Omega^1_{\mathbb{P}^1 - D_t}.$$

Let ρ be the Jordan equivalence class of its monodromy representation. Then the *Riemann-Hilbert correspondence* at time t is defined by the holomorphic map

$$\mathrm{RH}_t : \mathcal{M}_t \to \mathcal{R}_t, \qquad Q \mapsto \rho.$$

By Definition 4.3 there exists a commutative diagram of holomorphic maps

$$
\begin{array}{ccc}
\mathcal{M}_t & \xrightarrow{\mathrm{RH}_t} & \mathcal{R}_t \\
\pi_t \downarrow & & \downarrow \pi_t \\
\mathcal{K} & \xrightarrow{\mathrm{rh}} & A,
\end{array}
\tag{19}
$$

where $\pi_t : \mathcal{M}_t \to \mathcal{K}$ is the map sending each parabolic connection to its local exponents and the map $\pi_t : \mathcal{R}_t \to A$ is defined by (16). As the relative setting of (19) over T, we have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathrm{RH}} & \mathcal{R} \\
\pi \downarrow & & \downarrow \pi \\
T \times \mathcal{K} & \xrightarrow{\mathrm{id} \times \mathrm{rh}} & T \times A.
\end{array}
\tag{20}
$$

Since rh : $\mathcal{K} \to A$ is an infinite-to-one map, so is the base map of (20). This fact makes the analysis of (20) somewhat difficult. To avoid this, we consider instead the fiber product $\boldsymbol{\mathcal{R}}$ defined by

$$
\begin{array}{ccc}
\boldsymbol{\mathcal{R}} & \longrightarrow & \mathcal{R} \\
\pi \downarrow & & \downarrow \pi \\
T \times \mathcal{K} & \xrightarrow{\mathrm{id} \times \mathrm{rh}} & T \times A.
\end{array}
\tag{21}
$$

We now set up three versions of Riemann-Hilbert correspondence that will be used in what follows.

Définition 4.5 (Three Versions of Riemann-Hilbert Correspondence)

(1) From (20) and (21) we have the commutative diagram of holomorphic maps

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathrm{RH}} & \boldsymbol{\mathcal{R}} \\
\pi \downarrow & & \downarrow \pi \\
T \times \mathcal{K} & =\!=\!=\!= & T \times \mathcal{K},
\end{array}
\tag{22}
$$

which is called the *full-Riemann-Hilbert correspondence*.

(2) Fix an exponent $\kappa \in \mathcal{K}$ and put $a = \mathrm{rh}(\kappa) \in A$. Then (20) restricts to the diagram

$$
\begin{array}{ccc}
\mathcal{M}(\kappa) & \xrightarrow{\mathrm{RH}_\kappa} & \mathcal{R}(a) \\
\pi_\kappa \downarrow & & \downarrow \pi_a \\
T & =\!=\!= & T,
\end{array}
\tag{23}
$$

which is referred to as the *κ-Riemann-Hilbert correspondence*.

(3) Moreover, upon fixing a time $t \in T$, diagram (23) further restricts to the map

$$
\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{R}_t(a),
\tag{24}
$$

which is referred to as the *(t, κ)-Riemann-Hilbert correspondence*.

Among the three versions above, the importance of (23) and (24) is obvious: (23) will serve as a (semi-)conjugacy map of Painlevé flow to isomonodromic flow, while (24) will give a correspondence between the spaces of initial conditions for these two dynamics. On the other hand, although it is not yet clear, (22) will play an important part in constructing Painlevé flows based on "codimension-two argument" (see Lemma 4.16 and Remark 5.11).

The Riemann-Hilbert problem usually asks the surjectivity of Riemann-Hilbert correspondence. But the injectivity and properness are also important issues in our situation.

Problem 4.6 (**Riemann-Hilbert Problem**). — We formulate the problems for RH_κ in (23). Those for RH and $\mathrm{RH}_{t,\kappa}$ are formulated in similar manners.

(1) Is RH_κ surjective? This question is fundamental for the whole development of the story.

(2) To what extent RH_κ is injective? This question is important for the setup of RH_κ as a (semi-)conjugacy map between the Painlevé flow and the isomonodromic flow.

(3) Is RH_κ a proper map? This question is important because the properness of RH_κ leads to the geometric Painlevé property of the Painlevé flow (see Lemma 2.12).

In what follows, Riemann-Hilbert problem will often be abbreviated to RHP. Problems for RH, RH_κ and $\mathrm{RH}_{t,\kappa}$ will be referred to as full-RHP, κ-RHP and (t, κ)-RHP, respectively.

4.3. Affine Weyl Group of Type $D_4^{(1)}$. — Before stating our solution to the Riemann-Hilbert problem, we introduce an affine Weyl group of type $D_4^{(1)}$ acting on the parameter space $\mathcal{K}$ (see Definition 4.7) and characterize the singularities of $\mathcal{R}_t(a)$ in terms of the affine Weyl group structure (see Lemma 4.8). In connection with the singularity structure, we introduce the concept of Riccati loci (see Defintion 4.9).

Definition 4.7 (**Affine Weyl Group**). — The parameter space $\mathcal{K}$ in (2) is an affine space modeled on the four-dimensional linear space

$$K = \{k = (k_0, k_1, k_2, k_3, k_4) \in \mathbb{C}^5 : 2k_0 + k_1 + k_2 + k_3 + k_4 = 0\},$$

endowed with the inner product $\langle k, k' \rangle = k_1 k_1' + k_2 k_2' + k_3 k_3' + k_4 k_4'$. Let σ_i be the orthogonal affine reflection on $\mathcal{K}$ having $\{\kappa \in \mathcal{K} : \kappa_i = 0\}$ as its reflecting hyperplane. We observe that the group generated by $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4$ is an affine Weyl group of type $D_4^{(1)}$ (see Figure 7),

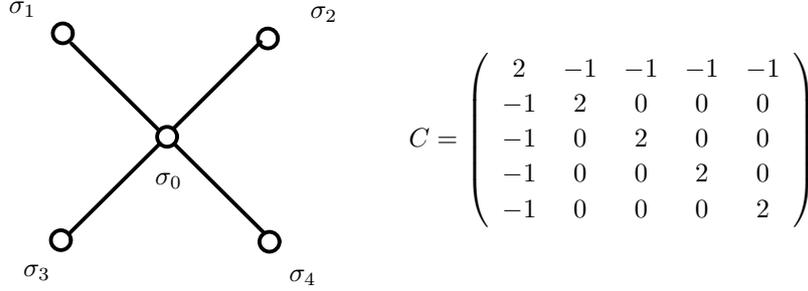$$W(D_4^{(1)}) = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle.$$

The i-th basic reflection σ_i is expressed as

$$\sigma_i(\kappa_j) = \kappa_j - \kappa_i c_{ij}, \tag{25}$$

where $C = (c_{ij})$ is the Cartan matrix of type $D_4^{(1)}$. Let **Wall** $\subset \mathcal{K}$ denote the union of the reflecting hyperplanes of all reflections in $W(D_4^{(1)})$.

We remark that a more intrinsic presentation of Definition 4.7 is possible along the line of Arinkin and Lysenko [**2**], as the Weyl group on the Picard lattice of the moduli space $\mathcal{M}_t(\kappa)$.

Let $\mathcal{R}_t^\mathrm{s}(a)$ be the singular locus and $\mathcal{R}_t^\circ(a) = \mathcal{R}_t(a) - \mathcal{R}_t^\mathrm{s}(a)$ be the smooth locus of $\mathcal{R}_t(a)$, respectively. The affine Weyl group structure allows us to describe the singularities of $\mathcal{R}_t(a)$.

$$C = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

FIGURE 7. Dynkin diagram and Cartan matrix of type $D_4^{(1)}$

Lemma 4.8 (Singularity). — *Let $\kappa \in \mathcal{K}$ and put $a = \mathrm{rh}(\kappa) \in A$.*

  (1) *The surface $\mathcal{R}_t(a)$ is smooth, that is, $\mathcal{R}_t^{\mathrm{s}}(a) = \varnothing$ if and only if $\kappa \notin \mathbf{Wall}$.*

  (2) *If $\kappa \in \mathbf{Wall}$, the singular locus $\mathcal{R}_t^{\mathrm{s}}(a)$ consists of at most four rational double points.*

The possible types of singularities on $\mathcal{R}_t(a)$ will be classified completely in Theorem 9.4. In connection with the singular loci of the surfaces $\mathcal{R}_t(a)$, we make the following definition.

Definition 4.9 (Riccati/Non-Riccati Loci). — The *Riccati loci* are defined by

$$\boldsymbol{\mathcal{R}}^{\mathrm{r}} = \coprod_{(t,\kappa) \in T \times \mathcal{K}} \mathcal{R}_t^{\mathrm{s}}(\mathrm{rh}(\kappa)), \qquad \mathcal{M}^{\mathrm{r}} = \mathrm{RH}^{-1}(\boldsymbol{\mathcal{R}}^{\mathrm{r}}).$$

By Lemma 4.8 the disjoint union may be taken only over $T \times \mathbf{Wall}$. The *non-Riccati loci*

$$\boldsymbol{\mathcal{R}}^{\circ} = \boldsymbol{\mathcal{R}} - \boldsymbol{\mathcal{R}}^{\mathrm{r}}, \qquad \mathcal{M}^{\circ} = \mathcal{M} - \mathcal{M}^{\mathrm{r}}$$

are the complements to the Riccati loci. These loci are restricted to the subspaces $\mathcal{R}(a)$, $\mathcal{R}_t(a)$, $\mathcal{M}(\kappa)$, $\mathcal{M}_t(\kappa)$ with $a = \mathrm{rh}(\kappa)$ in an obvious manner: The Riccati loci for them are defined by

$$\mathcal{R}^{\mathrm{r}}(a) = \coprod_{t \in T} \mathcal{R}_t^{\mathrm{s}}(a), \qquad \mathcal{M}^{\mathrm{r}}(\kappa) = \mathrm{RH}_{\kappa}^{-1}(\mathcal{R}^{\mathrm{r}}(a)),$$

$$\mathcal{R}_t^{\mathrm{r}}(a) = \mathcal{R}_t^{\mathrm{s}}(a), \qquad \mathcal{M}_t^{\mathrm{r}}(\kappa) = \mathrm{RH}_{t,\kappa}^{-1}(\mathcal{R}_t^{\mathrm{r}}(a)).$$

The corresponding non-Riccati loci are the complements to them:

$$\mathcal{R}^{\circ}(a) = \mathcal{R}(a) - \mathcal{R}^{\mathrm{r}}(a), \qquad \mathcal{M}^{\circ}(\kappa) = \mathcal{M}(\kappa) - \mathcal{M}^{\mathrm{r}}(\kappa),$$

$$\mathcal{R}_t^{\circ}(a) = \mathcal{R}_t(a) - \mathcal{R}_t^{\mathrm{r}}(a), \qquad \mathcal{M}_t^{\circ}(\kappa) = \mathcal{M}_t(\kappa) - \mathcal{M}_t^{\mathrm{r}}(\kappa).$$

It will turn out that Riccati loci are closely related to the so-called Riccati solutions of the Painlevé equation. This fact motivates the terminology *Riccati locus* (see §5.5).

4.4. Solution to Riemann-Hilbert Problem. — We are now in a position to state our solution to the Riemann-Hilbert problem [**29**].

***Theorem 4.10** (**Solution to Full-RHP**)*

(1) $\mathrm{RH} : \mathcal{M} \to \mathcal{R}$ *is a surjective proper holomorphic map, and*
(2) $\mathrm{RH} : \mathcal{M}^\circ \to \mathcal{R}^\circ$ *is a biholomorophism.*

Restricting this theorem to each $\kappa \in \mathcal{K}$, we have the following corollary.

***Corollary 4.11** (**Solution to κ-RHP**)*. — *Let $\kappa \in \mathcal{K}$ and put $a = \mathrm{rh}(\kappa) \in A$.*

(1) $\mathrm{RH}_\kappa : \mathcal{M}(\kappa) \to \mathcal{R}(a)$ *is a surjective proper holomorphic map, and*
(2) $\mathrm{RH}_\kappa : \mathcal{M}^\circ(\kappa) \to \mathcal{R}^\circ(a)$ *is a biholomorphism.*

Furthermore, at each (t, κ)-level we have the following theorem.

***Theorem 4.12** (**Solution to (t, κ)-RHP**)*. — *Let $(t, \kappa) \in T \times \mathcal{K}$ and put $a = \mathrm{rh}(\kappa) \in A$.*

(1) *If $\kappa \notin$* **Wall***, then $\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{R}_t(a)$ is a biholomorphic map, and*
(2) *if $\kappa \in$* **Wall***, then $\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{R}_t(a)$ is a minimal resolution of singularities having the Riccati locus $\mathcal{M}_t^{\mathrm{r}}(\kappa)$ as its exceptional divisor.*

These theorems can be generalized to stable parabolic connections of higher rank, with more regular singular points, and even on a curve of higher genus. We present an essence of the proof, focusing on the surjectivity of RH, which remains valid for such generalizations.

***Remark 4.13** (**How to Prove**)*. — Given a Jordan equivalence class of representations,

(1) choose a "good" representative from the given equivalence class and form the flat connection associated to it. Since we are working with Jordan equivalence, we can take a semisimple representation ρ_0 as the good representative.
(2) Extend the flat connection to a logarithmic connection by Deligne's canonical extension [**12**] and provide it with a parabolic structure. If the initial representation ρ_0 is irreducible, the resulting parabolic connection Q_0 is stable and so we are done. If ρ_0 is reducible, we cannot stop here because Q_0 may be unstable and we should proceed to step (3).
(3) If ρ_0 is reducible, take steps (1) and (2) relatively, so that we obtain a family of parabolic connections $\mathcal{Q} = \{Q_c\}_{c \in C}$ parametrized by some curve C, with $Q_{c_0} = Q_0$ at the reference point $c_0 \in C$, such that the monodromy of Q_c is irreducible for every $c \in C - \{c_0\}$. Then use Langton's technique in Theorem 4.14 in order to recast Q_0 to a *stable* parabolic connection.
(4) The family $\mathcal{Q}$ in step (3) is constructed as follows. Notice that reducible representations occur only on a Zariski-closed proper subset $B \subset A$. Let $c_0 \in B$ be the local monodromy data of ρ_0, take a curve $C \subset A$ that meets B only at c_0, and prolong the representation ρ_0 along the curve C. Taking steps (1) and (2) relatively, we obtain the desired family $\mathcal{Q}$.

Here is the version of Langton's technique [**42**] that is needed in the current situation.

Theorem 4.14 (**Langton's Technique**). — *Let $\mathcal{Q} = \{Q_c\}_{c \in C}$ be a family of parabolic connections parametrized by a curve C. By some applications of elementary transformations, $\mathcal{Q}$ can be transformed to a family of* <u>*stable*</u> *parabolic connections, if the monodromy of Q_c is* <u>*irreducible*</u> *for every $c \in C - \{c_0\}$. This means that the possible singularity of $\mathcal{Q}$ at c_0 can be removed by elementary tramsformations, provided that all the nearby connections are irreducible.*

Langton's theorem reminds us of the removable singularity theorem of Riemann in complex variable and that of Uhlenbeck in gauge theory. Riemann's classical theorem asserts that an isolated singularity of a holomorphic function can be removed, if the function is *bounded* around the singular point. Uhlenbeck's theorem [**78**] states that an isolated singularity of a Young-Mills connection can be removed by applying a gauge transformation, if the curvature of the connection is L^2-*bounded* around the singular point. Langton's theorem can be regarded as an algebraic-geometry version of such removable singularity principles, where the boundedness condition is replaced by the *irreducibility* of representations.

Remark 4.15 (**Family of (-2)-Curves**). — By Theorem 4.12, for any $(t, \kappa) \in T \times \mathbf{Wall}$, the (t, κ)-Riemann-Hilbert correspondence $\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{R}_t(a)$ gives a minimal resolution of singularities whose exceptional divisor is just the Riccati locus $\mathcal{M}_t^{\mathrm{r}}(\kappa)$. Each irreducible component of $\mathcal{M}_t^{\mathrm{r}}(\kappa)$ is a (-2)-*curve*, that is, a smooth curve $C \subset \mathcal{M}_t(\kappa)$ such that

$$C \simeq \mathbb{P}^1, \qquad C \cdot C = -2.$$

Conversely any (-2)-curve in $\mathcal{M}_t(\kappa)$ arises in this manner, since it must be sent to a singular point by $\mathrm{RH}_{t,\kappa}$. Considering this situation relatively for the family $\pi_\kappa : \mathcal{M}(\kappa) \to T$, we see that each irreducible component of the Ricatti locus $\mathcal{M}^{\mathrm{r}}(\kappa) \subset \mathcal{M}(\kappa)$ is a family of (-2)-curves over T, namely,

$$\pi_\kappa : \mathcal{C} \to T, \qquad C_t \subset \mathcal{M}_t(\kappa) \; : \; (-2)\text{-curve}. \tag{26}$$

To apply Hartog's extension theorem later, we state the following simple lemma.

Lemma 4.16 (**Codimension Two**). — *The Riccati locus $\mathcal{M}^{\mathrm{r}}$ is of codimension two in $\mathcal{M}$.*

This is intuitively clear: By Lemma 4.8 the Riccati locus $\mathcal{M}^{\mathrm{r}}$ can lie only over the codimension-one subset $T \times \mathbf{Wall} \subset T \times \mathcal{K}$ with respect to the fibration (9). On the other hand Remark 4.15 implies that for each $(t, \kappa) \in T \times \mathbf{Wall}$, the Riccati locus $\mathcal{M}_t^{\mathrm{r}}(\kappa)$ is of codimension one in $\mathcal{M}_t(\kappa)$. In total, $\mathcal{M}^{\mathrm{r}}$ is of codimension two in $\mathcal{M}$. Lemma 4.16 will be used in Remark 5.11.
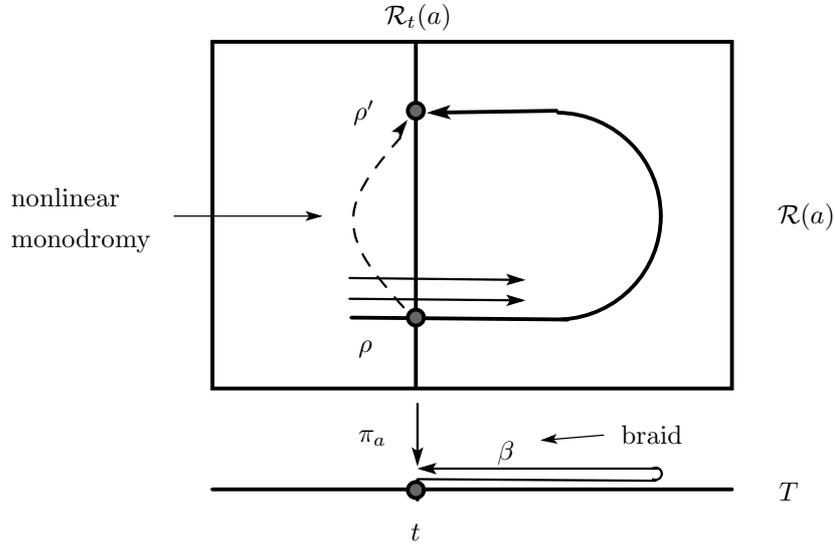
$$\mathcal{R}_t(a)$$

FIGURE 8. Isomonodromic flow

5. Isomonodromic Flow and Painlevé Flow

From our dynamical point of view, we should consiously distingush the Painlevé flow on the moduli space of stable parabolic connections from the isomonodromic flow on the moduli space of monodromy representations and throw a bridge between these two dynamics via the Riemann-Hilbert correspondence. We begin with the isomonodromic flow.

5.1. Isomonodromic Flow. — Fix a base point $t \in T$ and take the loops $\gamma_i \in \pi_1(\mathbb{P}^1 - D_t, *)$ as in Figure 6. Let U be a sufficiently small simply-connected neighborhood of t in T. Then, having $\{\gamma_i\}$ as common generators, all the fundamental groups $\pi_1(\mathbb{P}^1 - D_s, *)$ with $s \in U$ are identified with the reference group $\pi_1(\mathbb{P}^1 - D_t, *)$. Passing to moduli spaces of representations, we have isomorphisms

$$\psi_t^s : \mathcal{R}_t(a) \to \mathcal{R}_s(a) \qquad (s \in U). \tag{27}$$

This means that the fibration $\pi_a : \mathcal{R}(a) \to T$ is locally trivial, where a local trivialization over U is given by $\psi_t : \mathcal{R}_t(a) \times U \to \mathcal{R}(a)|_U, (\rho, s) \mapsto \psi_t^s(\rho)$. Then there exists the trivial foliation on $\mathcal{R}(a)|_U$ whose leaves are the slices $\psi(\{\rho\} \times U)$ parametrized by $\rho \in \mathcal{R}_t(a)$. These local foliations for various simply-connected open subsets $U \subset T$ are patched together to form a global foliation on $\mathcal{R}(a)$. Moreover, patching together various local isomorphisms of the form (27), we can associate to each path ℓ in T an isomorphism

$$\ell_* : \mathcal{R}_t(a) \to \mathcal{R}_s(a), \tag{28}$$

where t and s are the initial and terminal points of ℓ, respectively. Note that the isomorphism ℓ_* depends only on the homotopy class of the path ℓ.

Definition 5.1 (Isomonodromic Flow). — The foliation on $\mathcal{R}(a)$ induced from the local triviality of the fibration $\pi : \mathcal{R}(a) \to T$ is called the *a-isomonodromic flow* and is denoted by $\mathcal{F}_{\mathrm{IMF}(a)}$ (see Figure 8). It is a time-dependent Hamiltonian dynamics in the sense of Definition 2.5. Namely each fiber $\mathcal{R}_t(a)$ is a symplectic manifold possibly with singularities, whose symplectic structure $\Omega_{\mathcal{R}_t(a)}$ will be described in §5.2, and the isomorphism (28) is a symplectic isomorphism. The dynamical system $(\mathcal{R}(a), \mathcal{F}_{\mathrm{IMF}(a)})$ is denoted by $\mathrm{IMF}(a)$, whose fundamental 2-form $\Omega_{\mathcal{R}(a)}$ is defined by the following conditions:

(1) $\Omega_{\mathcal{R}(a)}$ is restricted to the symplectic structure $\Omega_{\mathcal{R}_t(a)}$ on $\mathcal{R}_t(a)$ for every $t \in T$.
(2) $\iota_v \Omega_{\mathcal{R}(a)} = 0$ for any $\mathcal{F}_{\mathrm{IMF}(a)}$-horizontal vector filed v.

We give relative versions of Definition 5.1, which will also be used later.

Definition 5.2 (Family of Isomonodromic Flows)

(1) There exists a (unique) family $\mathrm{IMF} = (\mathcal{R}, \mathcal{F}_{\mathrm{IMF}})$ of isomonodromic flows over A, where $\mathcal{F}_{\mathrm{IMF}}$ is a relative foliation on the fibration $\mathcal{R} \to A$ that restricts to the foliation $\mathcal{F}_{\mathrm{IMF}(a)}$ on each fiber $\mathcal{R}(a)$. Moreover there exists a relative 2-form $\Omega_{\mathcal{R}}$ on $\mathcal{R}$ that restricts to the fundamental 2-form $\Omega_{\mathcal{R}(a)}$ on $\mathcal{R}(a)$.
(2) By the fiber-product morphism (21), IMF is pulled back to a relative foliation $\mathbf{IMF} = (\boldsymbol{\mathcal{R}}, \mathcal{F}_{\mathbf{IMF}})$ on $\boldsymbol{\mathcal{R}}$, with the corresponding relative 2-form $\Omega_{\boldsymbol{\mathcal{R}}}$.

Although it is almost trivial from the purely topological nature of the isomonodromic flow, the following lemma is worth stating explicitly.

Lemma 5.3 (Geometric Painlevé Property). — *For each $a \in A$, the isomonodromic flow $\mathrm{IMF}(a)$ has geometric Painlevé property.*

It is clear from the construction that the Riccati locus $\mathcal{R}^{\mathrm{r}}(a)$ and the non-Riccati locus $\mathcal{R}^{\circ}(a)$ are stable under the isomonodromic flow $\mathrm{IMF}(a)$.

Definition 5.4 (Riccati/Non-Riccati Flow). — For each $a \in A$ (actually for each $a \in \mathrm{rh}(\mathbf{Wall})$),

(1) the isomonodromic flow $\mathrm{IMF}(a)$ restricted to the Riccati locus $\mathcal{R}^{\mathrm{r}}(a)$ is referred to as the *Riccati flow* and is denoted by $\mathrm{IMF}^{\mathrm{r}}(a)$, and
(2) the isomonodromic flow $\mathrm{IMF}(a)$ restricted to the non-Riccati locus $\mathcal{R}^{\circ}(a)$ is referred to as the *non-Riccati flow* and is denoted by $\mathrm{IMF}^{\circ}(a)$.

5.2. Symplectic Structure on $\mathcal{R}_t(a)$. — The symplectic nature of moduli spaces of monodromy representations was first discussed by Goldman [**21**]. It has been used to study Painlevé-type equations by Iwasaki [**31, 32**], Hitchin [**24**], Kawai [**40, 41**], Boalch [**6**] and others. Now we recall the topological description of the symplectic

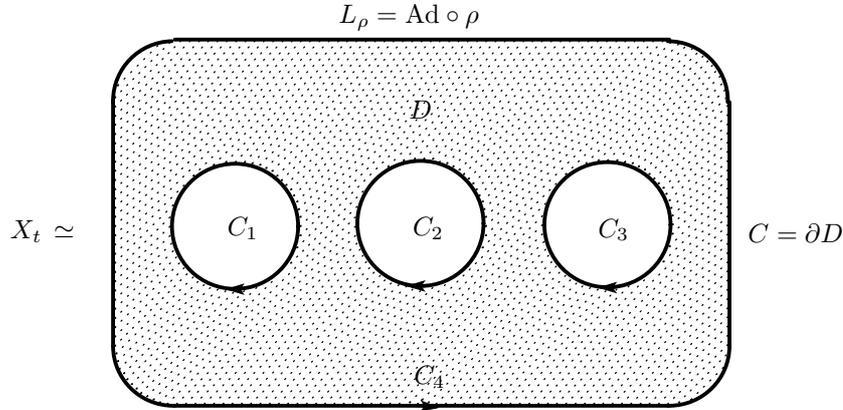$$L_\rho = \mathrm{Ad} \circ \rho$$

FIGURE 9. The Riemann sphere with four disks deleted

structure $\Omega_{\mathcal{R}_t(a)}$ on the smooth locus of $\mathcal{R}_t(a)$ (under some generic condition on a). A more comprehensive sheaf-cohomological description, which allows for every values of $a \in A$, can be found in Inaba, Iwasaki and Saito [**29**].

In general, given a topological space X, let $\mathcal{R}(X)$ denote the set of all Jordan equivalence classes of $SL_2(\mathbb{C})$-representations of $\pi_1(X)$. In stead of using the 4-punctured Riemann sphere $\mathbb{P}^1 - D_t$, we employ a homotopically equivalent domain[4] D obtained from $\mathbb{P}^1$ by deleting four disjoint open disks centered at t_1, t_2, t_3, t_4. The boundary C of D consists of four disjoint circles C_1, C_2, C_2, C_4, (see Figure 9). Then $\mathcal{R}(D)$ is identified with $\mathcal{R}_t = \mathcal{R}(\mathbb{P}^1 - D_t)$, while $\mathcal{R}(C)$ is identified with $A = \mathbb{C}_a^4$ by the isomorphism

$$\mathcal{R}(C) \to A, \qquad \rho \mapsto a = (\mathrm{Tr}\,\rho(C_1),\ \mathrm{Tr}\,\rho(C_2),\ \mathrm{Tr}\,\rho(C_3),\ \mathrm{Tr}\,\rho(C_4)).$$

Restricting representations of $\pi_1(D)$ to $\pi_1(C)$, we have the restriction mapping

$$r : \mathcal{R}(D) \to \mathcal{R}(C), \qquad \rho \mapsto \rho|_C.$$

Then $\mathcal{R}_t(a)$ is identified with the fiber $r^{-1}(a)$ over $a \in A = \mathcal{R}(C)$ and the Zariski tangent space to $\mathcal{R}_t(a)$ at a point $\rho \in \mathcal{R}_t(a)$ is given by

$$T_\rho \mathcal{R}_t(a) = \mathrm{Ker}\,[\,(dr)_\rho : T_\rho \mathcal{R}(D) \to T_{r(\rho)} \mathcal{R}(C)\,].$$

Let L_ρ be the locally constant system on D associated to the representation $\mathrm{Ad} \circ \rho$, where $\mathrm{Ad} : SL_2(\mathbb{C}) \to GL(\mathfrak{sl}_2(\mathbb{C}))$ is the adjoint representation of $SL_2(\mathbb{C})$. Then the standard infinitesimal deformation theory tells us that the Zariski tangent spaces $T_\rho \mathcal{R}(D)$ and $T_{r(\rho)} \mathcal{R}(C)$ are identified with the first cohomology groups $H^1(D, L_\rho)$

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[4]somewhat confusing notation: D should not be confused with D_t.

and $H^1(C; L_\rho)$, and that the tangent map $(dr)_\rho$ is identified with the homomorphism j^* in the cohomology long exact sequence

$$H^0(C; L_\rho) \xrightarrow{\delta^*} H^1(D, C; L_\rho) \xrightarrow{i^*} H^1(D; L_\rho) \xrightarrow{j^*} H^1(C; L_\rho)$$

for the pair of spaces (D, C) with coefficients in L_ρ. Thus we have an isomorphism

$$T_\rho \mathcal{R}_t(a) \cong \mathrm{Ker}\,[\,j^* : H^1(D; L_\rho) \to H^1(C; L_\rho)\,], \tag{29}$$

Moreover the cohomology long exact sequence yields another isomorphism induced by i^*,

$$T_\rho \mathcal{R}_t(a) \cong \frac{H^1(D, C; L_\rho)}{\delta^* H^0(C; L_\rho)}. \tag{30}$$

By the Poincaré-Lefschetz duality, there exists a nondegenerate bilinear form

$$H^1(D; L_\rho) \otimes H^1(D, C; L_\rho) \xrightarrow{\text{cup product}} H^2(D, C; L_\rho \otimes L_\rho)$$

$$\xrightarrow{\text{Killing form}} H^2(D, C; \mathbb{C}_D) \cong \mathbb{C},$$

which induces a nondegenerate pairing between the righthand sides of (29) and (30), and hence a nondegenerate skew-symmetric bilinear form on the tangent space $T_\rho \mathcal{R}_t(a)$ at ρ,

$$\Omega_{\mathcal{R}_t(a),\,\rho} : T_\rho \mathcal{R}_t(a) \times T_\rho \mathcal{R}_t(a) \to \mathbb{C}.$$

In this manner we have obtained an alomost symplectic structure $\Omega_{\mathcal{R}_t(a)}$ on $\mathcal{R}_t(a)$, which in fact is a symplectic structure. This fact, namely, the closedness of $\Omega_{\mathcal{R}_t(a)}$ is trivial in our 4-point case where $\mathcal{R}_t(a)$ is a surface. It can be proved in the general n-point situation on a Riemann surface of arbitrary genus ([**32**]).

5.3. Nonlinear Monodromy of Isomonodromic Flow. — Given a base point $t \in T$, we consider isomorphisms (28) when the ℓ's are loops in T with base point at t. Then they become automorphisms of $\mathcal{R}_t(a)$ and yield a group homomorphism

$$\pi_1(T, t) \to \mathrm{Aut}\,\mathcal{R}_t(a), \qquad \ell \mapsto \ell_*, \tag{31}$$

which is nothing but the nonlinear monodromy of the isomonodromic flow $\mathrm{IMF}(a)$ (see Definition 2.4). This homomorphism can be described in terms of braid groups on three strings (see Dubrovin and Mazzocco [**14**], Iwasaki [**34**] and Boalch [**5, 7**]). To recall this description, we put t_4 at infinity and redefine the time-variable space as the configuration space of distinct ordered three points in $\mathbb{C}$, that is,

$$T = \{\, t = (t_1, t_2, t_3) \in \mathbb{C}^3 \,:\, t_i \neq t_j \text{ for } i \neq j \,\}. \tag{32}$$

Then the fundamental group $\pi_1(T, t)$ is isomorphic to the pure braid group P_3 on three strings. If T is replaced by the configuration space of distinct *unordered* three points in $\mathbb{C}$, then $\pi_1(T, t)$ is isomorphic to the ordinary braid group B_3 on three strings (see e.g. Birman [**4**]). Recall that there exists the natural exact sequence $1 \to P_3 \to B_3 \to S_3 \to 1$, where S_3 represents the permutations of t_1, t_2, t_3. For later convenience, we employ the following terminology.

Definition 5.5 (**Full-Monodromy and Half-Monodromy**). — Monodromy in terms of pure braids are referred to as *full-monodromy*, while monodromy in terms of ordinary braids are referred to as *half-monodromy*, respectively.

Using half-monodromy will be convenient for shorter presentation and the full-monodromy is just obtained by restricting the ordinary braid group to its pure subgroup. The full-monodromy in (31) makes sense for each individual IMF(a), while the half-monodromy only makes sense for IMF. To describe the half-monodromy, we introduce the following natural action of B_3 on $\mathcal{R}_t$.

Definition 5.6 (**Action of Braids on Representations**). — The action of the braid group B_3 on the moduli $\mathcal{R}_t$ of monodromy representations,

$$B_3 \times \mathcal{R}_t \to \mathcal{R}_t, \qquad (\beta, \rho) \mapsto \rho^\beta,$$

is defined by the following condition, which we call the *global isomonodromy condition*,

$$\rho^\beta(\gamma^\beta) = \rho(\gamma). \tag{33}$$

Here $\gamma \mapsto \gamma^\beta$ is the natural action of $\beta \in B_3$ on $\pi_1(X_t, *)$ defined as in Definition 5.7, where we put

$$X_t = \mathbb{C} - \{t_1, t_2, t_3\}.$$

Definition 5.7 (**Action of Braids on Fundamental Group**). — Let β_i be the braid as indicated in Figure 10, where (i, j, k) is any cyclic permutation of $(1, 2, 3)$. Then the braid group B_3 is generated by the basic braids β_1, β_2, β_3. On the other hand the fundamental group $\pi_1(X_t, *)$ is the free group generated by the loops γ_1, γ_2, γ_3 in Figure 6. Thus we have

$$B_3 = \langle \beta_1, \beta_2, \beta_3 \rangle, \qquad \pi_1(X_t, *) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle.$$

In terms of these generators, the action of B_3 on $\pi_1(X_t, *)$ is given as in Figure 11. Namely the action of the i-th basic braid $\beta_i : (\gamma_i, \gamma_j, \gamma_k) \mapsto (\gamma_i', \gamma_j', \gamma_k')$ is expressed as

$$\gamma_i' = \gamma_i^{-1} \gamma_j \gamma_i, \qquad \gamma_j' = \gamma_i', \qquad \gamma_k' = \gamma_k.$$

where the composition of loops is taken from right to left.

The following theorem is clear from the manner in which the action is defined as in (33).

Theorem 5.8 (**Nonlinear Monodromy**). — *The half-monodromy of* IMF *is given by the* B_3*-action on* $\mathcal{R}_t$ *in Definition* 5.6 *and the full-monodromy of* IMF(a) *is the* P_3*-action on* $\mathcal{R}_t(a)$ *that is the restriction of the* B_3*-action above to* P_3 *(see Table* 4*).*

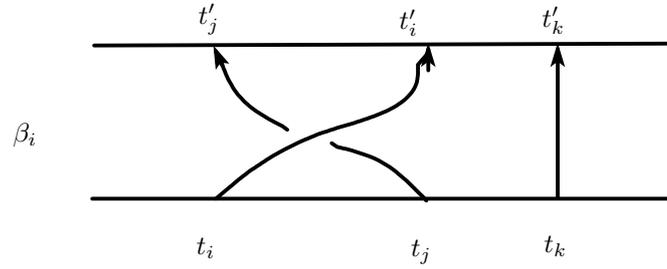As in Dubrovin and Mazzocco [**14**] and Iwasaki [**34**], we make the following remark.

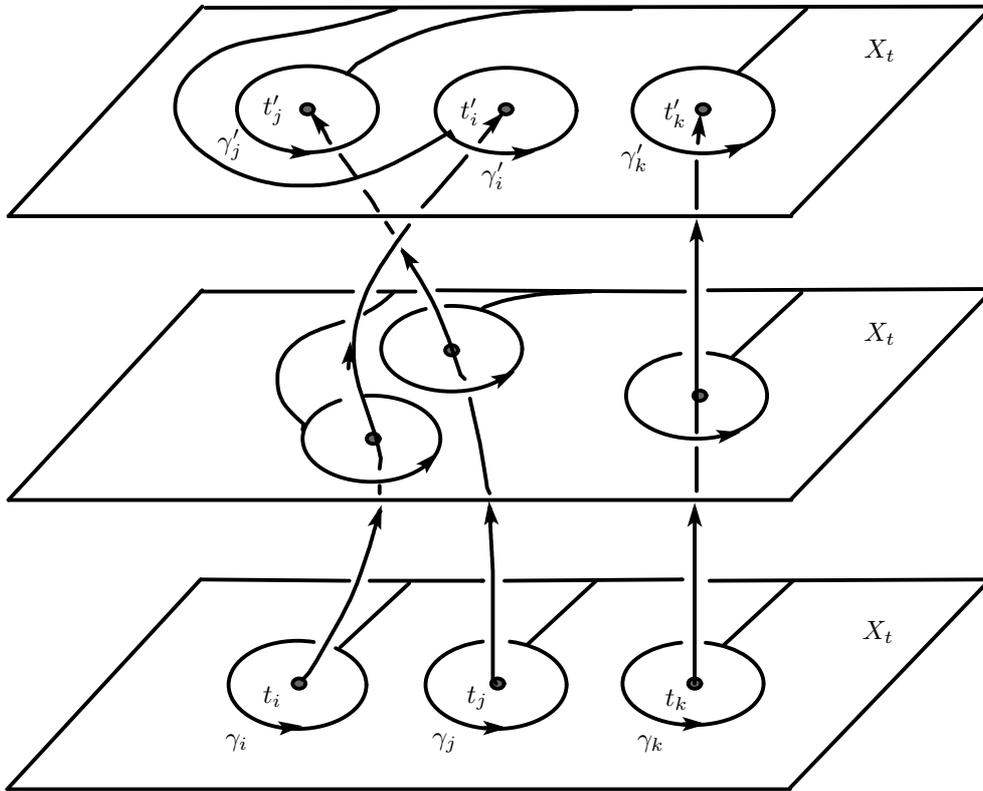FIGURE 10. Basic braid β_i, where (i, j, k) is a cyclic of $(1, 2, 3)$

FIGURE 11. The braid action $\beta_i : (\gamma_i, \gamma_j, \gamma_k) \mapsto (\gamma_i', \gamma_j', \gamma_k')$

half-monodromy	full-monodromy
$B_3 \curvearrowright \mathcal{R}_t$	$P_3 \curvearrowright \mathcal{R}_t(a)$

TABLE 4. Half-monodromy of IMF and full-monodromy of IMF(a)

Remark 5.9 (Reduction to Modular Group). — It is well known that the center $Z(B_3)$ of B_3 is the infinite cyclic group $\langle (\beta_i \beta_j)^3 \rangle$ generated by $(\beta_i \beta_j)^3$ and the quotient group $B_3/Z(B_3)$ is isomorphic to the full modular group $\Gamma \simeq PSL(2, \mathbb{Z})$. An inspection shows that our braid group action is trivial on the center $Z(B_3)$. Hence it is reduced to an action of the full modular group Γ on $\mathcal{R}_t$. In view of Remark 2.13, this reduction is quite possible since the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ is isomorphic to the level-two principal congruence subgroup $\Gamma(2)$ of Γ. The resulting modular group action will be described explicitly in Definition 8.1.

5.4. Painlevé Flow. — From our point of view, the Painlevé flow should be defined as the pull-back of the isomonodromic flow by the Riemann-Hilbert correspondence. This standpoint was first adopted by Iwasaki [**31, 32**], though things were still looked at locally. Currently a completely global formulation is feasible, now that we have such a neat result as in Theorem 4.10.

Theorem 5.10 (Painlevé Flow). — *For any $\kappa \in \mathcal{K}$, put $a = \mathrm{rh}(\kappa) \in A$.*

(1) *There exists a unique holomorphic foliation $\mathcal{F}_{\mathrm{P_{VI}}(\kappa)}$ on $\mathcal{M}(\kappa)$ such that the κ-Riemann-Hilbert correspondence $\mathrm{RH}_\kappa : \mathcal{M}(\kappa) \to \mathcal{R}(a)$ gives a semi-conjugacy*

$$\mathrm{RH}_\kappa : (\mathcal{M}(\kappa), \mathcal{F}_{\mathrm{P_{VI}}(\kappa)}) \to (\mathcal{R}(a), \mathcal{F}_{\mathrm{IMF}(a)}). \tag{34}$$

The dynamical system $\mathrm{P_{VI}}(\kappa) = (\mathcal{M}(\kappa), \mathcal{F}_{\mathrm{P_{VI}}(\kappa)})$ is called the κ-Painlevé flow.

(2) *The semi-conjugacy map (34) induces a conjugacy map*

$$\mathrm{RH}_\kappa : (\mathcal{M}^\circ(\kappa), \mathcal{F}_{\mathrm{P_{VI}}(\kappa)}) \to (\mathcal{R}^\circ(a), \mathcal{F}_{\mathrm{IMF}(a)}),$$

when restricted to the non-Riccati locus.

(3) *The fundamental 2-form $\Omega_{\mathcal{M}(\kappa)}$ for the κ-Painlevé flow $\mathrm{P_{VI}}(\kappa)$ is the unique holomorophic 2-form on $\mathcal{M}(\kappa)$ that satisfies the condition*

$$\Omega_{\mathcal{M}(\kappa)} = \mathrm{RH}_\kappa^* \Omega_{\mathcal{R}(a)} \qquad on \quad \mathcal{M}^\circ(\kappa).$$

The point of Theorem 5.10 is explained in the following manner.

Remark 5.11 (Codimension-Two Argument). — If $\kappa \in \mathcal{K} - \mathbf{Wall}$, this theorem immediately follows from Corollary 4.11, since in this case there is no Riccati locus and $\mathrm{RH}_\kappa : \mathcal{M}(\kappa) \to \mathcal{R}(a)$ is a biholomorphic map. However, if $\kappa \in \mathbf{Wall}$, things are not so simple because RH_κ fails to be injective on the Riccati locus $\mathcal{M}^\mathrm{r}(\kappa)$, which is of codimension one in $\mathcal{M}(\kappa)$. In this case it is not immediately clear as to whether the

Painlevé flow extends to the Riccati locus. To avoid this difficulty, we should consider the full-Riemann-Hilbert correspondence RH : $\mathcal{M} \to \mathcal{R}$ in (22). By Definition 5.2 we have relative foliation $\mathcal{F}_{\mathbf{IMF}}$ and relative 2-form $\Omega_{\mathcal{R}}$ on $\mathcal{R}$. Since RH : $\mathcal{M}^\circ \to \mathcal{R}^\circ$ is biholomorphic by Theorem 4.10, $\mathcal{F}_{\mathbf{IMF}}$ and $\Omega_{\mathcal{R}}$ can be pulled back to a holomorphic relative foliation $\mathcal{F}_{\mathrm{P_{VI}}}$ and to a holomorphic relative 2-form $\Omega_{\mathcal{M}}$ on $\mathcal{M}^\circ$. Since the complement $\mathcal{M}^{\mathrm{r}} = \mathcal{M} - \mathcal{M}^\circ$ is of codimension two in $\mathcal{M}$ (see Lemma 4.16), Hartog's extension theorem implies that $\mathcal{F}_{\mathrm{P_{VI}}}$ and $\Omega_{\mathcal{M}}$ can be extended to the whole space $\mathcal{M}$ holomorophically. Restricting these extensions to each $\mathcal{M}(\kappa)$ yields a holomorphic flow $\mathcal{F}_{\mathrm{P_{VI}}(\kappa)}$ and a holomorphic 2-form $\Omega_{\mathcal{M}(\kappa)}$ on $\mathcal{M}(\kappa)$. These are just what we have been seeking.

Theorem 5.12 (Geometric Painlevé Property). — *For any $\kappa \in \mathcal{K}$ the Painlevé flow $\mathrm{P_{VI}}(\kappa)$ enjoys geometric Painlevé property.*

This theorem readily follows from the geometric Painlevé property for the isomonodromic flow $\mathrm{IMF}(a)$ with $a = \mathrm{rh}(\kappa)$ (see Lemma 5.3) and from the fact that RH_κ is a semi-conjugacy map between $\mathrm{P_{VI}}(\kappa)$ and $\mathrm{IMF}(a)$, especially from the properness of the map RH_κ (see Lemma 2.12).

It is clear that the Riccati locus $\mathcal{M}^{\mathrm{r}}(\kappa)$ and the non-Riccati locus $\mathcal{M}^\circ(\kappa)$ are stable under the Painlevé flow $\mathrm{P_{VI}}(\kappa)$. As a counterpart of Definition 5.4 we make the following definition.

Definition 5.13 (Riccati/Non-Riccati Flow). — For each $\kappa \in \mathcal{K}$ (actually for each $\kappa \in$ **Wall**),

(1) the Painlevé flow $\mathrm{P_{VI}}(\kappa)$ restricted to the Riccati locus $\mathcal{M}^{\mathrm{r}}(\kappa)$ is referred to as the *Riccati flow* and is denoted by $\mathrm{P_{VI}^r}(\kappa)$, and

(2) the Painlevé flow $\mathrm{P_{VI}}(\kappa)$ restricted to the non-Riccati locus $\mathcal{M}^\circ(\kappa)$ is referred to as the *non-Riccati flow* and is denoted by $\mathrm{P_{VI}^\circ}(\kappa)$.

The assertion (2) of Theorem 5.10 is now restated as follows.

Theorem 5.14 (Conjugacy for Non-Riccati Flows). — *For any $\kappa \in \mathcal{K}$ put $a = \mathrm{rh}(\kappa) \in A$. The Riemann-Hilbert correspondence RH_k yields a conjugacy between the non-Riccati Painlevé flow $\mathrm{P_{VI}^\circ}(\kappa)$ and the non-Riccati isomonodromic flow $\mathrm{IMF}^\circ(a)$. In particular the nonlinear monodromy of $\mathrm{P_{VI}^\circ}(\kappa)$ is faithfully represented by that of $\mathrm{IMF}^\circ(a)$, where the latter is described by Theorem 5.8 restricted to the non-Riccati loci.*

The discussions of this subsection are summarized as follows. The Riemann-Hilbert correspondence gives an analytic semi-conjugacy between the Painlevé flow and the isomonodromic flow. It gives an analytic conjugacy in the strict sense *outside* the Riccati locus, while it collapses the Riccati locus to a family of singularities. Thus we have almost arrived at the situation described in the Guiding Diagram in Figure 3, though subtle details on the Riccati flow are not depicted there. One point yet to be

discussed in Figure 3 is the isomorphism $\mathcal{R}_t(\kappa) \simeq \mathcal{S}(\theta)$, which will be established in Theorem 6.5.

5.5. Riccati Flows and Hypergeometric Equations. — This subsection is devoted to the linearization of Riccati-Painlevé flows. This procedure will clearly explain why Riccati flows are called so. Throughout this subsection we fix $\kappa \in \mathbf{Wall}$.

The Riccati-Painlevé flow $\mathrm{P}_{\mathrm{VI}}^{\mathrm{r}}(\kappa)$ is confined in the Riccati locus $\mathcal{M}^{\mathrm{r}}(\kappa)$. By Remark 4.15 each irreducible component $\mathcal{C} \subset \mathcal{M}^{\mathrm{r}}(\kappa)$, which is stable under the flow, is a family of (-2)-curves over T as in (26). Thus $\mathrm{P}_{\mathrm{VI}}^{\mathrm{r}}(\kappa)$ restricts to a dynamical system on the $\mathbb{P}^1$-bundle $\pi_\kappa : \mathcal{C} \to T$. For this, we have the following theorem.

Theorem 5.15 (**Hypergeometric Equation**). — *On each irreducible component of* $\mathcal{M}^{\mathrm{r}}(\kappa)$ *the Riccati-Painlevé flow* $\mathrm{P}_{\mathrm{VI}}^{\mathrm{r}}(\kappa)$ *is linearizable in terms of a Gauss hypergeometric equation.*

To understand what this means, we should recall the following famous theorem.

Theorem 5.16 (**Fuchs-Poincaré**). — *Let* $F(x, y, z)$ *be a polynomial of* (y, z) *whose coefficients are meromorphic functions of* x *in a domain* $U \subset \mathbb{C}$. *Let* g *be the genus of the affine algebraic curve*

$$C_x = \{\, (y, z) \in \mathbb{C}^2 \,:\, F(x, y, z) = 0 \,\}$$

at a generic point $x \in U$. *If the first-order nonlinear differential equation*

$$F(x, y, y') = 0, \qquad y' = dy/dx, \tag{35}$$

has analytic Painlevé property, then there exists the following trichotomy:

(1) *if* $g = 0$ *then* (35) *can be reduced to a Riccati equation*

$$y' = a(x)\, y^2 + b(x)\, y + c(x), \tag{36}$$

(2) *if* $g = 1$ *then* (35) *can be reduced to the differential equation of an elliptic curve*

$$(y')^2 = 4\, y^3 - g_2\, y - g_3,$$

(3) *if* $g \geq 2$ *then* (35) *can be solved by algebraic quadratures.*

This theorem means that first-order dynamical systems with Painlevé property are classified by the genera of spaces of initial conditions. In the case of genus zero, it asserts that the dynamics is governed by a Riccati equation. The Riccati equation (36) is linearized as

$$y = -\frac{1}{a(x)} \frac{Y'}{Y}, \qquad a(x)\, Y'' - \{a'(x) + a(x)\, b(x)\}\, Y' + a^2(x)\, c(x)\, Y = 0. \tag{37}$$

Let us return to the situation in Theorem 5.15, where we were discussing the Riccati-Painlevé flow $\mathrm{P}_{\mathrm{VI}}^{\mathrm{r}}(\kappa)$ restricted to an irreducible component $\mathcal{C} \subset \mathcal{M}^{\mathrm{r}}(\kappa)$. Then

obviously we are in the genus-zero case of Theorem 5.16[5]. If we use the coordinate expression of $P_{VI}(\kappa)$, we can see that in our case the linear equation (37) is (essentially) a Gauss hypergeometric equation. Here the coordinate expression of $P_{VI}(\kappa)$ will be given in Theorem 10.10. Since using coordinate expressions is not beautiful, we may pose the following problem.

Problem 5.17 **(Linearization)**. — For each irreducible component $\mathcal{C} \subset \mathcal{M}^{\mathrm{r}}(\kappa)$ of the Riccati locus, show that there exist a rank-two vector bundle E on $(\mathbb{P}^1)^4$ and an integrable connection ∇ on it, having regular singularities along the diagonal $(\mathbb{P}^1)^4 - T$, such that the Riccati-Painlevé flow $P_{VI}^{\mathrm{r}}(\kappa)$ restricted to $\mathcal{C}$ is the flat projective connection induced from the flat linear connection $(E, \nabla)|_T$.

Of course we have to solve it conceptually without using coordinate expressions. In any case, it is now clear that classifying Riccati solutions amounts to classifying irreducible components of Riccati loci. We may consider this problem at a fixed $t \in T$. So the problem is to classify (-2)-curves on moduli spaces $\mathcal{M}_t(\kappa)$, $\kappa \in \mathbf{Wall}$. Originally, the relation between Riccati solutions to Painlevé equations and (-2)-curves on spaces of initial conditions was clarified by Saito and Terajima [**68**] and Sakai [**70**]. In particular Saito and Terajima gave a complete classification of (-2)-curves. We can now amplify their viewpoint by the picture of resolution of singularities by Riemann-Hilbert correspondence. To do so we should pose the following problem.

Problem 5.18 **(Classification of (-2)-Curves)**. — Given any $(t, \kappa) \in T \times \mathbf{Wall}$, classify all (-2)-curves on $\mathcal{M}_t(\kappa)$ in connection with the resolution of singularities by the Riemann-Hilbert correspondence $\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{R}_t(a)$.

This problem will be settled in Theorem 9.4. This subsection is closed with the following historical remark.

Remark 5.19 **(History)**. — Attempts at generalizing Theorem 5.16 to second-order equations led Painlevé to discover his famous equations.

6. Family of Affine Cubic Surfaces

All the constructions described so far can be made more explicit if we consider a family of affine cubic surfaces defined as a certain categorical quotient. We present the construction of the family, following Iwasaki [**34**]. Throughout this section we fix a time $t \in T$.

---

[5]To apply Theorem 5.16, we should recast the bundle $\pi_\kappa : \mathcal{C} \to T$ to a $\mathbb{P}^1$-bundle over $U = \mathbb{P}^1 - \{0, 1, \infty\}$ by using the symplectic reduction in Remark 2.13.

6.1. Categorical Quotient. — Let $\mathrm{Hom}_t = \mathrm{Hom}(\pi_1(\mathbb{P}^1 - D_t, *), SL_2(\mathbb{C}))$ be the set of all representations of $\pi_1(\mathbb{P}^1 - D_t, *)$ into $SL_2(\mathbb{C})$. Then Hom_t is naturally an affine algebraic variety and admits the adjoint action

$$\mathrm{Ad} : SL_2(\mathbb{C}) \times \mathrm{Hom}_t \to \mathrm{Hom}_t, \qquad (P, \rho) \mapsto \mathrm{Ad}(P)\rho,$$

defined by $(\mathrm{Ad}(P)\rho)(\gamma) = P\rho(\gamma)P^{-1}$ for $\gamma \in \pi_1(\mathbb{P}^1 - D_t, *)$. It is known (see e.g. Simpson [**73, 74**]) that the moduli space $\mathcal{R}_t$ of Jordan equivalence classes of representations is isomorphic to the categorical quotient

$$\mathrm{Hom}_t /\!/ \mathrm{Ad} = \mathrm{Spec}\left(\mathbb{C}[\mathrm{Hom}_t]^{\mathrm{Ad}}\right),$$

where $\mathbb{C}[\mathrm{Hom}_t]^{\mathrm{Ad}}$ is the Ad-invariant coordinate ring on Hom_t. If the generators γ_i of $\pi_1(X_t, *)$ are chosen as in Figure 6, then Hom_t can be identified with

$$\mathcal{R} = \{\, M = (M_1, M_2, M_3, M_4) \in SL_2(\mathbb{C})^4 \,:\, M_4 M_3 M_2 M_1 = I \,\},$$

through the map $\mathrm{Hom}_t \to \mathcal{R}$, $\rho \mapsto M$, defined by $M_i = \rho(\gamma_i)$. With this identification, the moduli space of representations $\mathcal{R}_t$ is isomorphic to the categorical quotient

$$\mathcal{R} /\!/ \mathrm{Ad} = \mathrm{Spec}\left(\mathbb{C}[\mathcal{R}]^{\mathrm{Ad}}\right), \tag{38}$$

where Ad represents the diagonal adjoint action of $SL_2(\mathbb{C})$ on $\mathcal{R}$.

The invariant ring $\mathbb{C}[\mathcal{R}]^{\mathrm{Ad}}$ has generators $(x, a) = (x_1, x_2, x_3, a_1, a_2, a_3, a_4)$ given by

$$\begin{cases} x_i &=& \mathrm{Tr}(M_j M_k) \quad (\{i, j, k\} = \{1, 2, 3\}), \\ a_i &=& \mathrm{Tr}\, M_i \quad\quad (i = 1, 2, 3, 4). \end{cases}$$

Note that $a = (a_1, a_2, a_3, a_4) \in A$ is just the local monodromy data defined in (16). We may refer to $x = (x_1, x_2, x_3) \in \mathbb{C}_x^3$ as the *global monodromy data*, since x_i comes from the monodromy matrix $M_j M_k$ along the global loop $\gamma_i \gamma_j$ surrounding the two points t_i and t_j simultaneously. The generators (x, a) have only one algebraic relation $f(x, \theta(a)) = 0$, where $f(x, \theta)$ is the cubic polynomial of x with coefficients $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ defined by

$$f(x, \theta) = x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4, \tag{39}$$

which can be found in the book [**16**] of Fricke and Klein. In terms of local monodromy data $a \in A$, the coefficients $\theta = \theta(a)$ are expressed as

$$\theta_i = \begin{cases} a_i a_4 + a_j a_k & (i = 1, 2, 3), \\ a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 & (i = 4). \end{cases} \tag{40}$$

Let $\Theta := \mathbb{C}_\theta^4$ denote the complex 4-space parametrizing the coefficients θ of $f(x, \theta)$.

6.2. Correspondences of Parameters. — So far, we have encountered three kinds of parameters, that is, the parameters $\kappa \in \mathcal{K}$ of $\mathrm{P_{VI}}$, which is nothing but the exponents of parabolic connections; the local monodromy data $a \in A$; and the coefficients $\theta \in \Theta$ of the cubic polynomial $f(x, \theta)$. Relations among them are depicted in Table 5, where $\kappa \mapsto a$ is given by (18) and $a \mapsto \theta$ is given by (40), respectively.

parameters of	local mono-	parameters
Painlevé VI	dromy data	of cubics
$\kappa \in \mathcal{K}$ $\longmapsto$	$a \in A$ $\longmapsto$	$\theta \in \Theta$

TABLE 5. Correspondences of parameters

In many respects the parameters $\theta \in \Theta$ of cubics surfaces are more essential than the local monodromy data $a \in A$. One reason for this lies in the following observation due to Terajima [**77**].

Lemma 6.1 (**Basis of $W(D_4^{(1)})$-Invariants**). — *As a function of exponents $\kappa \in \mathcal{K}$, the coefficients $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ of the cubic polynomial $f(x, \theta)$ form a basis of $W(D_4^{(1)})$-invariants.*

Here, by θ being a basis of $W(D_4^{(1)})$-invariants, we mean that any $W(D_4^{(1)})$-invariant entire functions on $\mathcal{K}$ is an entire function of θ. So far the map $\mathrm{rh} : \mathcal{K} \to A$ in (17) has been called the Riemann-Hilbert correspondence in the parameter level (see Definition 4.3). However, in view of Lemma 6.1 the following revised definition would be more appropriate.

Definition 6.2 (**Riemann-Hilbert Correspondence in Parameter Level**)

From now on the composite $\mathcal{K} \to \Theta$ of two maps $\mathcal{K} \to A$ in (17) and $A \to \Theta$ in (40) is referred to as the *Riemann-Hilbert correspondence in the parameter level*. Hereafter we write $\mathrm{rh} : \mathcal{K} \to \Theta$.

6.3. Family of Affine Cubic Surfaces. — The cubic equation $f(x, \theta) = 0$ defines a family of affine cubic surfaces, that is, the variety

$$\mathcal{S} = \{ (x, \theta) \in \mathbb{C}_x^3 \times \Theta \ : \ f(x, \theta) = 0 \},$$

together with the projection $\pi : \mathcal{S} \to \Theta$, $(x, \theta) \mapsto \theta$. The previous discussions imply that the categorical quotient $\mathscr{R}/\!/\mathrm{Ad}$ in (38) is realized as the fiber product of $\mathcal{S}$ and A over Θ relative to the natural projections $\pi : \mathcal{S} \to \Theta$ and $A \to \Theta$. Namely we have isomorphisms

$$\mathcal{R}_t \simeq \mathscr{R}/\!/\mathrm{Ad} \simeq \mathcal{S} \times_\Theta A. \tag{41}$$

We write $f_\theta(x) = f(x, \theta)$ regarding it as a polynomial of x depending on parameters θ. For each $\theta \in \Theta$ the fiber of $\pi : \mathcal{S} \to \Theta$ over θ is an affine cubic surface

$$\mathcal{S}(\theta) = \{\, x \in \mathbb{C}^3 \,:\, f_\theta(x) = 0 \,\}.$$

Now (41) means that $\mathcal{R}_t(a)$ is isomorphic to the cubic surface $\mathcal{S}(\theta)$ provided that θ is given by (40) in terms of a. Thus we have the following definition.

Definition 6.3 **(Reformulation of RH)**. — The isomorphism (41) and Definition 6.2 allow us to reformulate the t-Riemann-Hilbert correspondence (19) as the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_t & \xrightarrow{\ \mathrm{RH}_t\ } & \mathcal{S} \\
{\scriptstyle \pi_t}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathcal{K} & \xrightarrow[\ \mathrm{rh}\ ]{} & \Theta
\end{array}
$$

In a similar manner the (t, κ)-Riemann-Hilbert correspondence (24) is reformulated as

$$\mathrm{RH}_{t,k} : \mathcal{M}_t(\kappa) \to \mathcal{S}(\theta), \qquad \theta = \mathrm{rh}(\kappa). \tag{42}$$

Definition 6.4 **(Poincaré Residue)**. — A natural symplectic structure or an area form on the cubic surface $\mathcal{S}(\theta)$ is the *Poincaré residue* defined by

$$\omega_\theta = \frac{dx_i \wedge dx_j}{(\partial f_\theta / \partial x_k)},$$

where (i, j, k) is any cyclic permutation of $(1, 2, 3)$; it does not depends on the cyclic permutation chosen. The smooth and singular loci of $\mathcal{S}(\theta)$ are denoted by $\mathcal{S}^\circ(\theta)$ and $\mathcal{S}^{\mathrm{s}}(\theta)$, respectively. Then the Poincaré residue ω_θ is holomorphic on $\mathcal{S}^\circ(\theta)$, having singularities along $\mathcal{S}^{\mathrm{s}}(\theta)$.

A complete characterization of the singular locus $\mathcal{S}^{\mathrm{s}}(\theta)$ will be presented in Section 9. The following theorem is found in [**34**].

Theorem 6.5 **(Moduli of Representations and Cubic Surface)**. — *For any $(t, a) \in T \times A$, let $\theta = \theta(a)$ be defined by (40). Then there exists an identification of symplectic manifolds*

$$i \,:\, (\mathcal{R}_t(a), \Omega_{\mathcal{R}_t(a)}) \simeq (\mathcal{S}(\theta), \omega_\theta). \tag{43}$$

The main ingredient of the proof is the de Rham theorem. At the end of this section, the following remark would be of some interests.

Remark 6.6 **(Moduli of Cubic Surfaces)**. — It is well known in classical algebraic geometry that the isomorphism classes of cubic surfaces in $\mathbb{P}^3$ have a 4-dimensional moduli space and that there exists a 4-parameter family of general cubic surfaces, known as Cayley's normal form [**9**]. Some computations imply that our family $\mathcal{S}$ and

Cayley's normal form, as modified by Naruki and Sekiguchi [**53, 54**], have a common algebraic cover and hence our family captures general moduli (see Iwasaki [**33**] and Terajima [**77**]). Thus the family $\mathcal{S}$ can be taken as another normal form than Cayley's. It is remarkable that general sixth Painlevé equations are connected with general cubic surfaces through the Riemann-Hilbert correspondence.

7. Bäcklund Transformations — Symmetry

Symmetries of the Painlevé equation are called Bäcklund transformations. Naïvely, a Bäcklund transformation is a birational transformation that converts one Painlevé equation $\mathrm{P_{VI}}(\kappa)$ to another Painlevé equation $\mathrm{P_{VI}}(\kappa')$, where κ and κ' may differ. More precisely, it is a birational map from one phase space $\mathcal{M}(\kappa)$ to another phase space $\mathcal{M}(\kappa')$ that commutes with the Painlevé flows. There are at least two approaches to understand Bäcklund transformations.

Remark 7.1 (Two Approaches to Bäcklund Transformations)
  (1) birational canonical transformations,
  (2) covering transformations of the Riemann-Hilbert correspondence.

The first approach (1) has been employed by such authors as Lukashevich and Yablonski [**43**] Fokas and Ablowitz [**15**] and Okamoto [**63**] in the style of explicit calculations. In particular, Okamoto discovered that $\mathrm{P_{VI}}$ admits affine Weyl group symmetries of type $D_4^{(1)}$. He expressed them as birational canonical transformations of Hamiltonian systems; Noumi and Yamada [**58**] systematized them by a symmetric description in terms of a new Lax pair; Arinkin and Lysenko [**2**] geometrized them as isomorphisms between moduli spaces of $SL_2(\mathbb{C})$-connections; Sakai [**70**] also geometrized them in his Cremona framework; Saito and Umemura [**69**] characterized them as flops. In fact, these various viewpoints are too diverse to be tagged with the same label.

Nonetheless, they still have a common feature to the effect that they look only on a one-side of the Riemann-Hilbert correspondence, that is, the moduli space of parabolic connections or its relatives (spaces where $\mathrm{P_{VI}}$ is defined), with no attentions to the moduli space of representations. On the other hand, the second approach (2) is interested in the interaction between the source space and the target space of Riemann-Hilbert correspondence, asking what the Bäcklund transformations look like through the telescope of Riemann-Hilbert correspondence. In this section we take approach (2), following the exposition of Inaba, Iwasaki and Saito [**28**].

Take any $\kappa \in \mathcal{K}$ and put $\theta = \mathrm{rh}(\kappa) \in \Theta$. Given any element $\sigma \in W(D_4^{(1)})$, we consider the Riemann-Hilbert correspondences (42) for the parameter κ and for its σ-translate $\sigma(\kappa)$,

$$\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{S}(\theta), \qquad \mathrm{RH}_{t,\sigma(\kappa)} : \mathcal{M}_t(\sigma(\kappa)) \to \mathcal{S}(\sigma(\theta)).$$

$$\begin{array}{ccc}
\mathcal{M}_t(\kappa) & \xrightarrow{\;s_\sigma\;} & \mathcal{M}_t(\sigma(\kappa)) \\
{\scriptstyle \mathrm{RH}_{t,\kappa}} \downarrow & & \downarrow {\scriptstyle \mathrm{RH}_{t,\sigma(\kappa)}} \\
\mathcal{S}(\theta) & \xrightarrow[\text{identity}]{} & \mathcal{S}(\sigma(\theta))
\end{array}$$

FIGURE 12. Bäcklund transformations

By our solution to the Riemann-Hilbert problem (see Theorem 4.12), both of them are biholomorphic maps if $\kappa \notin \mathbf{Wall}$, and are minimal resolutions of singularities if $\kappa \in \mathbf{Wall}$, respectively; in any case they are bimeromorhpic morphisms. On the other hand, by the $W(D_4^{(1)})$-invariance of θ (see Lemma 6.1), we have $\theta = \sigma(\theta)$ and hence the cubic surfaces $\mathcal{S}(\theta)$ and $\mathcal{S}(\sigma(\theta))$ are identical. Therefore there exists a unique bimeromorphic map

$$s_\sigma : \mathcal{M}_t(\kappa) \to \mathcal{M}_t(\sigma(\kappa)) \tag{44}$$

that makes the diagram in Figure 12 commutative, that is, the unique lift that covers the *identity* on $\mathcal{S}(\theta) = \mathcal{S}(\sigma(\theta))$ through the Riemann-Hilbert correspondence. In the sprit of approach (2) it is natural to define the concept of Bäcklund transformations in the following manner.

Definition 7.2 **(Bäcklund Transformation)**. — By a *Bäcklund transformation*, we mean the lift s_σ of an element $\sigma \in W(D_4^{(1)})$ as in (44). The group of Bäcklund transformations is, by definition, the group consisting of all those lifts s_σ with $\sigma \in W(D_4^{(1)})$, that is,

$$G = \langle\, s_\sigma \,|\, \sigma \in W(D_4^{(1)}) \,\rangle = \langle s_0, s_1, s_2, s_3, s_4 \rangle \simeq W(D_4^{(1)}).$$

where s_i is the lift of the basic reflection σ_i for $i = 0, 1, 2, 3, 4$ (see (25) and Figure 7); we refer to s_i as the *i*-th *basic* Bäcklund transformation. For each $t \in T$ the group G acts on the moduli space $\mathcal{M}_t$ over $\mathcal{K}$ in such a manner that there exists the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}_t & \xrightarrow{\;G\;} & \mathcal{M}_t \\
{\scriptstyle \pi_t} \downarrow & & \downarrow {\scriptstyle \pi_t} \\
\mathcal{K} & \xrightarrow[W(D_4^{(1)})]{} & \mathcal{K}
\end{array}$$

Remark 7.3 **(Advantage and Disadvantage)**. — The following two advantages of Definition 7.2 are clear.

(1) The character of Bäcklund transformations is transparent, as the covering transformations of the Riemann-Hilbert correspondence, in the sense of Figure 12.

(2) The origin of the affine Weyl group structure of Bäcklund transformations is clear: It just comes from the fact that the Riemann-Hilbert correspondence in the parameter level rh : $\mathcal{K} \to \Theta$ is a branched $W(D_4^{(1)})$-covering.

There is also a disadvantage of this definition.

(3) The birational character of Bäcklund transformations is far from trivial, because our definition makes use of the Riemann-Hilbert correspondence which is highly transcendental. From Definition 7.2 we only know that Bäcklund transformations are bimeromorphic, while their birationality is *a priori* clear from the viewpoints of approach (1).

So we are obliged to discuss the relation between these two approaches and to unify them. In this respect, Inaba, Iwasaki and Saito [**28**] proved the following result.

Theorem 7.4 (Coincidence of Two Approaches). — *The two approaches in Remark* 7.1 *coincide. Namely the Bäcklund transformations in the sense of Definition* 7.2 *are exactly those which have been known as the birational canonical transformations.*

We remark that a different proof of this theorem was given later by Boalch [**7**]. There exists an explicit formula for the basic Bäcklund transformations s_i in terms of certain canonical coordinates on $\mathcal{M}(\kappa)$ (see Theorem 10.13). We can calculate the lift s_i of σ_i, overcoming the transcendental nature of the Riemann-Hilbert correspondence (see [**28**]). As a matter of fact, s_1, s_2, s_3, s_4 are easy to handle and the true difficulty lies in the treatment of s_0. As for this the following remark might be helpful.

Remark 7.5 (Gauge Transformations). — Let W' be the subgroup of $W(D_4^{(1)})$ stabilizing the local monodromy data $a = a(\kappa)$ as a function of κ (see (17) and (18)). The subgroup $G' \subset G$ corresponding to W' is called the group of *gauge transformations*. Note that a Bäcklund transformation is a gauge transformation if and only if it does not change monodromy.

(1) s_1, s_2, s_3, s_4 are very simple gauge transformations; see e.g. [**28**].
(2) s_0 is *not* a gauge transformation. Arinkin and Lysenko [**2**] described it in the level of *abstract* isomorphism between moduli spaces of $SL(2)$-connections. Boalch [**7**] was able to characterize it as a *concrete* transformation of 2×2 Fuchsian systems (11), passing through a 3×3 irregular singular systems via Fourier transformation. It is desirable to realize it as a natural transformation of moduli functors of stable parabolic connections.

As in [**63, 2, 58, 5**], the following remark should be made at this stage.

Remark 7.6 (Extended Affine Weyl Group). — The affine Weyl group symmetry can be enlarged to an *extended* affine Weyl group symmetry, if we allow some permutations of time variables $t = (t_1, t_2, t_3, t_4)$. Let Kl $\simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ be Klein's 4-group

of permutations Kl $= \{\, 1, (12)(34), (13)(24), (14)(23)\,\} \subset S_4$ acting on $T \times \mathcal{K}$ by permuting their components. Note that the semi-direct product

$$\mathrm{Kl} \ltimes W(D_4^{(1)}) = \widetilde{W}(D_4^{(1)}).$$

is the extended affine Weyl group of type $D_4^{(1)}$ acting on $T \times \mathcal{K}$. This action is lifted to the moduli space $\mathcal{M}$ over $T \times \mathcal{K}$ through the Riemann-Hilbert correspondence, as

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\ \widetilde{G}\ } & \mathcal{M} \\[2pt]
\pi \downarrow & & \downarrow \pi \\[2pt]
T \times \mathcal{K} & \xrightarrow[\widetilde{W}(D_4^{(1)})]{} & T \times \mathcal{K}.
\end{array}
$$

The original independent variable of $\mathrm{P_{VI}}$ in (1), namely, the cross ratio x in (6) is Kl-invariant and hence remains invariant under the $\widetilde{W}(D_4^{(1)})$-symmetry. Moreover, if one allows the full S_4-permutations of $t = (t_1, t_2, t_3, t_4)$, one gets the *full* $W(F_4^{(1)})$-*symmetry* of $\mathrm{P_{VI}}$. Its connection with the mapping class group of the 4-holed sphere is discussed in Remark 12 of Boalch [**7**].

8. Nonlinear Monodromy — Poincaré Return Maps

The nonlinear monodromy of the Painlevé flow, or more precisely, that of the non-Riccati Painlevé flow, can be represented explicitly in terms of a certain modular group action on cubic surfaces. In this seciton we are concerned with this description.

8.1. Modular Group Action. — The action is first defined on the ambient space $\mathbb{C}^7 = \mathbb{C}_x^3 \times \Theta$ and then restricted to $\mathcal{S}$. In what follows (i, j, k) stands for any cyclic permutation of $(1, 2, 3)$. We start with the symmetric group S_3 of degree 3 acting on Θ by permuting the first three components of $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$. If $\tau_i = (ij) \in S_3$ denotes the transposition[6] of θ_i and θ_j, then S_3 is generated by τ_1, τ_2, τ_3,

$$S_3 = \langle \tau_1, \tau_2, \tau_3 \rangle.$$

Next we introduce a lift of τ_i to $\mathbb{C}^7$ relative to the second projection $\mathbb{C}^7 = \mathbb{C}_x^3 \times \Theta \to \Theta$.

Definition 8.1 (**Group of Polynomial Automorphisms**). — For each $i = 1, 2, 3$, let

$$g_i : \mathbb{C}^7 \to \mathbb{C}^7, \quad (x, \theta) \mapsto (x', \theta')$$

be the polynomial automorphism defined by the formula

$$(x_i', x_j', x_k', \theta_i', \theta_j', \theta_k', \theta_4') = (\theta_j - x_j - x_k x_i, x_i, x_k, \theta_j, \theta_i, \theta_k, \theta_4).$$

---

[6] Note that τ_3 is not (34) but (31).

Moreover let G denote the transformation group$^{(7)}$ generated by g_1, g_2, g_3, that is,

$$G = \langle g_1, \, g_2, \, g_3 \rangle.$$

A direct check shows that the generators satisfy the three relations

$$g_i g_j g_i = g_j g_i g_j, \qquad (g_i g_j)^3 = 1, \qquad g_k = g_i g_j g_i^{-1},$$

which are exactly the defining relations of the full modular group

$$\Gamma = PSL_2(\mathbb{Z}) = \left\{ z \mapsto \frac{az + b}{cz + d} \; : \; a, \, b, \, c, \, d \in \mathbb{Z}, \; ad - bc = 1 \right\}.$$

So, there exists a group homomorphism $\Gamma \to G$, through which the modular group Γ acts on $\mathbb{C}^7$. This action is restricted to the principal congruence subgroup of level 2,

$$\Gamma(2) = \left\{ z \mapsto \frac{az + b}{cz + d} \in \Gamma \; : \; a \equiv d \equiv 1, \, b \equiv c \equiv 0 \; (\mathrm{mod} \, 2) \right\}.$$

The subgroup of G corresponding to $\Gamma(2) \subset \Gamma$ is given by

$$G(2) = \langle g_1^2, \, g_2^2, \, g_3^2 \rangle,$$

which is referred to as the transformation group of level 2. Note that there exists the natural homomorphism $G \to S_3$ defined by $g_i \mapsto \tau_i = (ij)$, whose kernel is just $G(2)$. Thus we have a 4-parameter family of $\Gamma(2)$-actions on $\mathbb{C}_x^3$ parametrized by $\theta \in \Theta$.

We observe that the polynomial $f(x, \theta)$ in (39) is g_i-invariant and hence the family of cubic surfaces $\mathcal{S}$ is stable under the action of G. Moreover, for each $\theta \in \Theta$, the cubic surface $\mathcal{S}(\theta)$ is stable under the action of $\Gamma(2)$. So the above action can be restricted to cubic surfaces.

Definition 8.2 (Modular Group Actions)

(1) The symmetric group S_3 of degree 3 acts on the base space Θ by permuting the first three components $(\theta_1, \theta_2, \theta_3)$ of θ, while keeping the fourth component θ_4 always fixed.

(2) The full modular group Γ acts on the family $\pi : \mathcal{S} \to \Theta$ of affine cubic surfaces through the transformation group G, covering the action of S_3 on Θ.

(3) The congruence subgroup $\Gamma(2)$ of level 2 acts on each cubic surface $\mathcal{S}(\theta)$ through the transformation group $G(2)$, *area-preservingly* with respect to the Poincaré residue ω_θ.

A full picture of these actions is presented in Figure 13. The identification of these actions with those in §5.3 (see Table 4) is stated in the following manner.

Lemma 8.3 (Braid Versus Modular Group Actions). — *With the isomorphisms* (41) *and* (43), *the braid group actions and the modular group actions are identified, including their symplectic structures, as in Table* 6.

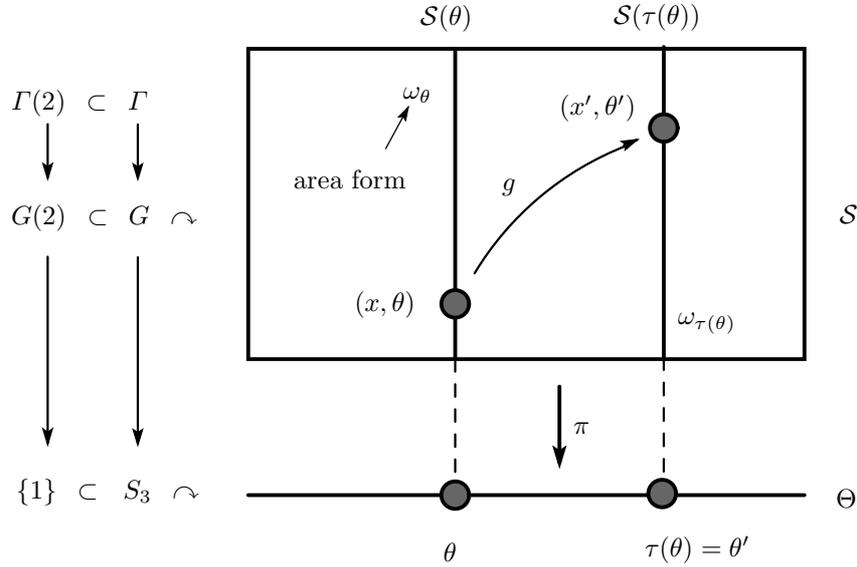$^{(7)}$The group of Bäcklund transformations is also denoted by G in §7, but no confusion might occur.

FIGURE 13. A full picture of the modular group action

	half-monodromy	full-monodromy	area form
braid group action	$B_3 \curvearrowright \mathcal{R}_t$	$P_3 \curvearrowright \mathcal{R}_t(a)$	$\Omega_{\mathcal{R}_t(a)}$
modular group action	$\Gamma \curvearrowright \mathcal{S}$	$\Gamma(2) \curvearrowright \mathcal{S}(\theta)$	ω_θ

TABLE 6. Identification of the braid and modular group actions

8.2. Nonlinear Monodromy of Painlevé Flow. — Having Theorem 5.14 and Lemma 8.3 in hands, we can easily describe the nonlinear monodromy of the non-Riccati part of P_{VI} in terms of the modular group action in Definition 8.2.

Theorem 8.4 (Nonlinear Monodromy). — *For any $\kappa \in \mathcal{K}$, put $\theta = \mathrm{rh}(\kappa) \in \Theta$. Then the nonlinear monodromy of the non-Riccati Painlevé flow $P_{VI}^\circ(\kappa)$ is faithfully represented by the $\Gamma(2)$-action on the smooth locus $\mathcal{S}^\circ(\theta)$ of the cubic surface $\mathcal{S}(\theta)$ through the Riemann-Hilbert correspondence $\mathrm{RH}_{t,\kappa}$. Namely we have the intertwining isomorphism*

$$[\,\mathrm{NM} \text{ of } P_{VI}^\circ(\kappa) \curvearrowright \mathcal{M}_t^\circ(\kappa)\,] \xrightarrow{\ \mathrm{RH}_{t,\kappa}\ } [\,\Gamma(2) \curvearrowright \mathcal{S}^\circ(\theta)\,], \qquad (45)$$

where NM *stands for the nonlinear monodromy. An image picture of (45) is given in Figure 14.*

Remark 8.5 (Dichotomy). — Theorem 8.4 means that the global nature of P_{VI} is well understood according to the *dichotomy* into the Riccati and non-Riccati components.

(1) On the Riccati component P_{VI}^r, the flows are linearizable in terms of Gauss hypergeometric equations (see Theorem 5.15), whose global nature is well understood classically.

(2) On the non-Riccati component P_{VI}°, the nonlinear monodromy is faithfully represented by an explicit modular group action on cubic surfaces, from which we can extract the global nature of P_{VI}°.

After giving a complete picture of Riccati solutions in §9, we shall give a more complete description of the nonlinear monodromy of $P_{VI}(\kappa)$, when it contains the Riccati component, in Theorem 9.9.
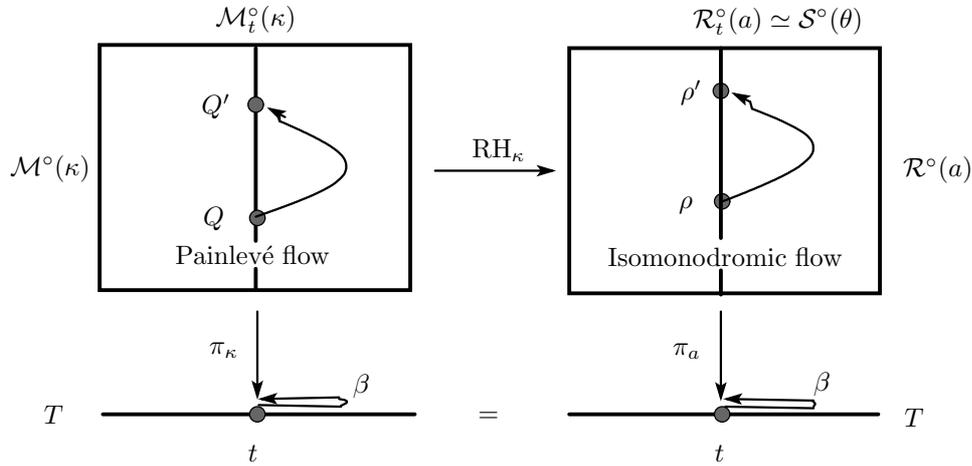
FIGURE 14. Nonlinear monodromy of $P_{VI}^\circ(\kappa)$

The monodromy problem for P_{VI} was discussed in Dubrovin and Mazzocco [**14**] for a special one-parameter family and in Iwasaki [**34**] for the full-family. Then the solution in Iwasaki [**34**] has been completed in Inaba, Iwasaki and Saito [**29**] by solving the Riemann-Hilbert problem precisely and is now presented in this article. The global nature of P_{VI} can also be investigated from a more analytical point of view, as the *connection problem*. For the latter subject we refer to the important papers by Jimbo [**36**] and Guzzetti [**23**]. We also remark that Jimbo's asymptotic formula was corrected by Boalch [**7**].

9. Singularities and Riccati Solutions — Classical Trajectories

We shall classify (-2)-curves on moduli spaces in terms of resolutions of singularities of cubic surfaces by Riemann-Hilbert correspondence. Together with the modular

group action on cubic surfaces, this makes it possible to get a full picture of Riccati solutions to P_{VI}.

9.1. Singularities of Cubic Surfaces. — For our family of cubic surfaces, the discriminant locus was calculated by Iwasaki [**33**].

Definition 9.1 **(Discriminant)**. — Let $\Delta(\theta)$ be the discriminant of the cubic surface $\mathcal{S}(\theta)$, which is an irreducible polynomial of $\theta \in \Theta$. Viewed as a function of $a \in A$ through the map $A \to \Theta$ in (40), the discriminant $\Delta(\theta)$ factors as

$$\Delta(\theta) \;=\; w(a)^2 \prod_{i=1}^{4} (a_i^2 - 4),$$

$$w(a) \;=\; \prod_{\varepsilon_1 \varepsilon_2 \varepsilon_3 = 1} (\varepsilon_1 a_1 + \varepsilon_2 a_2 + \varepsilon_3 a_3 + a_4) - \prod_{i=1}^{3} (a_i a_4 - a_j a_k),$$

where $\varepsilon_i = \pm 1$ and $\{i, j, k\} = \{1, 2, 3\}$.

Lemma 9.2 **(Discriminant Locus)**. — *The Riemann-Hilbert correspondence in the parameter level,* $\mathrm{rh} : \mathcal{K} \to \Theta$, *maps* **Wall** *onto the discriminant locus*

$$V = \{\, \theta \in \Theta \,:\, \Delta(\theta) = 0 \,\}$$

(see Figure 15*). For any* $\theta \in \Theta$, *the cubic surface* $\mathcal{S}(\theta)$ *is singular if and only if* $\theta \in V$.

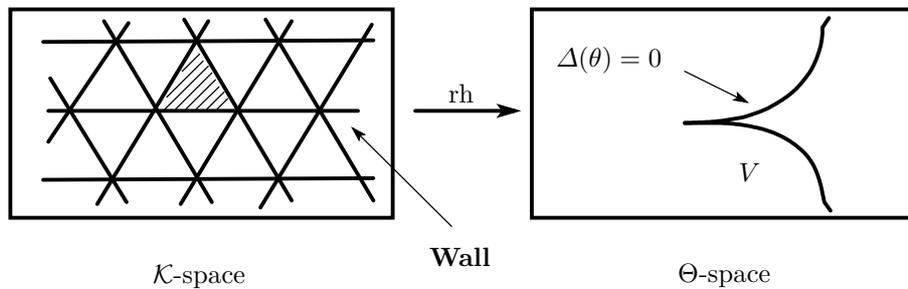
FIGURE 15. Riemann-Hilbert correspondence in the parameter level

In order to classify the singularities of $\mathcal{S}(\theta)$, we introduce a stratification of $\mathcal{K}$.

Definition 9.3 **(Stratification)**. — Let $\mathcal{I}$ denote the set of all *proper* subsets of $\{0, 1, 2, 3, 4\}$ including the empty set $\varnothing$. For each subset $I \in \mathcal{I}$ we put

$$\overline{\mathcal{K}}_I \;=\; W(D_4^{(1)})\text{-translates of the subset } \{\kappa \in \mathcal{K} \,:\, \kappa_i = 0 \ (i \in I)\},$$

$$D_I \;=\; \text{Dynkin subdiagram of } D_4^{(1)} \text{ that has nodes precisely in } I.$$

Let $\mathcal{K}_I$ be the set obtained from $\overline{\mathcal{K}}_I$ by removing the sets $\overline{\mathcal{K}}_J$ with $|J| = |I| + 1$. Then the parameter space $\mathcal{K}$ admits a stratification

$$\mathcal{K} = \bigcup_{I \in \mathcal{I}} \mathcal{K}_I, \qquad \mathcal{K}_\varnothing = \mathcal{K} - \mathbf{Wall} \quad \text{(the nonsingular locus)}.$$

Here either $\mathcal{K}_I = \mathcal{K}_{I'}$ or $\mathcal{K}_I \cap \mathcal{K}_{I'} = \varnothing$ holds for any I, $I' \in \mathcal{I}$. Those Dynkin diagrams which are realized as D_I for some $I \in \mathcal{I}$ are precisely the proper subdiagrams of $D_4^{(1)}$, tabulated in Table 7. We are interested not only in the subdiagram D_I but also in the inclusion pattern $D_I \hookrightarrow D_4^{(1)}$. Some typical patterns are illustrated in Table 8.

number of nodes	4	3	2	1	0
Dynkin diagram	D_4	A_3	A_2	A_1	$\varnothing$
	$A_1^{\oplus 4}$	$A_1^{\oplus 3}$	$A_1^{\oplus 2}$	–	–

TABLE 7. Dynkin types of singularities

Using the stratification in Definition 9.3, we can clearly describe all the possible singularities.

Theorem 9.4 (Classfication of Singularities). — *For any $I \in \mathcal{I} - \{\varnothing\}$ and any $\kappa \in \mathcal{K}_I$, the surface $\mathcal{S}(\theta)$ with $\theta = \mathrm{rh}(\kappa)$ has simple singularities of type D_I.*

In this situation the Riemann-Hilbert correspondence $\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{S}(\theta)$ gives a minimal resolution of singularities (Theorem 4.12). Thus the exceptional divisor of $\mathrm{RH}_{t,k}$, namely, the Riccati locus $\mathcal{M}_t^r(\kappa) \subset \mathcal{M}_t(\kappa)$ has the dual graph of Dynkin type D_I. We give an example for which a singularity of D_4-type occurs.

Example 9.5 (Singularity of Type D_4). — For $\theta = (8, 8, 8, 28)$ the cubic surface $\mathcal{S}(\theta)$ has a simple singularity of type D_4. If κ is a $W(D_4^{(1)})$-translate of $(0, 0, 0, 0, 1)$, then one has $\mathrm{rh}(\kappa) = \theta$ and the (t, κ)-Riemann-Hilbert correspondence $\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{S}(\theta)$ gives a minimal resolution of singularity as in Figure 16. In this case we have the Riccati locus of type D_4,

$$\mathcal{M}_t^r(\kappa) = e_0 \cup e_1 \cup e_2 \cup e_3. \tag{46}$$

9.2. A Complete Picture of Riccati Solutions. — As is mentioned in §5.5, Saito and Terajima [**68**] established the relation between (-2)-curves and Riccati solutions; this is an event on the Painlevé equation side. On the other hand, on the cubic surface side, Iwasaki [**33, 34**] pointed out that the singular points on $\mathcal{S}(\theta)$ are precisely the fixed points of the $\Gamma(2)$-action. Then Inaba, Iwasaki and Saito [**29**] added one more piece, namely, the aspect of resolution of singularities by Riemann-Hilbert
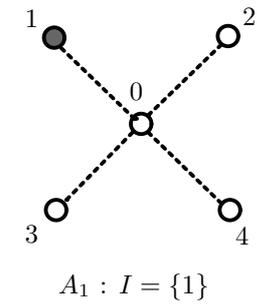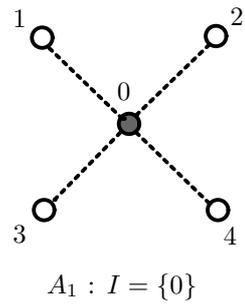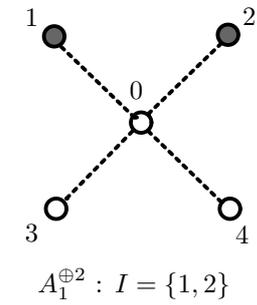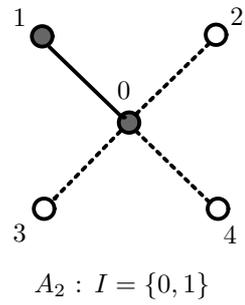
$D_4 : I = \{0, 1, 2, 3\}$

$A_1^{\oplus 4} : I = \{1, 2, 3, 4\}$

$A_3 : I = \{0, 1, 2\}$

$A_1^{\oplus 3} : I = \{1, 2, 3\}$
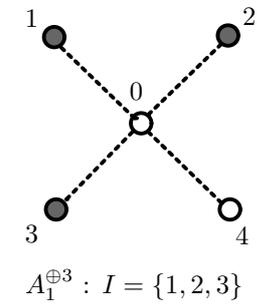
$A_2 : I = \{0, 1\}$

$A_1^{\oplus 2} : I = \{1, 2\}$
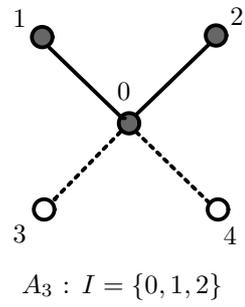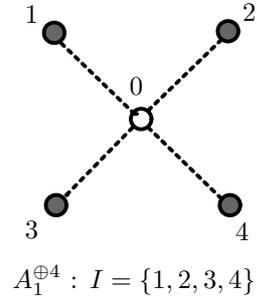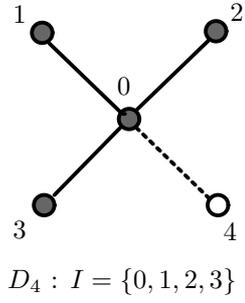
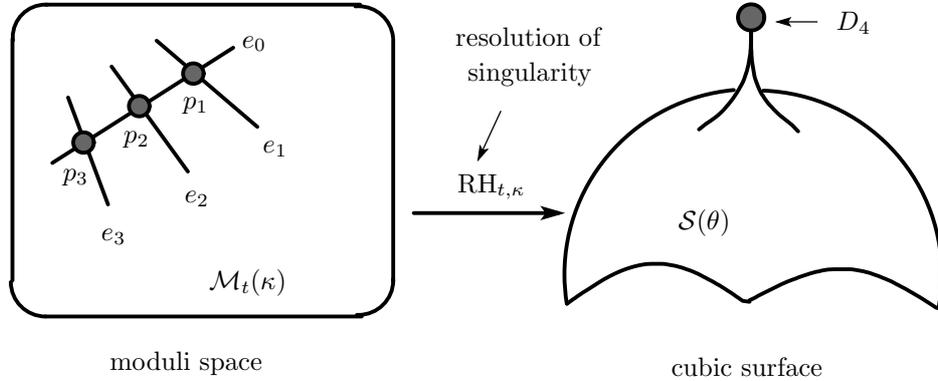$A_1 : I = \{0\}$

$A_1 : I = \{1\}$

TABLE 8. Examples of strata

FIGURE 16. A singularity of type D_4 for $\kappa = (0,0,0,0,1)$ and $\theta = (8,8,8,28)$

correspondence, which throws a bridge between the previous two aspects. Combining all these three, we are now able to get a full picture of Riccati solutions.

Theorem 9.6 (A Complete Picture). — *Let $(t, \kappa) \in T \times$ **Wall** and put $\theta = \mathrm{rh}(\kappa) \in \Theta$.*

(1) *The germs at t of Riccati solutions to $\mathrm{P_{VI}}(\kappa)$ are in one-to-one correspondence with the points on the (-2)-curves on the moduli space $\mathcal{M}_t(\kappa)$.*

(2) *Each (-2)-curve on the moduli space $\mathcal{M}_t(\kappa)$ is sent to a singular point on the cubic surface $\mathcal{S}(\theta)$ by the Riemann-Hilbert correspondence $\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{S}(\theta)$.*

(3) *Conversely, any (-2)-curve on the moduli space $\mathcal{M}_t(\kappa)$ arises as an irreducible component of the exceptional divisor of the minimal resolution of singularities $\mathrm{RH}_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{S}(\theta)$.*

(4) *The singular points on $\mathcal{S}(\theta)$ are exactly the fixed points of the $\Gamma(2)$-action on $\mathcal{S}(\theta)$.*

(5) *The singular points on $\mathcal{S}(\theta)$, as well as the (-2)-curves on $\mathcal{M}_t(\kappa)$, are completely classified as in Theorem 9.4.*

Theorem 9.6 is visualized as in Figure 17. The following is a simple application.

Corollary 9.7 (Single-Valued Solutions). — *Any single-valued solution to $\mathrm{P_{VI}}$ is a Riccati solution and moreover it is a rational solution.*

Proof. — The proof is very easy by now. Not a Riccati solution implies not a fixed point, implies not a single-valued solution, since the Riemann-Hilbert correspondence is one-to-one outside the Riccati locus. Hence any single-valued solution must be Riccati. Now recall that a Riccati solution is a logarithmic derivative of a hypergeometric function (see Theorem 5.15). Such a function of regular singular type can be single-valued only if it is a rational function. The proof is complete. □
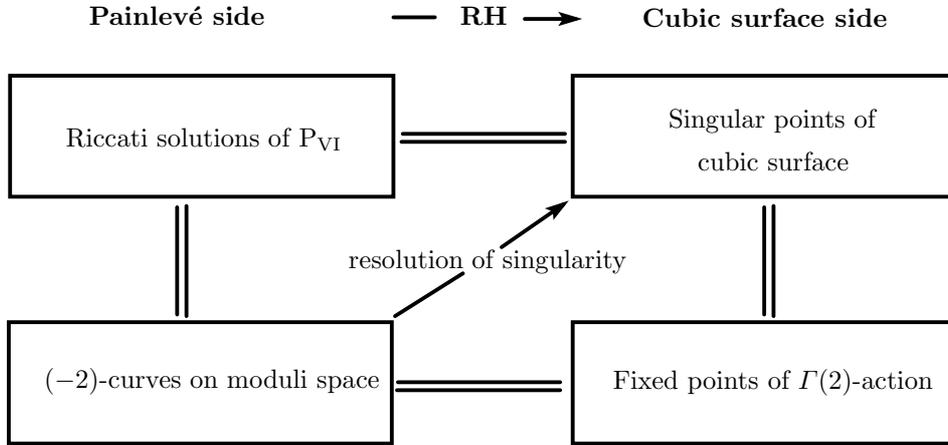
Painlevé side   —— **RH** ——▶   **Cubic surface side**

FIGURE 17. A complete picture of Riccati solutions

The rational solutions to P_{VI} were classified by Mazzocco [**48**].

Example 9.8 (**Some Rational Solutions**). — In Example 9.5 any solution on the (-2)-curve e_0 is rational. Indeed, any half-monodromy α preserves the Riccati configuration (46) and hence induces an automorphism (a Möbius transformation) β of $e_0 \simeq \mathbb{P}^1$ which just permutes the three points p_1, p_2, p_3 in Figure 16. If α is a full-monodromy, then the corresponding β fixes each of p_1, p_2, p_3 and hence is identity on e_0. This means that any solution on e_0 is single-valued. By Corollary 9.7 it is a rational solution.

We refer to Lukashevich and Yablonski [**43**], Fokas and Ablowitz [**15**], Okamoto [**63**], Watanabe [**79**], Gromak, Laine and Shimomura [**22**] and the references therein for explicit calculations of Riccati solutions.

Having established a complete picture of Riccati solutions, especially of their connection with Klein sigularities, we shall revisit the nonlinear monodromy of P_{VI} discussed in §8. For any $\kappa \in \mathcal{K} - \mathbf{Wall}$, Theorem 8.4 completely describes the nonlinear monodromy of $P_{VI}(\kappa)$, since $P_{VI}(\kappa)$ has no Riccati locus. We now deal with the case where $\kappa \in \mathbf{Wall}$. While the Riemann-Hilbert correspondence $RH_{t,\kappa} : \mathcal{M}_t(\kappa) \to \mathcal{S}(\theta)$ with $\theta = rh(\kappa)$ is an *analytic* minimal resolution of singularities, there exists such a standard *algebraic* minimal resolution $\widetilde{\mathcal{S}}(\theta) \to \mathcal{S}(\theta)$ that was constructed by Brieskorn [**8**] for Klein singularities. By the minimality, the Riemann-Hilbert correspondence lifts up to a biholomorphism

$$RH_{t,\kappa} : \mathcal{M}_t(\kappa) \to \widetilde{\mathcal{S}}(\theta), \tag{47}$$

and the action of $\Gamma(2)$ on $\mathcal{S}(\theta)$ can also be lifted to $\widetilde{\mathcal{S}}(\theta)$ uniquely. Combining these facts with Theorem 8.4, we have the following theorem.

Theorem 9.9 **(Nonlinear Monodromy Revisited)**. — *For any* $\kappa \in$ **Wall***, put* $\theta =$ $\mathrm{rh}(\kappa) \in \Theta$. *Then the nonlinear monodromy of* $\mathrm{P_{VI}}(\kappa)$ *is faithfully represented by the* $\Gamma(2)$-*action lifted on* $\widetilde{\mathcal{S}}(\theta)$.

An even more complete picture is obtained if one settles the following problem.

Problem 9.10 **(A Moduli Problem)**. — Construct $\widetilde{\mathcal{S}}(\theta)$ as a moduli space of monodromy representations with "parabolic structures" and set up the Riemann-Hilbert correspondence (47) directly without passing through the resolutions of singularities.

10. Canonical Coordinates

The moduli space $\mathcal{M}(\kappa)$ admits a natural canonical coordinate system whose local charts are laveled by the affine Weyl group $W(D_4^{(1)})$. In this section we shall construct such coordinates and write down the Painlevé dynamics explicitly in terms of them. The principle of producing canonical coordinates is the Wronskian construction that converts a stable parabolic connection to a second-order single Fuchsian differntial equation. In this section we mean by T the configuration space of distinct ordered three points in $\mathbb{C}$ as in (32) upon putting $t_4 = \infty$; hence

$$t = (t_1, t_2, t_3) = (t_1, t_2, t_3, \infty) \in T.$$

10.1. Space of Fuchsian Equations. — We start with spaces of Fuchsian equations from which local coordinates are to be extracted.

Definition 10.1 **(Fuchsian Equations)**. — For any $\kappa \in \mathcal{K}$, let $\mathcal{E}(\kappa)$ be the set of all second-order Fuchsian differential equations of the form

$$\frac{d^2 f}{dz^2} - v_1(z) \frac{df}{dz} + v_2(z) f = 0, \tag{48}$$

with four regular singular points $t = (t_1, t_2, t_3, t_4) \in T$ and an apparent singular point q, having Riemann scheme as in Table 9, where κ is fixed while t and q may vary in such a manner that q does not meet any of t_1, t_2, t_3, t_4.

singularity	t_1	t_2	t_3	$t_4 = \infty$	q
first exponent	0	0	0	κ_0	0
second exponent	κ_1	κ_2	κ_3	$\kappa_4 + \kappa_0$	2

TABLE 9. Riemann Scheme

The affine linear relation $2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1$ in (2) is exactly Fuchs' relation for Fuchsian differential equation (48). The classical Fuchs-Frobenius method in the

theory of Fuchsian differential equations allows us to determine the coefficients $v_1(z)$ and $v_2(z)$ as

$$v_1(z) = \frac{1}{z-q} + \sum_{i=1}^{3} \frac{\kappa_i - 1}{z - t_i}, \qquad v_2(z) = \frac{p}{z-q} + \sum_{i=1}^{3} \frac{H_i(\kappa)}{z - t_i}. \tag{49}$$

The condition that q is apparent with exponents 0 and 2 implies that $H_i(\kappa) = H_i(q, p, t; \kappa)$ is a function of (q, p, t, κ). This function, called the i-th *Hamiltonian*, is explicitly determined as follows.

***Lemma 10.2* (Hamiltonians)**. — *The i-th Hamiltonian $H_i(\kappa) = H_i(q, p, t; \kappa)$ is given by*

$$(t_{ij} t_{ik})\, H_i(\kappa) = (q_i q_j q_k) p^2 - \{(\kappa_i - 1) q_j q_k + \kappa_j q_k q_i + \kappa_k q_i q_j\} p + \kappa_0(\kappa_0 + \kappa_4) q_i, \tag{50}$$

with $\{i, j, k\} = \{1, 2, 3\}$, where $q_i := q - t_i$ and $t_{ij} := t_i - t_j$.

***Remark 10.3* (Polynomial Hamiltonians)**. — Note that Hamiltonians (50) are polynomials of (q, p). This is because one exponent at each finite singular point, t_1, t_2, t_3, q, is *zero* in the Riemann scheme of Table 9. We call (50) *polynomial Hamiltonians* (see Okamoto [**60, 61**]).

Formulas (49) and (50) tell us that for a fixed κ, the Fuchsian equation (48) is determined uniquely by the data (q, p, t). Thus the following definition is natural.

***Definition 10.4* (Canonical Coordinates)**. — The set $\mathcal{E}(\kappa)$ is identified with the affine variety

$$U = \{\, (q, p, t) \in \mathbb{C}_q \times \mathbb{C}_p \times \mathbb{C}_t^3 \, : \, q \neq t_i, \ t_i \neq t_j \ \text{for} \ i \neq j \,\}, \tag{51}$$

having coordinates (q, p, t), on which the fundamental 2-form is defined by

$$\Omega_{\mathcal{E}(\kappa)} = dq \wedge dp - \sum_{i=1}^{3} dH_i(q, p; \kappa) \wedge dt_i \tag{52}$$

10.2. Wronskian Construction. — First we shall define the concept of apparent singular point of a stable parabolic connection. Let $Q = (E, \nabla, \psi, l) \in \mathcal{M}(\kappa)$ be any stable parabolic connection. By the stability of Q, we can show that there exists a unique line subbundle $F \subset E$ of maximal degree. The line bundle F is called the maximal subbundle of E and the quotient bundle $L = E/F$ is called the minimal quotient bundle of E. Note that F and L are of degrees 0 and -1, respectively. Let

$$\pi : E \to L = E/F \tag{53}$$

be the canonical projection. We see that the composite $u : F \to L \otimes \Omega_{\mathbb{P}^1}^1(D_t)$ of the sequence

$$F \xrightarrow{\ \text{inclusion}\ } E \xrightarrow{\ \nabla\ } E \otimes \Omega_{\mathbb{P}^1}^1(D_t) \xrightarrow{\ \pi \otimes 1\ } L \otimes \Omega_{\mathbb{P}^1}^1(D_t)$$

is an $\mathcal{O}_{\mathbb{P}^1}$-homomorphism and gives a holomorphic section of the line bundle $\mathrm{Hom}(F, L) \otimes \Omega_{\mathbb{P}^1}^1(D_t)$, where $t \in T$ is the regular singular points of Q. Then

the stability of Q implies that u is a nontrivial section. Since the line bundle $\mathrm{Hom}(F, L) \otimes \Omega^1_{\mathbb{P}^1}(D_t)$ is of degree one, the nontrivial section u has a unique simple zero q. Since the construction so far is canonical, the point $q = q(Q) \in \mathbb{P}^1$ is uniquely determined by $Q \in \mathcal{M}(\kappa)$. Hence we have a well-defined morphism

$$q : \mathcal{M}(\kappa) \to \mathbb{P}^1, \qquad Q \mapsto q = q(Q). \tag{54}$$

Definition 10.5 **(Apparent Singular Point)**. — The point $q = q(Q) \in \mathbb{P}^1$ in (54) is called the *apparent singular point* of the stable parabolic connection $Q \in \mathcal{M}(\kappa)$.

Using the morphism (54), we can consider the locus $\mathcal{M}^{\mathrm{id}}(\kappa) \subset \mathcal{M}(\kappa)$ where the apparent singular point q does not meet any regular singular point t_i, $i = 1, 2, 3, 4$:

$$\mathcal{M}^{\mathrm{id}}(\kappa) = \{ Q \in \mathcal{M}(\kappa) : q(Q) \neq t_i(Q) \quad (i = 1, 2, 3, 4) \},$$

where $t_i = t_i(Q)$ denotes the i-th regular singular point of Q.

Next we proceed to the Wronskian construction that recast each stable parabolic connection in $\mathcal{M}^{\mathrm{id}}(\kappa)$ to a Fuchsian differential equation in $\mathcal{E}(\kappa)$. Given a stable parabolic connection $Q = (E, \nabla, \psi, l) \in \mathcal{M}(\kappa)$, we consider the locally constant sheaf

$$\mathcal{L}' = \mathrm{Ker}\, \big[\, \nabla|_{\mathbb{P}^1 - D_t} : E|_{\mathbb{P}^1 - D_t} \to E|_{\mathbb{P}^1 - D_t} \otimes \Omega^1_{\mathbb{P}^1 - D_t} \,\big]$$

of ∇-horizontal sections on $\mathbb{P}^1 - D_t$. By the stability of Q we can show that the canonical projection π in (53) induces an isomorphism of locally constant sheaves on $\mathbb{P}^1 - D_t$,

$$\pi : \mathcal{L}' \to \pi(\mathcal{L}') \subset L|_{\mathbb{P}^1 - D_t}$$

On the other hand, since the line bundle L^{-1} is of degree one, there exists a unique connection $\delta : L^{-1} \to L^{-1} \otimes \Omega^1_{\mathbb{P}^1}(D_t)$ whose residue at each singular point is given by

$$\mathrm{Res}_{t_i}(\delta) = \begin{cases} -\lambda_i & (i = 1, 2, 3), \\ \lambda_1 + \lambda_2 + \lambda_3 - 1 & (i = 4), \end{cases}$$

where λ_i is given by (7) in terms of κ_i. Let $\mathcal{L}''$ be the locally constant sheaf

$$\mathcal{L}'' = \mathrm{Ker}\, \big[\, \delta|_{\mathbb{P}^1 - D_t} : L^{-1}|_{\mathbb{P}^1 - D_t} \to L^{-1}|_{\mathbb{P}^1 - D_t} \otimes \Omega^1_{\mathbb{P}^1 - D_t} \,\big] \subset L^{-1}|_{\mathbb{P}^1 - D_t}.$$

of δ-horizontal sections on $\mathbb{P}^1 - D_t$. Tensoring $\pi(\mathcal{L}')$ with $\mathcal{L}''$, we have a locally constant sheaf

$$\mathcal{L}_Q = \pi(\mathcal{L}') \otimes \mathcal{L}'' \subset \mathcal{O}_{\mathbb{P}^1 - D_t}, \tag{55}$$

canonically associated to the stable parabolic connection $Q \in \mathcal{M}(\kappa)$ (see Remark 10.7 for the meaning of the tensoring with $\mathcal{L}''$). The construction so far is valid for any $Q \in \mathcal{M}(\kappa)$, but we have to put Q in $\mathcal{M}^{\mathrm{id}}(\kappa)$ to obtain the following theorem.

Theorem 10.6 **(Wronskian Isomorphism)**. — *For a stable parabolic connection $Q \in \mathcal{M}^{\mathrm{id}}(\kappa)$, with singular points at $t \in T$, we consider the second-order, single, monic differential equation on $\mathbb{P}^1 - D_t$ whose solution sheaf is given by the locally constant sheaf $\mathcal{L}_Q$ in (55). Then it is exactly such a Fuchsian differential equation that is*

formulated in Definition 10.1 *with apparent singular point at $q = q(Q)$ given by* (54). *Therefore there exists a well-defined morphism*

$$\Phi_\kappa : \mathcal{M}^{\mathrm{id}}(\kappa) \to \mathcal{E}(\kappa). \tag{56}$$

This morphism becomes an isomorphism.

Combined with the identification $\mathcal{E}(\kappa) \simeq U$ in Definition 10.4, where the set U is defined by (51), the isomorphism (56) yields a local coordinate mapping

$$\Psi_\kappa : \mathcal{M}^{\mathrm{id}}(\kappa) \to U, \quad Q \mapsto (q, p, t). \tag{57}$$

At the end of this subsection we emphasize that stability has been used many times in the Wronskian construction. Lastly the following technical remark may be helpful.

Remark 10.7 **(Shift of Exponents)**. — The essential factor in (55) is the rank-two local system $\pi(\mathcal{L}')$, which is tensored with the rank-one local system $\mathcal{L}''$ just for shifting the exponents. By the tensoring with $\mathcal{L}''$, the exponents in Table 3 are shifted to those in Table 9 by the vector $(\lambda_1, \lambda_2, \lambda_3, 1 - \lambda_1 - \lambda_2 - \lambda_3)$ at $t = (t_1, t_2, t_3, t_4)$. This process is needed to obtain polynomial Hamiltonians as in (50) (see Remark 10.3).

10.3. Canonical Coordinate System.
— Combined with Bäcklund transformations, Theorem 10.6 produces a canonical coordinate system on the moduli space $\mathcal{M}(\kappa)$. To see this, for each $\sigma \in W(D_4^{(1)})$, consider the open subset

$$\mathcal{M}^\sigma(\kappa) = s_\sigma^{-1}(\mathcal{M}^{\mathrm{id}}(\sigma(\kappa))) \subset \mathcal{M}(\kappa),$$

where $s_\sigma : \mathcal{M}(\kappa) \to \mathcal{M}(\sigma(\kappa))$ is the Bäcklund transformation corresponding to σ (see Figure 12). Then there exists an open covering of the moduli space $\mathcal{M}(\kappa)$,

$$\mathcal{M}(\kappa) = \bigcup_{\sigma \in W(D_4^{(1)})} \mathcal{M}^\sigma(\kappa)$$

On each open subset $\mathcal{M}^\sigma(\kappa)$ we have an isomorphism

$$\Phi_\kappa^\sigma : \mathcal{M}^\sigma(\kappa) \to \mathcal{E}(\sigma(\kappa)), \tag{58}$$

defined as the composite of the sequence of isomorphisms

$$\mathcal{M}^\sigma(\kappa) \xrightarrow{s_\sigma} \mathcal{M}^{\mathrm{id}}(\sigma(\kappa)) \xrightarrow{\Phi_{\sigma(\kappa)}} \mathcal{E}(\sigma(\kappa)).$$

To see that (58) is a Poisson isomorphism, we make use of the following theorem.

Theorem 10.8 **(Pull-Back Principle)**. — *Let $\kappa \in \mathcal{K}$ and put $a = \mathrm{rh}(\kappa) \in A$. We define the local Riemann-Hilbert correspondence $\mathrm{RH}_\kappa^\sigma : \mathcal{E}(\sigma(\kappa)) \to \mathcal{R}(a)$ as the composite of the sequence*

$$\mathcal{E}(\sigma(\kappa)) \xrightarrow{(\Phi_\kappa^\sigma)^{-1}} \mathcal{M}^\sigma(\kappa) \hookrightarrow \mathcal{M}(\kappa) \xrightarrow{\mathrm{RH}_\kappa} \mathcal{R}(a).$$

Then the fundamental 2-form $\Omega_{\mathcal{E}(\sigma(\kappa))}$ on $\mathcal{E}(\sigma(\kappa))$ is the pull-back of $\Omega_{\mathcal{R}(a)}$ by $\mathrm{RH}_\kappa^\sigma$,

$$\Omega_{\mathcal{E}(\sigma(\kappa))} = (\mathrm{RH}_\kappa^\sigma)^* \Omega_{\mathcal{R}(a)}.$$

This theorem is due to Iwasaki [**32**], where the map $\mathrm{RH}_\kappa^\sigma$ is defined[8] directly without passing through the moduli space $\mathcal{M}(\kappa)$. Hence we have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}^\sigma(\kappa) & \xrightarrow{\ \text{inclusion}\ } & \mathcal{M}(\kappa) \\
\Phi_\kappa^\sigma \downarrow & & \downarrow \mathrm{RH}_\kappa \\
\mathcal{E}(\sigma(\kappa)) & \xrightarrow[\mathrm{RH}_\kappa^\sigma]{} & \mathcal{R}(a),
\end{array}
$$

where RH_κ is Poisson by Theorem 5.10 while $\mathrm{RH}_\kappa^\sigma$ is also Poisson by Theorem 10.8 respectively. Therefore Φ_κ^σ becomes a Poisson isomorphism as desired.

Definition 10.9 (**Canonical Coordinate System**). — By the same procedure as in (57) the Poisson isomorphisms (58) induce local coordinate mappings

$$
\Psi_\kappa^\sigma : \mathcal{M}^\sigma(\kappa) \to U_\sigma, \quad Q \mapsto (q_\sigma, p_\sigma, t) \qquad (\sigma \in W(D_4^{(1)})), \tag{59}
$$

where U_σ is a copy of U endowed with the coordinates (q_σ, p_σ, t). Note that we have $\Psi_\kappa^\sigma = \Psi_{\sigma(\kappa)} \circ s_\sigma$ with Ψ_κ given by (57). The collection of maps (59) is referred to as the *canonical coordinate system* on the moduli space $\mathcal{M}(\kappa)$.

We are now in a position to derive a Hamiltonian system of differential equations for $\mathrm{P_{VI}}(\kappa)$ on each local chart $\mathcal{M}^\sigma(\kappa) \simeq U_\sigma$ based on the idea in Remark 2.6.

Theorem 10.10 (**Hamiltonian System**). — *In terms of the canonical coordinates* (q_σ, p_σ, t) *on* $\mathcal{M}^\sigma(\kappa) \simeq U_\sigma$, *the Painlevé flow* $\mathrm{P_{VI}}(\kappa)$ *is expressed as the Hamiltonian system* $\mathrm{H}_{\mathrm{VI}}^\sigma(\kappa)$ *of differential equations,*

$$
\frac{\partial q_\sigma}{\partial t_i} = \frac{\partial H_i(\sigma(\kappa))}{\partial p_\sigma}, \qquad \frac{\partial p_\sigma}{\partial t_i} = -\frac{\partial H_i(\sigma(\kappa))}{\partial q_\sigma}, \tag{60}
$$

with Hamiltonians $H_i(\sigma(\kappa)) = H_i(q_\sigma, p_\sigma, t; \sigma(\kappa))$ *where* $H_i(q, p, t; \kappa)$ *is given by* (50).

Proof. — The Painlevé flow $\mathrm{P_{VI}}(\kappa)$ is characterized by the condition that $\iota_v \Omega_{\mathcal{M}(\kappa)} = 0$ for every $\mathcal{F}_{\mathrm{P_{VI}}(\kappa)}$-horizontal vector field v. Since (58) is a Poisson isomorphism, this condition is equivalent to $\iota_v \Omega_{\mathcal{E}(\sigma(\kappa))} = 0$, from which system (60) readily follows. $\quad\square$

Remark 10.11 (**Malmquist Expression**). — Malmquist [**45**] obtained a Hamiltonian expression for $\mathrm{P_{VI}}$ as early as 1923. Our expression (60) is just a symmetric form of his expression that can be reduced to his original by the symplectic reduction in Remark 2.13. Malmquist's expression was rediscovered by Okamoto [**60, 61, 62**] in the isomonodromic context. Deriving Hamiltonian systems as in (60) by the pull-back principle in Theorem 10.8 is due to Iwasaki [**32**], where he works on a Riemann surface of arbitrary genus.

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[8]To be more precise, it is formulated only for $\sigma = \mathrm{id}$, but the modification for a general σ is obvious.

Theorem 10.12 (Analytic Painlevé Property). — *For any $\kappa \in \mathcal{K}$ and $\sigma \in W(D_4^{(1)})$ the Hamiltonian system $\mathrm{H}_{\mathrm{VI}}^{\sigma}(\kappa)$ has analytic Painlevé property.*

This theorem immediately follows from the geometric Painlevé property of the Painlevé flow $\mathcal{F}_{\mathrm{P}_{\mathrm{VI}}(\kappa)}$ (see Theorem 5.12) and the algebraicity of the phase space $\mathcal{M}(\kappa)$ (see Remark 2.8).

Theorem 10.13 (Basic Bäcklund Transformations). — *In terms of canonical coordinate charts in* (57),

$$\mathcal{M}^{\mathrm{id}}(\kappa) \simeq U \simeq \mathcal{M}^{\mathrm{id}}(\sigma_i(\kappa)) \qquad (i = 0, 1, 2, 3, 4),$$

the i-th basic Bäcklund transformation s_i is expressed as the birational canonical transformation

$$s_i(\kappa_j) = \kappa_j - \kappa_i\, c_{ij}, \qquad s_i(t_j) = t_j, \qquad s_i(q_j) = q_j + \frac{\kappa_i}{q_i}\, u_{ij}. \tag{61}$$

where $C = (c_{ij})$ is the Cartan matrix of type $D_4^{(1)}$ (see Figure 7) and

$$q_i = \begin{cases} p & (i = 0), \\ q - t_i & (i = 1, 2, 3, 4), \end{cases} \qquad u_{ij} = \{q_i, q_j\} = \frac{\partial q_i}{\partial p} \frac{\partial q_j}{\partial q} - \frac{\partial q_i}{\partial q} \frac{\partial q_j}{\partial p}.$$

As is mentioned after Theorem 7.4, it is not so straightforward to derive the formula (61) for s_0 from our definition of Bäcklund transformations in Definition 7.2. Our strategy is the coalescence of regular singular points along isomonodromic flow; see Inaba, Iwasaki and Saito [**28**].

Such coordinate expressions as in (60) and (61) have been the starting point of the traditional story. The other way round, in our story, we end up with coordinate expressions as concrete realizations of the abstract dynamical system P_{VI} that is defined conceptually.

Remark 10.14 (Gluing by Bäcklund Transformations). — The moduli space $\mathcal{M}(\kappa)$ is made up of local charts glued by Bäcklund transformations. Indeed it is clear from Definition 10.9 that for $\sigma, \sigma' \in W(D_4^{(1)})$ the transition function from $\mathcal{M}^{\sigma}(\kappa) \simeq U_{\sigma}$ to $\mathcal{M}^{\sigma'}(\kappa) \simeq U_{\sigma'}$ is just the Bäcklund transformation $s_{\sigma'\sigma^{-1}} = s_{\sigma'} s_{\sigma}^{-1}$. Noumi, Takano and Yamada [**57**] showed that their "manifold of Painlevé system" can be constructed in this way. Their empirical observation is trivial from our point of view, or even from the meta-physics: the phase space of a dynamical system should be made up of inertial coordinates glued together by symmetries of the system.

The construction of moduli spaces and that of canonical coordinates lead to the following remark.

Remark 10.15 (Systems or Single Equations?). — In doing isomonodromic deformations, some people work with first-order linear systems as in (11), while others work with second-order single equations as in (48). We may ask which approach is better. The answer is that both are important and necessary. Systems are sophisticated

to stable parabolic connections and are used to construct the phase space of the Painlevé dynamical system, while single equations are used to construct canonical coordinates of the phase space that make it possible to get a concrete realization of the dynamics. Therefore both are important and necessary.

11. Summary

In this article we have observed the natural manner in which the *continuous* Hamiltonian system P_{VI} induces two *discrete* Hamiltonian systems:

(1) **Bäcklund transformations** as convering transformations of the Riemann-Hilbert correspondence. They describe the symmetries of P_{VI}.

(2) **Poincaré return maps** (or the **nonlinear monodromy**). Through the Riemann-Hilbert correspondence, they are realized as an area-preserving action of the modular group on smooth affine cubic surfaces, or on the minimal desingularizations of singular affine cubic surfaces. They describe the global structures, especially the multi-valuedness, of the trajectories of P_{VI}.

Here we recall that the geometric Painlevé property is needed in order for the Poincaré return maps to be well-defined. In this respect we have shown by the conjugacy method that

(3) the **geometric Painlevé property** of the Painlevé flow follows from that of the isomonodromic flow, which holds trivially, through the Riemann-Hilbert correspondence.

As to the Riccati component of P_{VI} comprising the classical torajectories that can be linearized in terms of Gauss hypergeometric equations, we have given

(4) a **complete picture of Riccati solutions** in terms of resolutions of singularities by the Riemann-Hilbert correspondence.

Concerning the concrete realization of the Painlevé dynamics, we have constructed

(5) a **canonical coordinate system** via the Wronskian construction, in terms of which Hamiltonian systems and Bäcklund transformations are written down explicitly. The nonlinear monodromy is also made explicit in terms of **cubic surfaces** as in item (2).

We started the main body of this article with the Guiding Diagram in Figure 3. We wish to close the article with the Concluding Diagram in Figure 18. Located in the central position of the diagram, as well as of the development of our story, is the Riemann-Hilbert correspondence

$$\mathrm{RH}_\kappa : (\mathcal{M}(\kappa), \Omega_{\mathcal{M}(\kappa)}) \to (\mathcal{R}(a), \Omega_{\mathcal{R}(a)})$$

in the precise moduli-theoretical setting. The Painlevé dynamics encoded in this abstract object is concretized on both sides of the diagram: Hamiltonian systems and
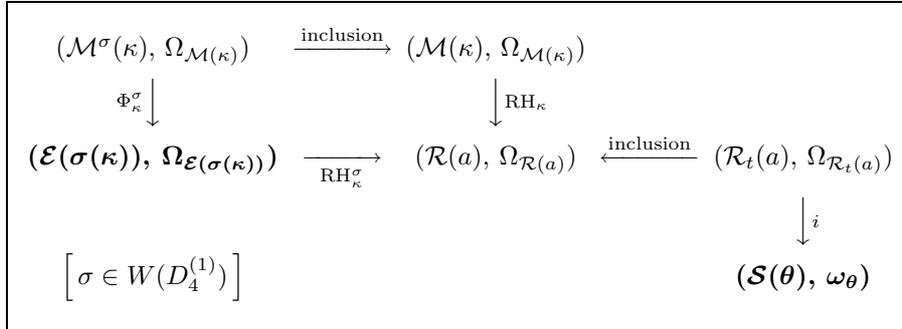
FIGURE 18. Concluding Diagram: bold-faced are concrete objects

Bäcklund transformations on the left-hand side, while nonlinear monodromy on the right-hand side, respectively.

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