REMARKS TOWARDS A CLASSIFICATION OF $RS_4^2(3)$-TRANSFORMATIONS AND ALGEBRAIC SOLUTIONS OF THE SIXTH PAINLEVÉ EQUATION

by

Alexander V. Kitaev

Abstract. — We introduce a special property, D-type, for rational functions of one variable and show that it can be effectively used for a classification of the deformations of dessins d'enfants related with the construction of algebraic solutions of the sixth Painlevé equation via the method of RS-transformations. In the framework of this classification we present a pure geometrical proof, based on the analysis of symmetry properties of the deformed dessins, of the nonexistence of some special rational coverings.

***Résumé* (Remarques pour une classification des transformations de type $RS_4^2(3)$ et des solutions algébriques de la sixième équation de Painlevé)**
    Nous introduisons une propriété spéciale, dite «de type D», pour les fonctions rationnelles d'une variable et nous montrons comment celle-ci pourrait être utilisée pour une classification des déformations de dessins d'enfants rattachée à la construction de solutions algébriques de l'équation de Painlevé VI via la méthode des RS-transformations. Dans le cadre de cette classification nous donnons une démonstration, purement géométrique et basée sur l'analyse des symétries des dessins déformés, de la non-existence de certains recouvrements rationnels.

1. Introduction

Recently the author introduced a general method of RS-transformations [**15**] for special functions of the isomonodromy type (SFITs) [**14**]. This method applies to SFITs defining isomonodromy deformations of linear $n \times n$-matrix ODEs of the first order with rational coefficients and with both regular and essential singular points.

RS-Transformations are just a proper combination of rational transformations (R-transformations) of the independent variable of the linear ODEs and Schlesinger transformations (S-transformation) of the dependent variable. Solutions of many different

and seemingly unrelated problems from various areas of the theory of functions get a unified and systematic approach in the framework of this method and can be reduced to the study, construction, and classification of different RS-transformations for matrix linear ODEs.

This method, e.g., allows one to prove the duplication formula for the Gamma-function (and most probably the general multiplication formula for the multiple argument [**3**]), build higher-order transformations for the Gauss hypergeometric function and reproduce the Schwarz table for it [**2, 17**], construct quadratic transformations for the Painlevé and classical transcendental functions [**13, 16**], and provide a systematical method for finding algebraic points at which transcendental SFITs attain algebraic values [**1**]. Without doubt, many other interesting problems can be approached via the method of RS-transformations. In this paper we apply this general method to the problem of construction and classification of algebraic solutions of the sixth Painlevé equation.

Recently scanning the literature, I realized that, possibly, the first serious profound result concerning RS-transformations was obtained by F. Klein [**19**], who proved that any scalar Fuchsian equation of the second order with finite monodromy group is a "pull-back" (R-transformation) of the Euler hypergeometric equation. In this context instead of the S-transformations the notion of "projective equivalence" is used. The latter is more restrictive than general S-transformations because in terms of the matrix ODEs it corresponds to triangular Schlesinger transformations, that finally results in a more restrictive special choice of the exponent differences (formal monodromy) of the hypergeometric equation, than when more general S-transformations are allowed.

Klein's result immediately implies that any solution of the Garnier system and, in particular the sixth Painlevé equation that corresponds to a finite monodromy group of the associated Fuchsian equation, is algebraic. It is important to mention that the converse statement is not true.

In the context of the sixth Painlevé equation the first person who could, theoretically, apply the "pull-back ideology" was R. Fuchs because it was he who found that the sixth Painlevé equation governs isomonodromy deformations of the certain scalar second order Fuchsian ODE and, moreover, received an informative letter from F. Klein. He actually did it, in a study of algebraic solutions in the so-called Picard case of the sixth Painlevé equation [**10, 11**][(1)].

Recently appeared a paper by Ch. Doran [**8**] who formulated a more general scheme (than that used by R. Fuchs) for construction of algebraic solutions of the sixth Painlevé equation from the pull-back point of view. A more detailed account of the last work is given in Introduction of [**17**]. In the following two paragraphs we explain

---

[(1)]These works were not known to me and, possibly, to most modern researchers until very recently, when Yousuke Ohyama called our attention to them.

why the method of RS-transformations for construction of the algebraic solutions is more general than the pull-back back one.

For a given R-transformation one can normally associate a few different RS-transformations, due to the possibility of choosing different (not related by the contiguity transformations) initial hypergeometric equations, which suffer this R-transformation and, by further application of proper S-transformations, are mapped into the Fuchsian ODE with four regular points. Each of these RS-transformations generate an algebraic solution of the sixth Painlevé equation, which sometimes depends on a complex parameter. Thus we have a finite number of algebraic solutions associated with each rational function (R-transformation). On the other hand it is well known that on the set of algebraic solutions acts the subgroup of RS-transformations with $\deg R = 1$: it is just a subgroup of compositions of Möbius transformations interchanging three points 0, 1, and ∞, and those Schlesinger transformations that does not add singularities to the Fuchsian ODE with four singular points. Thus the subset of algebraic solutions associated with the same R-transformation generate a finite number of orbits of the algebraic solutions with respect to the action of the subgroup mentioned above. The minimal subset of algebraic solutions that generate these orbits are called the subset of *seed algebraic solutions*, and RS-transformations that generate them – the *seed RS-transformations*. The seed algebraic solutions corresponding to the same rational covering (R-transformation) are different, by definition; however, the seed solutions associated with different rational coverings can coincide. Furthermore, the seed solutions, even corresponding to the same rational covering, can sometimes be related by some compositions of the quadratic transformations and/or Bäcklund transformations. Since the quadratic transformations are generated by the RS-transformations with $\deg R = 2$, and one of the Bäcklund transformations has no realization as the Schlesinger transformation of the 2×2-matrix Fuchsian ODE; we call this special transformation the Okamoto transformation (see [**20**] and Appendix [**17, 18**]).

We call attention of the reader that the possibility of construction of different RS-transformations starting from the same rational covering mentioned in the previous paragraph is not considered by the successors of the "pull-back ideology" because of the projective invariance property which assumes only one particular choice of the formal monodromy of the initial hypergeometric equation. Therefore, the "pull-back results" in many cases, namely in those ones where the property of projective equivalence can be changed to a less restrictive condition of the existence of S-transformation, can be extended or completed. We discuss this opportunity for construction of higher-order transformations of the Gauss hypergeometric functions in the Remarks in Sections 4 and 5. However, it seems that the pull-back from the hypergeometric equation, due to specific properties of the hypergeometric functions, is equivalent to the formally more general method of RS-transformations. This fact we are planning to discuss in a separate paper.

This paper is a continuation of author's previous work [**17**]. In [**17**] we give a general definition of the one-dimensional deformations of dessins d'enfants and their relation to the algebraic solutions of the sixth Painlevé equation, construct by this method numerous examples of different algebraic solutions, and discuss different features of this technique, e.g., a mechanism of appearance of genus-1 algebraic solutions. In Section 2 we recall the facts from [**17**] which are necessary for understanding of this work. Here we put this technique onto a systematic footing. A new idea we use here is symmetry preserving and symmetry breaking deformations of the dessins d'enfants and their relation to uniqueness of the corresponding rational covering.

More precisely, in Section 3 we introduce a notion of the *divisor type* (*D*-type) of rational functions and classify all *D*-types of the rational functions that generate algebraic solutions of the sixth Painlevé equation via the method of *RS*-transformations (*R*₄(3)-*functions*). The divisor type represents a special numerical property of the critical values of rational functions, more precisely, a property of the set of multiplicities of preimages (ramification patterns) of the critical values. This set we call the type (*R*-type) of a rational function. Note that because of our normalization (0 and ∞ are also the critical values) a specification of the divisor type also means a special property of the divisor of zeroes and poles of our rational functions.

We call the *D-series* the set of all $R_4(3)$-functions having the same *D*-type. Among these *D*-series there are two ones with finitely many, actually a few, members. This fact is proved and the corresponding rational functions are explicitly constructed in Sections 4 and 5. Each of the other *D*-series, corresponding to the *D*-types specified in the classification theorem of Section 3, are infinite.

It is worth noticing that modern personal computers (PC) allows one to construct all rational coverings that are presented here and in [**17**] without any advanced algorithms just by the natural method explained in Remark 2.1 of [**17**]. The time of calculation with MAPLE code on a relatively powerful PC does not exceed 1 second for any of these functions. Of course, finding the concise parametrization requires much more additional time. It is interesting to note that in 1998-2000, when we used exactly the same calculational scheme but on the Pentium 2 based PC with about 256 Mb RAM, we were not able to construct many interesting functions, even some Belyi function of degree 8, see [**2**], we have found only numerically. This remark, however, does not mean that we do not need any advanced calculational algorithms; explicit construction of most of the rational coverings with the degree > 12 still represent substantial difficulties.

To each $R_4(3)$-function we also indicate the number of the seed RS-transformations and present one algebraic solution whose construction does not require explicit form of the related Schlesinger transformation. It is exactly the "pull-back" solution, to get explicitly the other seed solutions one has to construct (explicitly) corresponding S-transformations. This procedure is absolutely straightforward and does not require

any advance computer algorithms and we do not consider it here. Numerous examples of the complete constructions of RS-transformations are given in [**1**].

This paper is a far-going extension of the second part of my talk in Angers, where I have only explained some simplest ideas concerning the concept of deformations of the dessins d'enfants and announced the construction of the solution presented in Section 4.

In the proofs of sections 4 and 5 we substantially use a graphical representation of the rational functions introduced in [**17**], which we call the *deformation dessins*. The reader should consult this work for a better understanding of these proofs, however I hope that the general idea and the scheme of these proofs can be understood even with the help of the following comments. In case, $R_4(3)$-function exists there is at least one graph, constructed according the rules given in [**17**], which represents it. In the proofs of nonexistence of some rational functions we use the evident fact that if the graph (the deformation dessin) does not exist, then clearly the rational function does not exist. In case some deformation dessin exists, it defines R-type, the conjecture, which is made in [**17**], says, that in this case rational function also exists. So, the statement, of *existence* of certain rational mappings which is based on existence of the deformation dessins is conventional and assumes the validity of this conjecture. In fact, for all rational functions, which existence we claim, we give either explicit formulae, or prove that they can be presented as the composition of explicitly known functions. So, all our proofs of existence of rational functions are based on explicit constructions and therefore also does not rely on any hypothesis.

Every deformation dessin can be obtained from a proper Grothendieck's dessin d'enfant as a result of the so-called *face deformations*: the *join* and *cross*. We consider also one more face deformation which is called the *twist*, however the latter can be treated as a special case of the join. We also consider vertex deformations, however, they can be avoided, more precisely instead we can always consider proper face deformations of an equivalent rational function transformed under a proper the Möbius transforms.

Suppose that for a given R-type there exists a corresponding rational function. Such rational function normally is not unique. Say, rational functions corresponding to $R_4(3)$-types often depends on one (sometimes on a few (!)) additional parameters. Moreover, there always exists a parametrization of these functions that they become rational functions of these additional parameters. Clearly, the latter parametrization is not unique: we can make Möbius transformations of the independent variable of our rational function with the coefficients depending on the additional parameters and also substitutions of the additional parameters by rational functions of additional parameters. However, even modulo such transformations the rational functions are not uniquely defined by their R-types. Some light on this problem is brought by the

discussion of symmetry preserving and symmetry breaking deformations of dessins d'enfants considered in sections 4 and 5.

During the preparation of this paper there appeared two papers by P. Boalch [**5, 6**], who is classifying algebraic solutions of the sixth Painlevé equation by developing the method (or, perhaps, more precisely to say following the trend) suggested by B. A. Dubrovin and M. Mazzocco [**9**]. The author has conjectured in [**15**] that all the algebraic solutions of PVI can be obtained via the quadratic and Bäcklund transformations. In [**17**], the author has already shown that all genus zero algebraic solutions of PVI presented in [**9**] can be constructed with the help of the RS-transformations. Some of the solutions obtained in [**5, 6**] are equivalent or related to the solutions already published in [**1, 17**]. In the forthcoming publication, which will be devoted to the systematic study of the infinite D-series, we will show how the rest of the solutions found in [**5, 6**] can be derived in the framework of the RS-method.

2. Deformation Dessins

In [**17**] the author introduced tricolour graphs that will be intensively used in the following sections, therefore for convenience of the reader we review some important facts concerning these graphs.

The tricolour graph is a connected graph on the Riemann sphere with black and white vertices and one blue vertex. The edges connect vertices of different colours. The faces are homeomorphic to the circles. The boundary of every face should contain at least one black vertex. The circles should contain at least one black or blue vertex. The loops are not allowed. The valency of the blue vertex equals 4. More precisely, the blue vertex is connected by edges with two white and two black vertices[2]

When all valencies of the white vertices equal 2 we are not indicating them and instead of a tricolour graph get a bicolour graph with all black vertices and only

---

[2]This property is not clearly indicated in [**17**]. However, it implicitly presented there too, because in that work we first define dessins d'enfants and than consider triclour graphs as bicolour with the additional blue point, in that case the blue point is exactly an intersection of two edges of the bicolour graph.

one blue vertex. The latter bicolour graph of course, has nothing to do with the black-white bicolour graphs of dessin d'enfants, which are also widely used below. In particular, the black-blue bicolour graph may contain loops. In case, dessin d'enfant has all valencies of the white vertices equal 2, they are also not indicated and we get a graph with only black vertices.

We use the notion of the black edge for both dessin's d'enfants and the tricolour graphs, as the edge connecting to consecutive black vertices. That means that each black edge contains exactly one white vertex and may additionally contain the blue vertex. The black order of face is the number of black edges in the face boundary.

The tricolour graphs can be viewed as obtained from the bicolour ones (dessins d'enfants) as a result of simple "*deformations*", see examples, in [**17**] and in Sections 4 and 5. Therefore we call them the *deformation dessins*, or very often, when there is no cause for a confusion, just the dessins.

Consider these deformations in more details. We consider three major types of the deformations: *face deformations*, *W- and B-splits*. It is convenient to consider three types of the face deformations: Twist, Join and Cross. The face deformations are continuous of the dessin d'enfant, it is enough to "move" only one edge of the graph until it "touches" or crosses some other edge with the appearance at the intersection point the blue point. In fact, the twist can be viewed as a special case of join and instead of the cross one can consider the join of some other dessin d'enfant.

The B- and W-splits are just a special procedure of splitting of black and white vertices, respectively, with the appearance of two vertices of the same colour instead of the splitting ones. One can find a numerous examples giving actually the precise definition of these deformations in [**17**].

In fact, although it is not evident from the very beginning, for the deformation dessins one can always consider only one type of deformations, say, "face deformations" and thus essentially we need only one deformation: the join. However, this is technically inconvenient. The fact that all deformations can be reduced to the deformation of one type only, can be established for the graphs related with the rational coverings with the help of the fractional-linear transformations (see below). This fact is actually not used neither in this paper, nor in [**17**].

Now suppose we have a tricolour graph. From the definition it is clear that there are two white (and two black) vertices connected with the blue point by the edges. If we make the reverse process to W-split (B-split), i.e. continuously merge these two white (black) points together with their connecting edges into the blue point and give to, thus obtained new point the white (black) colour, then we get nothing but dessin d'enfant. Therefore every tricolour graph can be obtained from some dessin d'enfant as a W- or B-split. One can consider of course also the surgery of tricolour graphs: cutting of the blue point such that one pair of black and white points would be on one side and another on the other side of the cut, but sometimes this procedure leads

to a disconnected graph, two dessins d'enfants. Therefore below we use the names: tricolour graphs and deformation dessins as synonyms.

It is well known that with dessins d'enfants on the Riemann sphere are related the Belyi functions on the Riemann sphere,i.e., the rational functions mapping the Riemann sphere onto itself with no more than three critical values normalized at 0, 1, and ∞. With the deformation dessins one can also relate rational functions. These functions also mapping the Riemann sphere onto itself but has got no more than four critical values. These functions (see Proposition 2.1 of [**17**]) depend on auxiliary complex parameter, one of its critical points has the partition of preimages $|2 + 1 + 1 \cdots + 1|$. The other critical values can be also normalized at 0, 1, ∞. Let us call for brevity such rational functions as the *proper rational functions*.

The relation between deformation dessins and the rational coverings were described in [**17**] by the following conjectures:

Conjecture 2.1. — *For any proper rational function, R, whose first three critical values are 0, 1 and ∞, there exists a tricolour graph such that:*
1. There is a one-to-one correspondence between its faces, white, and black vertices and critical points of the function R for the critical values 0, 1, and ∞, respectively.
2. Black orders of the faces and valencies of the vertices coincide with multiplicities of the corresponding critical points.

Remark 2.1. — We can add to the formulation of Conjecture 2.1 that this tricolour graph can be obtained as a deformation of some dessin d'enfant it was also stated and assumed in [**17**], however, not in Conjecture 2.1

We call attention of the reader that there is no uniqueness statement, analogous to the one known for the dessins d'enfants: uniqueness of R modulo fractional-linear transformations.The reason is that R actually is the function of two variables, $R(z_1) = R(z_1, y)$, where $z_1, y \in \mathbb{CP}^1$, z_1 is the main variable with respect to which we consider it as a rational function, and y is a parameter, say, location of the forth critical point. As a function of y, R has different branches. The different tricolour graphs that homotopic equivalent, can be continuously deformed to each other, constitute these different branches. In [**17**] as well as in this work we present examples of non-homotopic tricolour graphs with that define different functions with the same R-types. In this connection we discuss symmetric and symmetry breaking deformations of dessins d'enfants.

The converse statement is given by the following conjecture.

Conjecture 2.2. — *For any tricolour graph there exists a function $z = R(z_1, y)$, which is a rational function of $z_1 \in \mathbb{CP}^1$ with four critical values. Three of them are 0, 1, and ∞, and the corresponding critical points are related with the tricolour graph as stated in Conjecture 2.1. The variable y denotes the unique second order critical point*

corresponding to the fourth critical value of $z = R(z_1, y)$. R is an algebraic function of y of the zero genus.

Corollary 2.3. — *In the conditions of Conjecture 2.2 There is a representation of the function R as the ratio of coprime polynomials of z_1 such that its coefficients and y allow a simultaneous rational parametrization.*

Conjecture 2.2 actually contains two parts: "existence" and "rational parametriza-tion". The "existence" part can be formulated as follows:

Conjecture 2.4. — *For any deformation dessin there exists a proper rational function with four critical values. Three of them can be placed at 0, 1, and ∞, they are related to the dessin as stated in Conjecture 2.1.*

In the proofs of Propositions in Sections 4 and 5 we assume that Conjecture 2.1 is true. Below we provide the proof of Conjecture 2.1.

Proof. — Consider a simple curve on the Riemann sphere connecting 1 and infinity and passing through the fourth critical point (different from 0) of a given proper rational function R. Then R- preimage of this curve will be the tricolour graph on the Riemann sphere, where the blue point is the preimage of the fourth critical point and the relation of the other critical points with the tricolour graph are defined in Conjecture 2.1. □

Conjecture 2.2 is actually not used in this work and will be proved in a forthcoming paper.

3. D-Type of Rational Functions

We begin this part of the lecture with the canonical form of the sixth Painlevé equation, because we are going to present a few algebraic solutions of this equation in the explicit form.

$$
\begin{aligned}
\frac{d^2 y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\
&\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha_6 + \beta_6\frac{t}{y^2} + \gamma_6\frac{t-1}{(y-1)^2} + \delta_6\frac{t(t-1)}{(y-t)^2}\right),
\end{aligned}
\tag{1}
$$

where α_6, β_6, γ_6, $\delta_6 \in \mathbb{C}$ are parameters. For a convenience of comparison of the re-sults obtained here with the ones from the other works we will use also parametrization of the coefficients in terms of the formal monodromies $\hat{\theta}_k$:

$$
\alpha_6 = \frac{(\hat{\theta}_\infty - 1)^2}{2}, \quad \beta_6 = -\frac{\hat{\theta}_0^2}{2}, \quad \gamma_6 = \frac{\hat{\theta}_1^2}{2}, \quad \delta_6 = \frac{1 - \hat{\theta}_t^2}{2}.
$$

It is well known that every solution of this equation defines an isomonodromy deformation of the 2×2 matrix Fuchsian ODE on the Riemann sphere with four singular points.

As a first step in construction of algebraic solutions of Equation (1) via the method of RS-transformations one has to construct a proper rational covering of the Riemann sphere. The corresponding rational function has four critical values. Three of them are supposed to be placed at 0, 1, and ∞. To specify proper rational functions we use the symbol of their *R-type*, which consists of three boxes. In these boxes we consecutively write partitions of multiplicities of preimages of the points 0, 1, and ∞, correspondingly. The fourth critical point has a standard partition of its multiplicities $2 + 1 + \cdots + 1$ which is not indicated in the R-type.

According to [**17**] the numbers in each box can be presented as a union of two non-intersecting sets: the *apparent* set and *nonapparent* one. The characteristic property of the apparent set is that g.c.d. of its members is ≥ 2. It might be that nonapparent set has also nontrivial g.c.d., thus in general a presentation of the box as a union of the apparent and nonapparent sets is not unique. Moreover, nonapparent set may contain a number which is divisible by the g.c.d. of the apparent set. When the subdivision of the boxes in the apparent and nonapparent sets is chosen we have an ordered triplet of three integer numbers, $< m_0, m_1, m_\infty >$, the divisors of the apparent sets in the corresponding boxes, which we call the *divisor type (D-type) of the rational function*.

Remark 3.1. — In our notation of the D-types we always assume that $m_0 \leq m_1 \leq m_\infty$, clearly this can always be achieved by rearranging the points 0, 1, and ∞ by a fractional-linear transformation. However, in the notation of R-types we do not follow this agreement, and in most cases we have $m_0 > m_\infty > m_1$. Actually, we can speak of the two types of numbering of the boxes in R-types: the natural one, i.e., according to their position in the R-symbol; and the D-consistent numbering, i.e., according to the rule: the larger g.c.d., the larger number of the box. Below, in the statements we always assume the D-consistent numbering, while in the proofs – the natural one.

Another important parameter of the proper rational functions is the total number of members in all three nonapparent sets. In our case this number is 4. We put this number as the subscript in the notation of the R-type: $R_4(\ldots|\ldots|\ldots)$ or in short $R_4(3)$. To simplify notation we omit the subscript in situations where it cannot course a confusion.

Theorem 3.1. — *$R_4(3)$-Rational functions have one of the following eight D-types:*
$< 2, 2, m >$, $< 2, 3, 3 >$, $< 2, 3, 4 >$, $< 2, 3, 5 >$, $< 2, 3, 6 >$, $< 2, 3, 7 >$, $< 2, 3, 8 >$,
$< 2, 4, 4 >$. Where $m - 1 \in \mathbb{N}$.

Proof. — Let $n \geq 2$ be the degree of some $R_4(3)$ function of D-type $< m_0, m_1, m_\infty >$. Denote the sum of numbers in nonapparent sets in the consecutive boxes of a $R_4(3)$ function as σ_0, σ_1, and σ_∞, respectively.

From the Riemann-Hurwitz formula, with a help of Proposition 2.1 of [**17**], one deduces the following "master" inequality,

$$(2) \qquad \frac{n - \sigma_0}{m_0} + \frac{n - \sigma_1}{m_1} + \frac{n - \sigma_\infty}{m_\infty} \geq n - 1.$$

Clearly the numbers σ_k satisfy one more inequality

$$(3) \qquad \sigma_0 + \sigma_1 + \sigma_\infty \geq 4.$$

We begin with the proof that $m_0 = 2$. Suppose that all numbers $m_k \geq 3$, then from the master inequality we deduce,

$$3n - \sigma_0 - \sigma_1 - \sigma_\infty \geq 3n - 3 \quad \Rightarrow \quad \sigma_0 + \sigma_1 + \sigma_\infty \leq 3,$$

which contradicts Inequality (3).

Now, suppose that $m_1 \geq 5$. In this case from the master inequality we get,

$$\frac{n - \sigma_0}{2} + \frac{n - \sigma_1}{5} + \frac{n - \sigma_\infty}{5} \geq n - 1 \quad \Rightarrow \quad 10 \geq n + 5\sigma_0 + 2\sigma_1 + 2\sigma_\infty \quad \Rightarrow$$
$$2 \geq n + 3\sigma_0 \quad \Rightarrow \quad \sigma_0 = 0, \quad n = 2.$$

Since $n = 2$ the apparent sets in the second and third boxes are empty, thus the corresponding $R_4(3)$-type reads $R(2|1 + 1|1 + 1)$. The latter transformation can be treated as belonging to any of the D-types mentioned in the Proposition. Explicit form of the corresponding $RS_4^2(3)$-transformation can be found in [**1**] (Section 2).

Consider D-type $< 2, 4, m_\infty >$ with $m_\infty \geq 5$. the master inequality implies:

$$\frac{n - \sigma_0}{2} + \frac{n - \sigma_1}{4} + \frac{n - \sigma_\infty}{5} \geq n - 1 \quad \Rightarrow \quad 20 \geq n + 10\sigma_0 + 5\sigma_1 + 4\sigma_\infty \quad \Rightarrow$$
$$4 \geq n + 6\sigma_0 + \sigma_1 \quad \Rightarrow \quad n = 4, \ \sigma_0 = \sigma_1 = 0, \ \sigma_\infty = 4, \quad \text{or}$$
$$n = 2, \ \sigma_0 = 0, \ \sigma_1 = \sigma_\infty = 2.$$

The logical case $n = 3$ is excluded because it contradicts the condition $\sigma_0 = 0$ which holds for all n. Thus we get two $R_4(3)$-types: $R(2|1 + 1|1 + 1)$ and $R(2 + 2|4|\underbrace{1 + \cdots + 1}_{4})$. The last R-type has an empty apparent set in the last box and, thus can also be treated as belonging to the D-type $< 2|4|4 >$. The rational function with this R-type exists and the corresponding $RS_4^2(3)$-transformation is explicitly constructed in [**1**] (Section 4, Subsection 4.1.4).

Consider finally D-types $< 2, 3, m_\infty >$ with $m_\infty \geq 9$. From the master inequality we find:

$$\frac{n - \sigma_0}{2} + \frac{n - \sigma_1}{3} + \frac{n - \sigma_\infty}{9} \geq n - 1 \quad \Rightarrow \quad 18 \geq n + 9\sigma_0 + 6\sigma_1 + 2\sigma_\infty \quad \Rightarrow$$

(4) $\qquad\qquad 10 \geq n + 7\sigma_0 + 4\sigma_1 \quad \Rightarrow \quad \sigma_0 = 1, \quad \sigma_1 = 0, \quad n = 3, \quad \sigma_\infty = 3,$

(5) $\qquad\qquad\qquad\qquad\qquad \sigma_0 = 0, \quad \sigma_1 = 0, \quad n = 2, \dots, 10;$

(6) $\qquad\qquad\qquad\qquad\qquad \sigma_0 = 0, \quad \sigma_1 = 1, \quad n = 2, \dots, 6;$

(7) $\qquad\qquad\qquad\qquad\qquad \sigma_0 = 0, \quad \sigma_1 = 2, \quad n = 2.$

In the solution given by Equation (4) we excluded the case $n = 2$, which agrees with the master inequality, because it contradicts to the condition $\sigma_0 = 1$. There is only one $R_4(3)$-type corresponding to solution (4), namely $R(2 + 1|3|1 + 1 + 1)$. Because the apparent set in the last box is empty this R-type can be associated with any D-type of the form $< 2, 3, m >$ with arbitrary $m \geq 3$, in particular, with $m < 9$. The corresponding rational mapping exists and explicit form of the $RS_4^2(3)$-transformations is given in [**1**] (Section 3, Subsection 3.1.2).

In the solution (5) n should be divisible by 2 and 3, thus $n = 6$, the apparent set in the last box is empty and hence $\sigma_\infty = 6$. There are two corresponding $R_4(3)$-types: $R(2 + 2 + 2|3 + 3|2 + 2 + 1 + 1)$ and $R(2 + 2 + 2|3 + 3|3 + 1 + 1 + 1)$. In both cases the corresponding rational functions exist, see their explicit forms and corresponding solutions of Equation (1) in [**17**] (Section 3, Subsection 3.3, Examples 1 and 2). Again by the analogous reasoning as in the previous case to both rational functions we can assign the same D-type $< 2, 3, m >$, with $m \leq 8$. Note that in this case we can also assign to the first function D-type $< 2, 2, 3 >$, because in this case we can choose the apparent set in the first box consisting of one number 2, and the nonapparent one – with two numbers, both equal 2, dividing the rest boxes into the apparent and nonapparent sets into the natural way we still get the function of $R_4(3)$-type.

Turning to the solution (6). We see that n should be even and has the form $1 + 3k$ with some integer k. Thus the only possibility is $n = \sigma_\infty = 4$ and the apparent set in the last box is empty. The only $R_4(3)$-type is $R(2 + 2|3 + 1|2 + 1 + 1)$. The corresponding rational function exists and related $RS_4^2(3)$-transformation are constructed in [**1**] (Section 4, Subsection 4.1.7).

Finally, the only $R_4(3)$-type corresponding to Equation (7) is $R(2|1 + 1|1 + 1)$ is already discussed above. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 3.2. — To each of the D-types, except $< 2, 3, 7 >$ and $< 2, 3, 8 >$, in Theorem 3.1 correspond infinite series of rational functions of $R_4(3)$-types. There is a finite number of rational functions of $R_4(3)$-type corresponding to the two exceptional D-types. The latter D-types are studied in the subsequent Sections 4 and 5, respectively. It is also not too complicated to describe explicitly the infinite series, we plan to do it in further publications.

4. Classification of RS-Transformations of D-Type $< 2, 3, 7 >$

Proposition 4.1. — *There are only three $R_4(3)$-types, with the nonempty apparent set in the third box, corresponding to the D-type $< 2, 3, 7 >$, namely[3],*

$$(8) \qquad \deg R_4 = 10: \quad R_4(7 + 1 + 1 + 1 | \underbrace{2 + \cdots + 2}_{5} | 3 + 3 + 3 + 1),$$

$$(9) \qquad \deg R_4 = 12: \quad R_4(7 + 2 + 1 + 1 + 1 | \underbrace{2 + \cdots + 2}_{6} | \underbrace{3 + \cdots + 3}_{4}),$$

$$(10) \qquad \deg R_4 = 18: \quad R_4(7 + 7 + \underbrace{1 + \cdots + 1}_{4} | \underbrace{2 + \cdots + 2}_{9} | \underbrace{3 + \cdots + 3}_{6}).$$

Proof. — Put in the master inequality $m_0 = 2$, $m_1 = 3$, and $m_\infty = 7$, then we can rewrite it as follows:

$$42 - 21\sigma_0 - 14\sigma_1 - 6\sigma_\infty \geq n.$$

Taking into account that $n \geq m_\infty \geq 7$ and Inequality (3) we obtain,

$$(11) \qquad\qquad 18 - 8\sigma_1 - 15\sigma_0 \geq n \geq 7.$$

The solution of Diophantine Inequality (11) reads:

$$(12) \qquad\qquad \sigma_0 = 0, \quad \sigma_1 = 0, \quad \sigma_\infty \geq 4, \quad 7 \leq n \leq 18,$$

$$(13) \qquad\qquad \sigma_0 = 0, \quad \sigma_1 = 1, \quad \sigma_\infty \geq 3, \quad 7 \leq n \leq 10.$$

Note that Solutions (12) and (13) completely define the second and third boxes of the possible R-types.

Consider Solution (12). In this case, n is divisible by $2 \cdot 3 = 6$. Thus the only possibilities are $n = 12$ or $n = 18$.

If $n = 12$ the only possibility is that the apparent set of the first box of the R-type contains only one element. Thus there is only one R-type in this case, namely, (9). The corresponding covering and algebraic solution was constructed in my work [**17**] Section 3, Subsection 3.4, Example 3 (Cross). The same algebraic solution was also constructed in about the same time by P. Boalch [**7**] by an elaboration of the method of B. Dubrovin and M. Mazzocco [**9**].

If $n = 18$ there are two main possibilities:

1. The apparent set of the first box consists of two elements the only possible R-type is (10), because the second and third boxes are completely defined. Below we show the deformation dessin for this R-type confirming that the corresponding covering really exists.

2. The apparent set of the first box consists of one element. There are several logical possibilities corresponding to the partitions of $18 - 7 = 11$ into four

---

[3] In the numbering of boxes we follow the convention of Remark 3.1.

natural numbers. No one of these R-types corresponds to a rational covering. Actually, recall that Euler characteristics of the sphere is 2,

$$(14) \qquad V - E + F = 2.$$

Suppose that there exists a deformation dessin on the sphere corresponding to some of these R-types: V is the number of black points plus the blue one; F is the number of faces which is counted as the four faces, corresponding to the non-apparent set, plus one face from the apparent set; and, finally, the valencies of the black points are 3 and valency of the blue one is 4, each edge is incidental to two vertices:

$$V = 6 + 1 = 7, \qquad F = 4 + 1 = 5. \quad \text{and} \quad E = \frac{3 \cdot 6 + 4}{2} = 11.$$

Now we have $7 - 11 + 5 = 1$, this contradicts Equation (14).

Consider now Solution (13). Since $\sigma_0 = 0$ we have that $(n|2) > 1$, therefore the only logical possibilities are $n = 8$, and $n = 10$. For $n = 8$ we must have $8 = 3 \cdot k + 1$ for some integer k, which is a contradiction. In the case $n = 10$ we have $\sigma_\infty = 10 - 7 = 3$; together with the facts that $\sigma_0 = 0$, $\sigma_1 = 1$, and that the total number of points in the non-apparent set is 4, this implies that there is only one R-type corresponding to this case, namely, (8). $\qquad \square$

Now we turn to the discussion of existence and explicit constructions of rational functions with the R-types (8) and (10), as is mentioned in the proof the function with R-type (9) is already constructed in [**17**].

Consider R-type (10), to confirm the existence of the corresponding covering we have yet to present the corresponding deformation dessin. Note that the type is reducible,

$$(15) \qquad R(7 + 7 + \underbrace{1 + \cdots + 1}_{4} | \underbrace{2 + \cdots + 2}_{9} | \underbrace{3 + \cdots + 3}_{6}) =$$

$$(16) \qquad R(7 + 1 + 1 | \underbrace{2 + \cdots + 2}_{4} + \underset{\wedge}{1} | 3 + 3 + 3) \circ R(1 + 1 | \underset{\wedge}{2} | 1 + 1)$$

Remark 4.1. — This is a digression to the theory of the Gauss hypergeometric functions. The irreducible Belyi function $R(7 + 1 + 1 | \underbrace{2 + \cdots + 2}_{4} + 1 | 3 + 3 + 3)$ is of $R_3(3)$-type and defines the following "seed" RS-transformations [4],

$$RS_3^2 \left(\begin{array}{c|c|c} k/7 & 1/2 & 1/3 \\ 7{+}1{+}1 & \underbrace{2 + \ldots + 2}_{4} + 1 & 3 + 3 + 3 \end{array} \right), \qquad k = 1, 2, 3,$$

---

[4] The extended notation for RS-transformations that we use below is explained in [**17, 2**].

which are equivalent to three seed transformations of the Gauss hypergeometric functions of order 9 and in terms of the θ-triples[5] read:

$$\left(\frac{k}{7}, \frac{1}{2}, \frac{1}{3}\right) \to \left(\frac{k}{7}, \frac{k}{7}, \frac{1}{2}\right), \qquad k = 1, 2, 3.$$

Each of these three transformations allows one to enlarge (by two lines) one of the corresponding Octic Clusters introduced in Section 5 of [**17**], because to the resulted hypergeometric function one can apply a proper quadratic transformation.

One of the simplest forms of the first R-function in Equation (16) is

$$\lambda_1 = \frac{3^{10}(16\lambda^2 + 7\lambda + 49)}{8(32\lambda^3 + 84\lambda - 35)^3},$$

$$\lambda_1 - 1 = -\frac{(512\lambda^4 + 256\lambda^3 + 2208\lambda^2 + 328\lambda + 1799)^2(\lambda - 1)}{8(32\lambda^3 + 84\lambda - 35)^3}.$$

For application to the theory of algebraic solutions as well as for the Gauss hypergeometric functions we need the following cumbersome looking normalizations of this function:

(17)

$$\lambda_1 = \frac{27(13 + 7i\sqrt{7})(\lambda - 1)(\lambda - \frac{7^3}{2^8} - \frac{7 \cdot 13}{2^8}i\sqrt{7})^7}{2\left(\lambda^3 + \frac{7}{2^4}(49 + 29i\sqrt{7})\lambda^2 - \frac{7^3}{2^{10}}(129 + 29i\sqrt{7})\lambda + \frac{3^2 \cdot 7^3}{2^{15}}(\frac{7^2 \cdot 13}{3} - 29i\sqrt{7})\right)^3},$$

(18)

$$\lambda_1 = \frac{\lambda(\lambda - 1)(\lambda - \frac{1}{2} + \frac{13}{98}i\sqrt{7})^7}{\left(\lambda^3 - (\frac{3}{2} + \frac{29}{42}i\sqrt{7})\lambda^2 + (\frac{121}{294} + \frac{29}{42}i\sqrt{7})\lambda + \frac{13}{294} + \frac{29}{4802}i\sqrt{7}\right)^3}$$

We factorized integers in Equation (17) only for the purpose of fitting on one line.

Remark 4.2. — While this paper was under preparation I got an information from R. Vidūnas about his recent paper [**22**] on classification of pull-back transformations for the Gauss hypergeometric functions. This paper is giving a nice and quite profound account of this subject, in particular, one finds there Equation (18) in a slightly different notation. Some of the other Belyi functions of $R_3(3)$-type that we discuss in this work were constructed by R. Vidūnas with the help of the method developed in his earlier work [**23**]. The previous Remark 4.1 gives also an illustration to the statement made in Introduction that RS-transformations seems to be a more general ones than the algebraic pull-back transformations considered in [**22**]. Because the local exponent differences, in the language of [**22**], for the RS-transformations should not necessarily be equal to inverse integers as is assumed in [**22**]: with each rational covering, in general, we associate a few *independent* (seed) transformations of the Gauss hypergeometric function, see Remark 4.1 above and exact examples in [**2**].

---

[5] θ-triples, the set of formal monodromies for the matrix form of the hypergeometric equation (see, [**2, 17**]), which differ from the standard triples of the local exponents for the canonical (scalar) form of the Gauss hypergeometric equation by the shift 1 in one of the elements.

The situation in this respect is similar with the construction of algebraic solutions for Equation (1). There are also some intersections of [**22**] with Sections 4 and 5 work [**17**].

To get an explicit realization of Equation (15), (16) we have to present a rational function of the R-type $R(1+1|21+1)$ in a suitable normalization:

$$(19) \qquad\qquad \lambda = \frac{(1-2s)(\lambda_2 - t/s)^2}{(\lambda_2 - t)}, \qquad t = \frac{s^2}{(2s-1)}.$$

Note that the rational function $\hat{\lambda} = \hat{\lambda}(\lambda_2) \equiv \lambda - 1$, where λ is given by Equation (19), is correctly normalized in the sense of Theorem 2.1 [**17**]: $\hat{\lambda} = \frac{(1-2s)\lambda_2(\lambda_2 - 1)}{(\lambda_2 - t)}$. Applying this Theorem we calculate algebraic solution of Equation (1),

$$(20) \qquad\qquad t = \frac{s^2}{2s-1}. \qquad y(t) = s = t + \sqrt{t^2 - t},$$

corresponding to the following θ-tuple,

$$(21) \qquad\qquad\qquad \theta_0 = \theta_1, \qquad \theta_t = 1 - \theta_\infty,$$

with two parameters θ_0 and $\theta_t \in \mathbb{C}$. Substituting λ given by Equation (19) into Equation (17) one gets a rational function of the R-type (15) correctly normalized in the sense of Theorem 2.1 of [**17**]. Clearly the only critical points of such composed function which depends on s should coincide with the critical points of the function (19), thus the algebraic solution defined by the composition exactly coincide with the solution (20), however, now Theorem 2.1 gives for this solution a more restricted θ-tuple: $\theta_0 = \theta_1 = \theta_t = 1 - \theta_\infty = 1/7$.
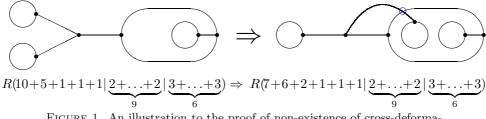
Are there any other rational functions of the R-type (10)? To answer the question let's study the following problem: how one can get the functions of this type via deformations of dessins d'enfants?
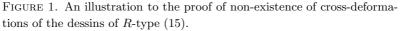
It will be convenient to define a notion of *symmetric dessins*. We call a dessin d'enfant or deformation dessin symmetric if it is homeomorphic to a graph on the Riemann sphere which is invariant under the involution $\lambda \to -\lambda$. In this case the rational function corresponding to such dessin can be presented as a composition of a quadratic rational function with a rational function with the twice lower degree than the original one. In the case of dessins d'enfants both rational functions are, of course, the Belyi functions, while for the deformation dessins the first function of the composition is the Belyi function, while the second one is a one dimensional deformation of the quadratic Belyi function. The latter is unique modulo fractional linear transformation of the critical points. Suppose we consider deformations of a symmetric dessin d'enfant. If the deformation dessin is symmetric, then we call it the *symmetry preserving deformation*; if the deformation dessin is not symmetric – the *symmetry breaking deformations*.

Proposition 4.2. — *Deformation dessins corresponding to the R-type* (10) *can be obtained only as the symmetry preserving deformations of dessins d'enfants.*

Proof. — We begin with the "face" deformations. There are two types of such deformations: the cross and join, since the twist can be regarded as a special case of the join). First consider the cross. Such deformation is dividing one face of a dessin on two faces and can increase the black order of a face neighboring with the divided one (if the latter exists). All in all a dessin before the cross-deformation should have 5 faces. We call "heads" the faces with the black order 1. In case the dessin contains already four heads its R-type can be only $R(14 + \underbrace{1 + \cdots + 1}_{4} | \underbrace{2 + \cdots + 2}_{9} | \underbrace{3 + \cdots + 3}_{6})$

(see Figure 3). Each head contains on the boundary exactly one black point of valency 3 and therefore looks like a balloon on a rope or, as we say, the head on the "neck". In this case the only possibility is to cross with a chosen neck one of the heads, the necks, or the edge connecting the "dumbbells". The edge and the neck cannot cross themselves because in this case we have an "illegal" deformation which contains a face surrounded with this edge and, therefore, having its black order equal 0. Because the dessins are located on the sphere we have only two different possibilities of crossing each neck or the edge, all in all $(8 = 2 \cdot 4)$ variants. One checks that none of them leads to the right face distribution (15). If the dessin before the deformation contains exactly three heads, then at least two of them located in one large face, because we cannot have more then 5 faces to get after the deformation 6 ones. The black order of face with two heads is at least 9, so that the remaining face has the black order ≤ 6. So, after the deformation the black order of the latter face should be increased to 7. The only way of such increase is when one of the heads entering into it, so that the neck of this head crosses the boundary of the face. This deformation increases the black order of the face by 2. This means that the face distribution of the dessin before the deformation is $18 = 10 + 5 + 1 + 1 + 1$. Such dessin really exists, but its cross deformation of the type we discuss leads to the face distribution $18 = 7 + 6 + 2 + 1 + 1 + 1$, see Figure 1.

$$R(10+5+1+1+1| \underbrace{2+\ldots+2}_{9} | \underbrace{3+\ldots+3}_{6}) \Rightarrow R(7+6+2+1+1+1| \underbrace{2+\ldots+2}_{9} | \underbrace{3+\ldots+3}_{6})$$

FIGURE 1. An illustration to the proof of non-existence of cross-deformations of the dessins of R-type (15).

Suppose now that a dessin before the cross-deformation contains only two heads. In this case two more heads should appear as a result of the deformation. The only

way it can happen is if the dessin consists of the circle with one black point on it. The rest of the dessin should be located inside of the circle and "live" on the "trunk" which "grows" on this black point. If there would be a part of the dessin outside the circle, then the circle should contain one more black point, because the valencies of all black points equal 3. The deformation in this case is a crossing of the circle by the trunk such that inside the circle remains only a part of the trunk while all other parts of the dessin move outside the circle. Because our pictures are drawn on the Riemann sphere, the dessin before the deformation actually should contain 3 heads rather than 2! In fact, the face outside the circle where the whole dessin is located is the head. If the rest of the dessin contains only one more head then, we would have only three heads as the result of the deformation.

Clearly a dessin before the cross should contain at least two heads, because there are no one-dimensional deformations that can reduce black orders of three faces.

Deformation of the join type affects only one face and does not change the black order of other faces. The affected face is divided by two ones. So the only possible face distributions of the dessins that can be deformed by a join to R-type (15) are: $18 = 7 + 7 + 2 + 1 + 1$, $18 = 14 + 1 + 1 + 1 + 1$, $18 = 8 + 7 + 1 + 1 + 1$. The dessin with the last face distribution does not exist. In fact, suppose that the last dessin exists, then it contains three heads. There are two large faces with the black order equal 8 and 7, therefore two heads are located inside one of them. Their "necks" are connected either with each other at some point and then this point connected to the boundary of the surrounding face or with the boundary of the face. Calculating the black order of such "minimal construction" we get 9 in the first case and 8 in the second. However, there is one more head. This head should be located in the other large face because, otherwise it cannot have a black order more than 3. The last head should be connected with its "neck" to the joint boundary of the large faces at some point different from the connection points of the other heads, because the valencies of all connection points equal 3. Therefore, the minimal black order of the face containing two heads would be 9.
Figures 2 and 3 proves that the first two face deformations really exist.

$$R(7+7+2+1+1\,|\,\underbrace{2+\ldots+2}_{9}\,|\,\underbrace{3+\ldots+3}_{6})\quad\Rightarrow\quad R(7+7+\underbrace{1+\ldots+1}_{4}\,|\,\underbrace{2+\ldots+2}_{9}\,|\,\underbrace{3+\ldots+3}_{6})$$
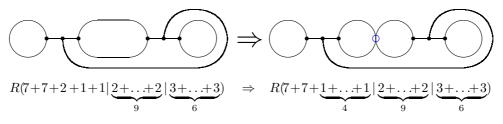
FIGURE 2. A symmetry preserving twist of the reducible symmetric dessin.

Note that the deformation dessins in the r.-h.s. of Figures 2 and 3 are homeomorphic on the Riemann sphere.

$$R(14+\underbrace{1+\ldots+1}_{4}|\underbrace{2+\ldots+2}_{9}|\underbrace{3+\ldots+3}_{6}) \Rightarrow R(7+7+\underbrace{1+\ldots+1}_{4}|\underbrace{2+\ldots+2}_{9}|\underbrace{3+\ldots+3}_{6})$$
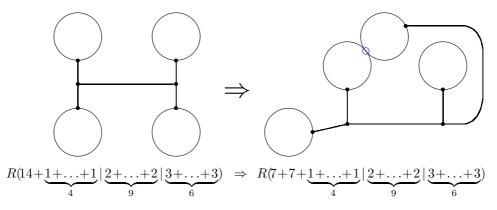
FIGURE 3. A symmetry preserving join of the reducible symmetric dessin.

Besides the face deformations, there are also deformations which we call "splits", or, more specifically, B- and W-splits, depending on the color (black or white) of the splitted vertex. As we will see below not all such splits are equivalent. To distinguish different splits we use notation LB- or,say, CW-split to denote location of the blue point after the split, in the first case the blue point belongs to the crossing of two lines, in the second – of two circles, the last letter means, of course, the color of the splitted vertex. If the blue vertex belongs to a circle and line we denote such deformation as CLB-split, if B-vertex is splitted.

In our case we obviously have only two splits: W-split $(4=2+2)$ and B-split $(6=3+3)$. These deformations are shown on Figures 4 and 5.

$$R(7+7+\underbrace{1+\cdots+1}_{4}|\underbrace{2+\ldots+2+4}_{7}|\underbrace{3+\ldots+3}_{6}) \Rightarrow R(7+7+\underbrace{1+\ldots+1}_{4}|\underbrace{2+\ldots+2}_{9}|\underbrace{3+\ldots+3}_{6})$$
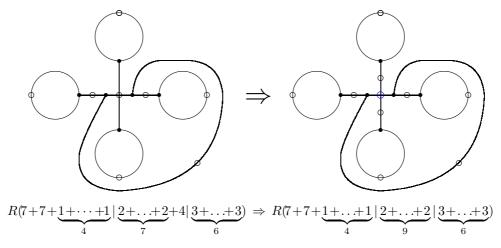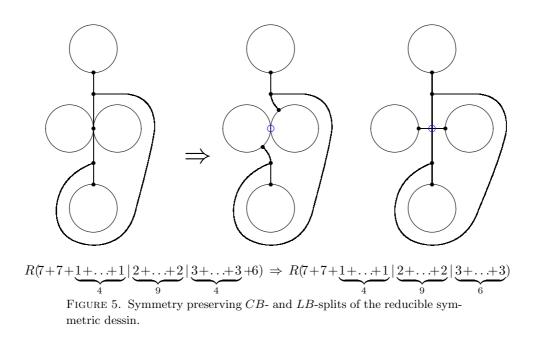
FIGURE 4. A symmetry preserving W-split of the reducible symmetric dessin.

Note that W-split on Figures 4 is homeomorphic in the Riemann sphere to LB-split on Figure 5. Also CB-split on Figure 5 is homeomorphic on the Riemann sphere

$$R(7+7+\underbrace{1+\ldots+1}_{4}|\underbrace{2+\ldots+2}_{9}|\underbrace{3+\ldots+3}_{4}+6) \Rightarrow R(7+7+\underbrace{1+\ldots+1}_{4}|\underbrace{2+\ldots+2}_{9}|\underbrace{3+\ldots+3}_{6})$$

FIGURE 5. Symmetry preserving CB- and LB-splits of the reducible symmetric dessin.

to the twist on Figures 2. Moreover, both deformation dessins on Figure 5 represent two branches of the same rational covering, because, clearly, they are homotopic, continuously deformable one into another through the dessin in the l.-h.s. of this picture.

$$R(7+7+\underbrace{1+\ldots+1}_{4}|\underbrace{2+\ldots+2}_{9}|\underbrace{3+\ldots+3}_{4}+6) \Rightarrow R(8+6+\underbrace{1+\ldots+1}_{4}|\underbrace{2+\ldots+2}_{9}|\underbrace{3+\ldots+3}_{6})$$
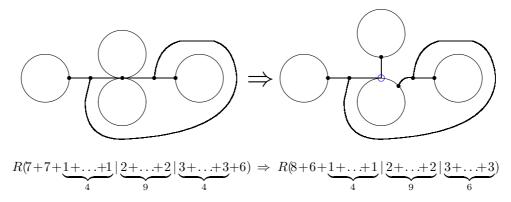
FIGURE 6. "Symmetry breaking" CLB-split of the reducible symmetric dessin.

There is also CLB-split of the dessin on Figure 5 with the right valencies of black (and, of course, white) vertices (see Figure 6). However the resulted deformation dessin does not belong to R_4-type.   □

Finally, consider R-type (8) of Proposition 4.1. It can be obtained as: (1) face deformations of the following dessins, $R(8+1+1|\underbrace{2+\ldots+2}_{5}|3+3+3+1)$ and $R(7+2+1|\underbrace{2+\ldots+2}_{5}|3+3+3+1)$; (2) W- split of $R(7+1+1+1|4+\underbrace{2+\ldots+2}_{4}|3+3+3+1)$; or (3) B-splits of $R(7+1+1+1|\underbrace{2+\ldots+2}_{5}|3+3+4)$ and $R(7+1+1+1|\underbrace{2+\ldots+2}_{5}|6+3+1)$. We leave to the interested reader to prove that all these dessins are homotopic in the Riemann sphere so that the corresponding deformation dessins represent different branches of one and the same algebraic function. Instead of studying the dessins we present below an explicit form of the corresponding rational covering:

$$(22) \quad \lambda_1 = \frac{1728 s^{12}(3s^2 - 4s + 4)}{(s+2)^{14}(s-1)^8} \frac{(\lambda_2 - 1)(\lambda_2^2 + a_1 \lambda_2 + a_0)}{\lambda_2(\lambda_2^3 + c_2 \lambda_2^2 + c_1 \lambda_2 + c_0)^3}, \qquad \lambda_2 = \frac{\lambda A}{\lambda - 1 + A},$$

where

$$a_0 = \frac{27 s^4(2s^2 - 3s + 2)^2}{(s+2)^4(s-1)^4(3s^2 - 4s + 4)}, \quad a_1 = -\frac{2(14 s^5 - 25 s^4 + 20 s^3 + 8s^2 - 16s + 8)}{(s+2)^2(s-1)^2(3s^2 - 4s + 4)},$$

$$c_0 = -\frac{24 s^4(4s^3 - s^2 - 4s + 4)}{(s+2)^6(s-1)^4}, \quad c_1 = \frac{60 s^6 - 84 s^5 - 15 s^4 + 72 s^3 - 8s^2 - 32s + 16}{(s+2)^4(s-1)^4},$$

$$c_2 = -\frac{2(6s^3 - 3s^2 - 4s + 4)}{(s+2)^2(s-1)^2}, \qquad \text{and}$$

$$A = \frac{14 s^5 - 25 s^4 + 20 s^3 + 8(s-1)^2 + 8(s-1)(s^2 - s + 1)w}{(s+2)^2(s-1)^2(3s^2 - 4s + 4)}, \quad \text{where}$$

$$(23) \qquad\qquad w^2 = (2s+1)(1-s)(s^2 - s + 1),$$

is a solution of the quadratic equation, $\lambda_2^2 + a_1 \lambda_2 + a_0 = 0$. Note that the function $\lambda_1 = \lambda_1(\lambda_2)$ has a rational parametrization, however it is not correctly normalized. After a normalization, the fractional-linear transformation $\lambda_2 = \lambda_2(\lambda)$, we get the function $\lambda_1 = \lambda_1(\lambda)$, which has an elliptic parametrization. Applying now Theorem 2.1 of [**17**], we get an algebraic solution of Equation (1),

$$(24) \qquad y(t) = 1 + \frac{(3s-2)(s^2 - 2s + 4)^2}{4(s+2)(s-1)^2(s^2 - s + 1)(3s^2 - 4s + 4)} \times$$

$$\frac{-14 s^5 + 25 s^4 - 20 s^3 - 8s^2 + 16s - 8 - 8(s-1)(s^2 - s + 1)w}{(2s+1)(3s^3 - 10 s^2 + 6s - 2) - 14(s-1)w},$$

$$(25) \quad t = \frac{1}{2} - \frac{14 s^9 - 105 s^8 + 252 s^7 - 392 s^6 + 420 s^5 - 336 s^4 + 112 s^3 + 72 s^2 - 96s + 32}{16(s+2)^2(s-1)^3(s^2 - s + 1)w},$$

with w defined in Equation (23), for the following set of θ-parameters:

$$\theta_0 = \frac{1}{3}, \qquad \theta_1 = \frac{1}{7}, \qquad \theta_t = \frac{1}{7}, \qquad \theta_\infty = \frac{6}{7}.$$

There are a few other suitable normalizations of the function $\lambda_1(\lambda_2)$, clearly all of them can be parameterized only by algebraic curves of genus 1. Theorem 2.1 [**17**] allows to find an algebraic solution (of genus 1) to each such normalization. However, it is easy to check that all these solutions are related to each other via so-called Bäcklund transformations for Equation (1). Thus, Equations (24) and (25) represent the only "pull-back" seed algebraic solution. The list of the "*RS*" seed algebraic solutions is given below in Proposition 4.4. We can summarize our study as the following Propositions.

Proposition 4.3. — *For all R-types specified in Proposition 4.1: (8), (9), and (10), there exist rational functions with these R-types. Each of these rational functions can be rationally parameterized by a "deformation" parameter $s \in \mathbb{CP}^1 \setminus \mathcal{B}$, where $\mathcal{B}$ is a finite set. The resulting birational functions: $\lambda_1 = \lambda_1(\lambda, s)$ (Equation (22)), $\lambda_1 = \lambda_1(\lambda_2, s)$ (Equations (19) and (17)), and $z = z(z_1, s)$ in [**17**] Section 3, Subsection 3.4, Example 3 (Cross), are unique up to fractional-linear transformations of the first argument and reparametrization of s.*

Proposition 4.4. — *There are three seed RS-transformations related with R-type (8):*

$$RS_4^2 \left(\begin{array}{c|c|c} k/7 & 1/2 & 1/3 \\ 7+1+1+1 & \underbrace{2+\ldots+2}_{5} & 3+3+3+1 \end{array} \right) \text{ for } k = 1, 2, \text{ and } 3. \text{ Each of these}$$

transformations produces one algebraic genus 1 solution for the following sets of the θ-parameters:

$$
\begin{array}{ccccc}
k = 1, & \theta_0 = \dfrac{1}{3}, & \theta_1 = \dfrac{1}{7}, & \theta_1 = \dfrac{1}{7}, & \theta_\infty = \dfrac{6}{7}, \\[2mm]
k = 2, & \theta_0 = \dfrac{1}{3}, & \theta_1 = \dfrac{2}{7}, & \theta_1 = \dfrac{2}{7}, & \theta_\infty = \dfrac{2}{7}, \\[2mm]
k = 3, & \theta_0 = \dfrac{1}{3}, & \theta_1 = \dfrac{3}{7}, & \theta_1 = \dfrac{3}{7}, & \theta_\infty = \dfrac{4}{7}.
\end{array}
$$

Proposition 4.5. — *There are four seed RS-transformations related with R-type (9):*

$$RS_4^2 \left(\begin{array}{c|c|c} k/7 & 1/2 & 1/3 \\ 7+2+1+1+1 & \underbrace{2+\ldots+2}_{6} & \underbrace{3+\ldots+3}_{4} \end{array} \right) \text{ for } k = 1, 2, 3, \text{ and } 7/2. \text{ Each of}$$

these transformations produces one algebraic genus 0 solution for the following sets of the θ-parameters:

$$
\begin{array}{ccccc}
k = 1, & \theta_0 = \dfrac{1}{7}, & \theta_1 = \dfrac{1}{7}, & \theta_1 = \dfrac{1}{7}, & \theta_\infty = \dfrac{5}{7}, \\[2mm]
k = 2, & \theta_0 = \dfrac{2}{7}, & \theta_1 = \dfrac{2}{7}, & \theta_1 = \dfrac{2}{7}, & \theta_\infty = \dfrac{4}{7}, \\[2mm]
k = 3, & \theta_0 = \dfrac{3}{7}, & \theta_1 = \dfrac{3}{7}, & \theta_1 = \dfrac{3}{7}, & \theta_\infty = \dfrac{1}{7}, \\[2mm]
k = \dfrac{7}{2}, & \theta_0 = \dfrac{1}{2}, & \theta_1 = \dfrac{1}{2}, & \theta_1 = \dfrac{1}{2}, & \theta_\infty = -\dfrac{5}{2}.
\end{array}
$$

5. Classification of RS-Transformations of D-Type $< 2, 3, 8 >$

Proposition 5.1. — *There is only one $R_4(3)$-type, with the nonempty apparent set in the third box* [6], *corresponding to the D-type $< 2, 3, 8 >$, namely,*

$$(26) \qquad R_4(8 + \underbrace{1 + \cdots + 1}_{4} | \underbrace{2 + \cdots + 2}_{6} | \underbrace{3 + \cdots + 3}_{4})$$

Proof. — We again refer to Master Inequality (2). For our particular divisors after simple manipulations it can be rewritten as follows:

$$24 - 8\sigma_0 - 5\sigma_1 - 3(\sigma_0 + \sigma_1 + \sigma_\infty) \geq n.$$

Now, taking into account Inequality (3) and the fact that $n \geq 8$, we find that $12 - 8\sigma_0 - 5\sigma_1 \geq 8$. Therefore, $\sigma_0 = \sigma_1 = 0$ and thus $\sigma_\infty \geq 4$. Again returning to Master Inequality (2) and substituting in it $\sigma_0 = \sigma_1 = 0$, we obtain, $24 - 3\sigma_\infty \geq n$ which implies that $n \leq 12$. On the other hand $n \geq m_\infty + \sigma_\infty \geq 12$. Therefore, $n = 12$, the only possible $R_4(3)$-type with four non-apparent entries and non-empty apparent set in the third box is equivalent to (26). $\qquad\square$

Because the degree of the function (26) is $12 = 2 \cdot 6 = 3 \cdot 4 = 4 \cdot 3 = 6 \cdot 2$, we have to examine whether this rational function can be presented as a composition of rational functions of the lower degree. Clearly, that one of these functions should be the Belyi function and the other a one-dimensional deformation of (another) Belyi function. The latter generates an algebraic solution of the sixth Painlevé equation. It is easy to see that such composition defines exactly the same algebraic solution of P6 as its member, the deformed Belyi function.

In the above factorizations of 12 into the divisors we assume that the first function is the Belyi one, while the second is a deformation, therefore in this sense these decompositions are not commutative. By a straightforward analysis, just an examination of a few possibilities, we find that there is actually only one such composition (see also Remark 5.2) below) corresponding to the factorization $12 = 6 \cdot 2$, namely,

$$(27) \quad R(8 + \underbrace{1 + \ldots + 1}_{4} | \underbrace{2 + \ldots + 2}_{6} | \underbrace{3 + \ldots + 3}_{4}) = R(\underset{\wedge}{4} + 1 + 1 | 2 + 2 + 2 | 3 + 3) \circ R(\underset{\wedge}{2} | 1 + 1 | 1 + 1).$$

The function $R(4+1+1|2+2+2|3+3)$ itself is also reducible, $R(4+1+1|2+2+2|3+3) = R((\underset{\wedge}{2} + 1|2 + \underset{\wedge}{1}|3) \circ R(\underset{\wedge}{2}|\underset{\wedge}{2}|1 + 1)$, however it is not important in the following. Explicit form of the functions in the r.-h.s. of Equation (27) is as follows:

$$\lambda_2 = \frac{108\lambda_1^4(\lambda_1 - 1)}{(\lambda_1^2 - 16\lambda_1 + 16)^3}, \qquad \lambda_1 = (1 - 2s)\frac{(\lambda - t/s)^2}{(\lambda - t)},$$

---

[6] In the numbering of boxes we follow the convention of Remark 3.1. Here the third box comes first in the natural counting.

where

$$(28) \qquad t = \frac{s^2}{2s - 1}, \quad \text{and} \quad y(t) = s$$

is the solution of Equation (1) for the θ-tuple,

$$(29) \qquad \theta_0 = \theta_1, \qquad \theta_t = \theta_\infty - 1,$$

for arbitrary θ_0 and $\theta_\infty \in \mathbb{C}$ (see Theorem 2.1 of [**17**]). The r.-h.s. now is easy to find,

$$(30) \qquad \lambda_1 = \frac{108\lambda(\lambda - 1)(\lambda - t)(\lambda - t/s)^8}{(2s - 1)(\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0)^3},$$

$$c_3 = -\frac{4(s - 4)}{(2s - 1)}, \quad c_2 = -\frac{2(5s^2 + 16s - 8)}{(2s - 1)^2}, \quad c_1 = \frac{4(7s - 4)s^2}{(2s - 1)^3}, \quad c_0 = \frac{s^4}{(2s - 1)^4},$$

where t is the same as in (28). Applying Theorem 2.1 of [**17**] we again arrive at the solution $y(t)$ defined in Equations (28) but now for a particular choice of the θ-tuple (29), $\theta_0 = \theta_t = 1/8$.

It is instructive to confirm the above mentioned analysis that leads to Equation (27) graphically (see Figure 7).

$$R(8 + 2 + 1 + 1 \mid \underbrace{2 + \ldots + 2}_{6} \mid \underbrace{3 + \ldots + 3}_{4}) \quad \Rightarrow \quad R(8 + \underbrace{1 + \cdots + 1}_{4} \mid \underbrace{2 + \ldots + 2}_{6} \mid \underbrace{3 + \ldots + 3}_{4})$$

FIGURE 7. A symmetry preserving twist of the reducible dessin for the Belyi function

There are two more reducible dessins with the symmetry preserving deformations:

$$R(8 + \underbrace{1 + \ldots + 1}_{4} \mid \underbrace{2 + \ldots + 2}_{4} + 4 \mid \underbrace{3 + \ldots + 3}_{4}) = R(\underset{\wedge}{4} + 1 + 1 \mid 2 + 2 + \underset{\wedge}{2} \mid 3 + 3) \circ R(\underset{\wedge}{2} \mid \underset{\wedge}{2} \mid 1 + 1),$$

$$R(8 + \underbrace{1 + \cdots + 1}_{4} \mid \underbrace{2 + \cdots + 2}_{6} \mid 3 + 3 + 6) = R(\underset{\wedge}{4} + 1 + 1 \mid 2 + 2 + 2 \mid 3 + \underset{\wedge}{3}) \circ R(\underset{\wedge}{2} \mid 1 + 1 \mid \underset{\wedge}{2}).$$

Their deformation dessins exactly coincide and correspond to another branch of the solution (28).

The covering constructed above is not the only possible for this R-type. To get an idea why there should be another solution, we observe that there is the following deformation:

Remark 5.1. — This is a digression to the theory of the Gauss hypergeometric functions. The Belyi function in the r.-h.s. of Figure 8, as well as the Belyi function from

$$R(9{+}1{+}1{+}1\,|\,\underbrace{2{+}\ldots{+}2}_{6}\,|\,\underbrace{3{+}\ldots{+}3}_{4}) \quad \Rightarrow \quad R(8{+}\underbrace{1{+}\ldots{+}1}_{4}\,|\,\underbrace{2{+}\ldots{+}2}_{6}\,|\,\underbrace{3{+}\ldots{+}3}_{4})$$

FIGURE 8. A symmetry breaking "face" deformation (LC-Join) of the dessin for the reducible Belyi function
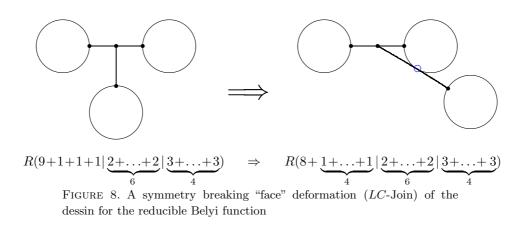
Figure 7, have in our terminology R_3-type and therefore define transformations of order 12 for the Gauss hypergeometric function. Both of these functions are reducible:

$$(31) \qquad R(9{+}1{+}1{+}1\,|\,\underbrace{2{+}\ldots{+}2}_{6}\,|\,\underbrace{3{+}\ldots{+}3}_{4}) = R(\underset{\wedge}{3}{+}1\,|\,2{+}2\,|\,3{+}\underset{\wedge}{1}) \circ R(\underset{\wedge}{3}\,|\,1{+}1{+}1\,|\,\underset{\wedge}{3}),$$

$$(32) \quad R(8{+}2{+}1{+}1\,|\,\underbrace{2{+}\ldots{+}2}_{6}\,|\,\underbrace{3{+}\ldots{+}3}_{4}) = R(2{+}1\,|\,2{+}1\,|\,3) \circ R(2\,|\,2\,|\,1{+}1) \circ R(2\,|\,1{+}1\,|\,2).$$

In most cases reducibility of R-function means that the corresponding higher order transformation for the Gauss hypergeometric function is a composition of transformations of the lower order, namely those that correspond to the R-functions of lower degree from the decomposition of the original R-function. This happens when all rational functions in the corresponding decomposition have R_3-type. Like it is in the case of the function (32), for $k = 1, 3$:

$$(33) \quad RS_3^2 \begin{pmatrix} k/8 & 1/2 & 1/3 \\ 8{+}2{+}1{+}1 & \underbrace{2{+}\ldots{+}2}_{6} & \underbrace{3{+}\ldots{+}3}_{4} \end{pmatrix} =$$

$$RS_3^2 \begin{pmatrix} k/8 & 1/2 & 1/3 \\ 2{+}1 & 2{+}1 & 3 \end{pmatrix} \circ RS_3^2 \begin{pmatrix} k/4 & 1/2 & k/8 \\ 2 & 2 & 1{+}1 \end{pmatrix} \circ RS_3^2 \begin{pmatrix} k/2 & k/8 & k/8 \\ 2 & 2 & 1{+}1 \end{pmatrix}$$

The last transformation is possible because in this case $k/2 = 1/2 \mod (1)$. The resulting θ-triple[7] is $\big(k/8, k/8, k/4 - 1\big)$. The other two "seed" transformations (33) for $k = 2.4$ concerns elementary functions, for them the chain of transformations 33 is also working.

As for the Belyi function from Figure 8, the higher order transformations of the Gauss hypergeometric function associated with it cannot be presented (in general) as a composition of transformations of the lower order, because the first term of the

---

[7]see Footnote 5 on page 213.

composition (31) does not have R_3-type. In terms of the θ-triples the corresponding seed transformations for the Gauss function are as follows:

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{k}{9}\right) \leftarrow \left(\frac{k}{9}, \frac{k}{9}, \frac{k}{9} - \epsilon_k\right), \quad k = 1, 2, 3, 4, \quad \epsilon_0 = \epsilon_2 = 1 - \epsilon_1 = 1 - \epsilon_3 = 0.$$

Because of the composition character both Belyi functions discussed here are easy to construct. We present here the function (31), because as follows from the above discussion it is a useful function in the theory of the Gauss hypergeometric functions:

$$\lambda_1 = \frac{64(\lambda^3 - 1)}{\lambda^3(9\lambda^3 - 8)^3}, \qquad \lambda_1 = -\frac{192i\sqrt{3}\lambda(\lambda - 1)(\lambda - \frac{1 - i\sqrt{3}}{2})^9}{(\lambda - \frac{1 + i\sqrt{3}}{2})^3(\lambda^3 - \frac{3(1 + 17i\sqrt{3})}{2}\lambda^2 - \frac{3(1 - 17i\sqrt{3})}{2}\lambda + 1)^3}$$

The first formula shows how this function is constructed via the composition given in Equation (31) and the second one - its much more sophisticated form after a normalization suitable for the construction of the higher order transformation of the Gauss hypergeometric function.

Turning back to the discussion of Figure 8 it is interesting to notice that although Equation (27) formally still holds for the R-types the deformation function is indecomposibale, see Remark 5.2 below. A direct analysis of the system of algebraic equations defining this rational covering (see [**17**], Remark 2.1) reveals another solution:

$$(34) \qquad \lambda_1 = \frac{27(s^2 + s + is - i)^8}{8s^2(s^4 + 1)^3} \frac{\lambda(\lambda - 1)(\lambda - t)(\lambda - a)^8}{(\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0)^3},$$

$$(35) \qquad a = \frac{(1 + i)(s^2 + 1)(s^2 + is - s + i)(s^2 + 2s - 1)^3}{8s(s^2 + s + is - i)(s^2 + i)^3},$$

$$c_3 = \frac{(s^2 + 2s - 1)}{2s^2(s^2 - i)^2(s^2 + i)^3}(s^{12} + (5 + 7i)s^{11} + (31i - 13)s^{10} + (5i - 29)s^9$$
$$+ (32 + 17i)s^8 + (62i - 2)s^7 + (28 - 28i)s^6 + (62 - 2i)s^5 - (17 + 32i)s^4$$
$$+ (5 - 29i)s^3 + (13i - 31)s^2 + (7 + 5i)s - i),$$

$$c_2 = \frac{(s^2 + 2s - 1)^4}{64s^4(s^2 - i)^2(s^2 + i)^6}(s^{16} + 8is^{15} - 68s^{14} - 296is^{13} + 252s^{12} - 184is^{11} + 420s^{10}$$
$$+ 472is^9 + 454s^8 + 472is^7 + 420s^6 - 184is^5 + 252s^4 - 296is^3 - 68s^2 + 8is + 1),$$

$$c_1 = -\frac{(s^2 + 1)^2(s^2 + 2s - 1)^7}{64s^4(s^2 - i)^2(s^2 + i)^9}(s^{12} + (7i - 5)s^{11} - (13 + 31i)s^{10} + (29 + 5i)s^9$$
$$+ (32 - 17i)s^8 + (2 + 62i)s^7 + (28 + 28i)s^6 - (62 + 2i)s^5 + (32i - 17)s^4$$
$$- (5 + 29i)s^3 - (31 + 13i)s^2 + (5i - 7)s + i),$$

$$c_0 = \frac{(s^2 - 2s - 1)^2(s^2 + 1)^4(s^2 + 2s - 1)^{10}}{1024s^4(s^2 - i)^2(s^2 + i)^{10}},$$

The solution obtained from the above function via Theorem 2.1 [**17**] reads,

$$(36) \qquad t = -\frac{(s^2+1)^2(s^2+2s-1)^3(s^2-2s-1)^3}{32s^2(s^2+i)^3(s^2-i)^3},$$

$$(37) \qquad y(t) = \frac{(1+i)(s^2+s-is+i)(s^2-2s-1)(s^2+1)(s^2+2s-1)^2}{8s(s^2+i)(s^2-i)^2(s^2-s-is-i))},$$

It solves Equation (1) for the following θ-tuple

$$\theta_0 = \theta_1 = \theta_t = \frac{1}{8}, \quad \theta_\infty = \frac{7}{8}.$$

Proposition 5.2. — *With each rational function* (30) *and* (34) *associated four seed RS-transformations:*

$$RS_4^2 \left(\begin{array}{c|c|c} k/8 & 1/2 & 1/3 \\ 8 + \underbrace{1+\ldots+1}_{4} & \underbrace{2+\ldots+2}_{6} & \underbrace{3+\ldots+3}_{4} \end{array} \right) \text{ for } k = 1, 2, 3, \text{ and } 4. \text{ Each of these}$$

*transformations produces one algebraic genus-*0 *solution of Equation* (1) *for the following sets of the θ-parameters:*

$$k = 1, \qquad\qquad \theta_0 = \theta_1 = \theta_t = 1 - \theta_\infty = \frac{1}{8},$$

$$k = 2, \qquad\qquad \theta_0 = \theta_1 = \theta_t = \theta_\infty = \frac{1}{4},$$

$$k = 3, \qquad\qquad \theta_0 = \theta_1 = \theta_t = 1 - \theta_\infty = \frac{3}{8}, \quad and$$

$$k = 4, \qquad\qquad \theta_0 = \theta_1 = \theta_t = \theta_\infty = \frac{1}{2}.$$

The solutions for $k = 1$ corresponding to the functions (30) *and* (34) *are given by Equations* (28) *and* (36), (37), *respectively.*

Remark 5.2. — All solutions of Equation (1) which can be produced via Proposition 5.2 with the help of the function (30) are rational functions of $\sqrt{t}$ and $\sqrt{t-1}$.

The situation with the solutions that are generated via the function (34) is more interesting: most probably, these solutions (it is not checked yet) coincides with the solution that can be obtained via a successive compositions of $RS_4^2 \left(\begin{array}{c|c|c} 1/4 & 1/2 & 1/3 \\ 4+1+1 & 2+2+1+1 & 3+3 \end{array} \right)$ (See, Subsection 3.3, Example 5 (*CW*-split) of [**17**]) the Okamoto transformation in the sense of Appendix of [**18**], and one of the quadratic transformations for Equation (1) at the last page of Appendix of [**17**]), such transformations also can be described as the simple quadratic Belyi functions. Appearance of the Okamoto transformation in this composition makes impossible to lift it on the level of rational coverings.

The last statement can be easily observed in this particular case. Because, In case we suppose that these two examples are related with some RS-transformation, then it would mean that two hypergeometric functions with the θ-triples $(1/2, 1/3, 1/8)$, from

the above example, and $(1/2, 1/3, 1/4,$ from Subsection 3.3, Example 5 (CW-split) of [**17**]), are related with some algebraic transformation, which is impossible, because the first function does not belong to the Schwarz list [**21**] while the second is in its fourth row.

This, possibly, explains the appearance of i in the parametrization (34). Therefore, although the function (34) defines an algebraic solution which (most probably) can be obtained from the already known simplier ones by the certain transformations, the explicit formula for the covering (34) has an independent value. In particular, if we are interesting not only in the solutions of the sixth Painlevé equation but also in the solutions of the associated linear ODE the function (34) gives us an additional opportunity to provide an explicit construction of the latter function in terms of the hypergeometric ones.

References

[1] F. V. ANDREEV & A. V. KITAEV – Transformations $RS_4^2(3)$ of the ranks ≤ 4 and algebraic solutions of the sixth Painlevé equation, *Comm. Math. Phys.* **228** (2002), no. 1, p. 151–176.

[2] _____, Some examples of $RS_3^2(3)$-transformations of ranks 5 and 6 as the higher order transformations for the hypergeometric function, *Ramanujan J.* **7** (2003), no. 4, p. 455–476, `http://xyz.lanl.gov`, `nlin.SI/0012052`, 1-20, 2000.

[3] G. E. ANDREWS, R. ASKEY & R. ROY – *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.

[4] G. V. BELYĬ – Galois extensions of a maximal cyclotomic field, *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), no. 2, p. 267–276, 479.

[5] P. P. BOALCH – The fifty-two icosahedral solutions to Painlevé VI, to appear in *J. Reine Angew. Math.* (`math.AG/0406281`).

[6] _____, Some explicit solutions to the Riemann–Hilbert problem, to appear in *IRMA Lect. Math. Theor. Phys.*, (`math.DG/0501464`).

[7] _____, From Klein to Painlevé via Fourier, Laplace and Jimbo, *Proc. London Math. Soc.* **90** (2005), no. 3, p. 167–208, (`math.AG/0308221`).

[8] C. F. DORAN – Algebraic and geometric isomonodromic deformations, *J. Differential Geom.* **59** (2001), no. 1, p. 33–85.

[9] B. DUBROVIN & M. MAZZOCCO – Monodromy of certain Painlevé-VI transcendents and reflection groups, *Invent. Math.* **141** (2000), no. 1, p. 55–147.

[10] R. FUCHS – Ueber lineare homogene Differentialgleichungen zweiter Ordnung mit vier wesentlich singulären Stellen, *Nachr. d. Kgl. Ges. d. Wiss. zu Göttingen, Math.-Phys. Klasse* (1910), p. 146–153.

[11] _____, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen, *Math. Ann.* **70** (1911), no. 4, p. 525–549.

[12] M. JIMBO & T. MIWA – Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, *Phys. D* **2** (1981), no. 3, p. 407–448.

[13] A. V. KITAEV – Quadratic transformations for the sixth Painlevé equation, *Lett. Math. Phys.* **21** (1991), no. 2, p. 105–111.

[14] _____, Special functions of the isomonodromy type, *Acta Appl. Math.* **64** (2000), no. 1, p. 1–32.

[15] _____, Special functions of isomonodromy type, rational transformations of the spectral parameter, and algebraic solutions of the sixth Painlevé equation, *Algebra i Analiz* **14** (2002), no. 3, p. 121–139, English translation in *St. Petersburg Math. J.* **14**, (2003), no. 3, p. 453–465, `http://xxx.lanl.gov`, `nlin.SI/0102020`, 1–13, 2000.

[16] _____, Quadratic transformations for the third and fifth Painlevé equations, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **317** (2004), p. 105–121.

[17] _____, Grothendieck's dessins d'enfants, their deformations, and algebraic solutions of the sixth Painlevé and Gauss hypergeometric equations, *Algebra i Analiz* **17** (2005), no. 1, p. 224–275, `http://xxx.lanl.gov`, `nlin.SI0309078`, 1-35, 2003.

[18] A. V. KITAEV & D. A. KOROTKIN – On solutions of the Schlesinger equations in terms of Θ-functions, *Internat. Math. Res. Notices* (1998), no. 17, p. 877–905.

[19] F. KLEIN – *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Birkhäuser Verlag, Basel, 1993, reprint of the 1884 original, edited, with an introduction and commentary by Peter Slodowy.

[20] K. OKAMOTO – Studies on the Painlevé equations. I. Sixth Painlevé equation P_{VI}, *Ann. Mat. Pura Appl. (4)* **146** (1987), p. 337–381.

[21] H. A. SCHWARZ – Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elements darstellt, *J. Reine Angew. Math.* **75** (1873), p. 292–335.

[22] R. VIDŪNAS – Algebraic transformations of Gauss hypergeometric functions, preprint, `http://xyz.lanl.gov`, `math.CA/0408269`, 2004, v.1, p. 1–24.

[23] _____, Transformations of some Gauss hypergeometric functions, *J. Comput. Appl. Math.* **178** (2005), no. 1-2, p. 473–487.

A. V. KITAEV, Steklov Mathematical Institute, Fontanka 27, St.Petersburg, 191023, Russia and School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia • *E-mail :* `kitaev@pdmi.ras.ru` and `kitaev@maths.usyd.edu.au`
Url : `http://www.pdmi.ras.ru`