THE LAX PAIR FOR THE MKDV HIERARCHY

by

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Abstract. — In this paper we give an algorithmic method of deriving the Lax pair for the modified Korteweg-de Vries hierarchy. For each n, the compatibility condition gives the n-th member of the hierarchy, rather than its derivative. A direct consequence of this is that we obtain the isomonodromy problem for the second Painlevé hierarchy, which is derived through a scaling reduction.

Résumé (La paire de Lax de la hiérarchie mKdV). — Dans cet article, nous présentons une méthode algorithmique pour le calcul de la paire de Lax de la hiérarchie de Korteweg-de Vries modifiée. Pour tout n, la condition de compatibilité fournit le $n^{\text{ième}}$ membre de la hiérarchie lui-même et non pas sa dérivée. Grâce à une réduction par l'action du groupe de similarité, nous en déduisons un problème d'isomonodromie pour la deuxième hiérarchie de Painlevé.

1. Introduction

There has been considerable interest in partial differential equations solvable by inverse scattering, the so-called *soliton equations*, since the discovery in 1967 by Gardner, Greene, Kruskal and Miura [**8**] of the method for solving the initial value problem for the Korteweg-de Vries (KdV) equation

$$(1) \qquad u_t + 6uu_x + u_{xxx} = 0.$$

In the inverse scattering method, which can be thought of as a nonlinear analogue of the Fourier transform method for linear partial differential equations, the nonlinear

PDE is expressed as the compatibility of two **linear** equations (the celebrated Lax Pair). Typically, this has the form

$$\Phi_x = \mathcal{L}\Phi, \tag{2}$$

$$\Phi_t = \mathcal{M}\Phi, \tag{3}$$

where Φ is a vector, an eigenfunction, and $\mathcal{L}$ and $\mathcal{M}$ are matrices whose entries depend on the solution $u(x,t)$ of the associated nonlinear partial differential equation. Given suitable initial data $u(x,0)$, one obtains the associated scattering data $S(0)$ by solving the spectral problem (2). The scattering data $S(t)$ is then obtained by solving the temporal problem (3), and finally the solution $u(x,t)$ of the partial differential equation is obtained by solving an inverse problem, which is usually expressed as a Riemann-Hilbert problem and frequently the most difficult part (see, for example, [**1, 4**] and the references therein).

Solutions of the modified Korteweg-de Vries (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0, \tag{4}$$

are related to solutions of the KdV equation (1) through the Miura transformation $u = v_x - v^2$ [**19**].

Soliton equations all seem to possess several remarkable properties in common including, the "elastic" interaction of solitary waves, i.e. multi-soliton solutions, Bäcklund transformations, an infinite number of independent conservation laws, a complete set of action-angle variables, an underlying Hamiltonian formulation, a Lax representation, a bilinear representation à la Hirota, the Painlevé property, an associated linear eigenvalue problem whose eigenvalues are constants of the motion, and an infinite family of equations, the so-called *hierarchy*, which is our main interest in this manuscript (cf. [**1, 4**]).

The standard procedure for generating the mKdV hierarchy is to use a combination of the Lenard recursion operator for the KdV hierarchy and the Miura transformation, as we shall briefly explain now.

The KdV hierarchy is given by

$$u_{t_{n+1}} + \frac{\partial}{\partial x}\mathcal{L}_{n+1}[u] = 0, \qquad n = 0, 1, 2, \ldots, \tag{5}$$

where $\mathcal{L}_n$ satisfies the Lenard recursion relation [**15**]

$$\frac{\partial}{\partial x}\mathcal{L}_{n+1} = \left(\frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2u_x\right)\mathcal{L}_n. \tag{6}$$

Beginning with $\mathcal{L}_0[u] = \frac{1}{2}$, this gives

$$\mathcal{L}_1[u] = u, \qquad \mathcal{L}_2[u] = u_{xx} + 3u^2, \qquad \mathcal{L}_3[u] = u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3,$$

and so on. The first four members of the KdV hierarchy are

$$u_{t_1} + u_x = 0,$$
$$u_{t_2} + u_{xxx} + 6uu_x = 0,$$
$$u_{t_3} + u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = 0,$$
$$u_{t_4} + u_{xxxxxxx} + 14uu_{xxxxx} + 42u_x u_{xxxx} + 70u_{xx} u_{xxx}$$
$$+ 70u^2 u_{xxx} + 280uu_x u_{xx} + 70u_x^3 + 140u^3 u_x = 0.$$

The mKdV hierarchy is obtained from the KdV hierarchy via the Miura transformation $u = v_x - v^2$ (see [**3**, **5**, **7**]) and can be written as

$$(7) \qquad v_{t_{n+1}} + \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_n \left[v_x - v^2 \right] = 0, \qquad n = 1, 2, 3, \ldots$$

The first three members of the mKdV hierarchy are

$$v_{t_1} + v_{xxx} - 6v^2 v_x = 0,$$
$$v_{t_2} + v_{xxxxx} - 10v^2 v_{xxx} - 40v_x v_{xx} - 10v_x^3 + 30v^4 v_x = 0,$$
$$v_{t_3} + v_{xxxxxxx} - 14v^2 v_{xxxxx} - 84vv_x v_{xxxx} - 140vv_{xx} v_{xxx}$$
$$- 126v_x^2 v_{xxx} - 182v_x v_{xx}^2 + 70v^4 v_{xxx} + 560v^3 v_x v_{xx}$$
$$+ 420v^2 v_x^3 - 140v^6 v_x = 0.$$

This procedure generates the mKdV hierarchy. We show how to derive a Lax pair for this hierarchy from the one of the KdV hierarchy in the appendix of this paper. However, this procedure gives rise to a hierarchy which is the derivative of the mKdV hierarchy.

Our interest is in the mKdV hierarchy rather than its derivative. We overcome this by generating the Lax pair for the (undifferentiated) mKdV hierarchy in a straightforward, algorithmic way, by using the AKNS expansion technique [**2**]. We call the result the *natural* Lax pair for the mKdV hierarchy. A direct consequence of this is that we also obtain the isomonodromic problem for the second Painlevé hierarchy. Our natural Lax pair for the mKdV hierarchy yields a natural isomonodromy problem that contains the Flaschka-Newell linear problem as the $n = 1$ case.

We derive the natural Lax pair for the mKdV hierarchy in §2 and the natural isomonodromy problem for the second Painlevé hierarchy in §3. In §4 we discuss our results. The Lax pair arising from that for the KdV hierarchy is derived in the appendix.

2. The Natural Lax Pair for the mKdV Hierarchy

The well known Lax pair for the mKdV equation is

$$(8a) \qquad \frac{\partial \Phi}{\partial x} = \mathcal{L}\Phi = \begin{pmatrix} -i\zeta & v \\ v & i\zeta \end{pmatrix} \Phi$$

$$(8b) \qquad \frac{\partial \Phi}{\partial t} = \mathcal{M}\Phi$$

$$= \begin{pmatrix} -4i\zeta^3 - 2i\zeta v^2 & 4\zeta^2 v + 2i\zeta v_x - v_{xx} + 2v^3 \\ 4\zeta^2 v - 2i\zeta v_x - v_{xx} + 2v^3 & 4i\zeta^3 + 2i\zeta v^2 \end{pmatrix} \Phi$$

This Lax pair was first given by Ablowitz, Kaup, Newell, and Segur (AKNS) [**2**]. In the same paper it is suggested that higher order equations in the mKdV hierarchy could be generated by considering higher degree expansions in the entries of $\mathcal{M}$. We follow this procedure here.

Proposition 1. — *For each integer $n \geq 1$, the Lax pair for that n-th equation (7) of the mKdV hierarchy is*

$$(9a) \qquad \frac{\partial \Phi}{\partial x} = \mathcal{L}\Phi = \begin{pmatrix} -i\zeta & v \\ v & i\zeta \end{pmatrix} \Phi$$

$$(9b) \qquad \frac{\partial \Phi}{\partial t_{n+1}} = \mathcal{M}\Phi = \begin{pmatrix} \displaystyle\sum_{j=0}^{2n+1} A_j (i\zeta)^j & \displaystyle\sum_{j=0}^{2n} B_j (i\zeta)^j \\ \displaystyle\sum_{j=0}^{2n} C_j (i\zeta)^j & -\displaystyle\sum_{j=0}^{2n+1} A_j (i\zeta)^j \end{pmatrix} \Phi$$

where

$$(10a) \quad A_{2n+1} = 4^n, \qquad A_{2k} = 0, \qquad \forall k = 0, \dots, n,$$

$$(10b) \quad A_{2k+1} = \frac{4^{k+1}}{2} \left\{ \mathcal{L}_{n-k} \left[v_x - v^2 \right] - \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_{n-k-1} \left[v_x - v^2 \right] \right\},$$
$$k = 0, \dots, n-1,$$

$$(10c) \quad B_{2k+1} = \frac{4^{k+1}}{2} \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_{n-k-1} \left[v_x - v^2 \right], \quad k = 0, \dots, n-1,$$

$$(10d) \qquad B_{2k} = -4^k \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_{n-k} \left[v_x - v^2 \right], \quad k = 0, \dots, n,$$

$$(10e) \quad C_{2k+1} = -B_{2k+1}, \quad k = 0, \dots, n-1,$$

$$(10f) \qquad C_{2k} = B_{2k}, \quad k = 0, \dots, n.$$

Proof. — The compatibility $\Phi_{xt} = \Phi_{tx}$ of equations (9) is guaranteed by the conditions

$$(11a) \qquad vC - vB \;=\; \frac{\partial A}{\partial x},$$

$$(11b) \qquad v_t - 2i\zeta B - 2vA \;=\; \frac{\partial B}{\partial x},$$

$$(11c) \qquad v_t + 2i\zeta C + 2vA \;=\; \frac{\partial C}{\partial x}.$$

At the order $\mathcal{O}(1)$ in ζ we obtain

$$v_t = \frac{\partial B_0}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_n\left[v_x - v^2\right],$$

that is (7). We have to show that at each order in ζ^j the compatibility conditions (11) are satisfied. At each order $\mathcal{O}(\zeta^j)$ the conditions (11) give

$$(12a) \qquad \frac{\partial A_j}{\partial x} \;=\; v(C_j - B_j),$$

$$(12b) \qquad \frac{\partial B_j}{\partial x} \;=\; -2B_{j-1} - 2vA_j,$$

$$(12c) \qquad \frac{\partial C_j}{\partial x} \;=\; 2C_{j-1} + 2vA_j.$$

We proceed by induction. At the order $\mathcal{O}(\zeta^{2n+1})$, since by assumption, B_{2n+1} and C_{2n+1} are null, the compatibility conditions give

$$\frac{\partial}{\partial x}A_{2n+1} = 0$$

and by assuming $B_{2n} = -4^n\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_0\left[v_x - v^2\right]$, (12b) gives

$$-4^n\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_0 + vA_{2n+1} = 0.$$

Assuming $A_{2n+1} = 4^n$ the compatibility condition is satisfied because $\mathcal{L}_0 = \frac{1}{2}$. We now assume

$$(13) \qquad \begin{aligned} B_{2k+1} &= \frac{4^{k+1}}{2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + 2v\right)\mathcal{L}_{n-k-1}\left[v_x - v^2\right], \\ C_{2k+1} &= -B_{2k+1}, \end{aligned}$$

for a fixed $0 \leq k < n$ and prove

$$(14a) \quad A_{2k+1} = \frac{4^{k+1}}{2} \left\{ \mathcal{L}_{n-k} \left[v_x - v^2 \right] - \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_{n-k-1} \left[v_x - v^2 \right] \right\},$$

$$(14b) \quad B_{2k} = -4^k \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_{n-k} \left[v_x - v^2 \right],$$

$$(14c) \quad C_{2k} = B_{2k},$$

$$(14d) \quad \frac{\partial}{\partial x} A_{2k} = 0,$$

$$(14e) \quad B_{2k-1} = \frac{4^k}{2} \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_{n-k} \left[v_x - v^2 \right],$$

$$(14f) \quad C_{2k-1} = -B_{2k-1}.$$

For $j = 2k + 1$, (12a) gives

$$\frac{\partial}{\partial x} A_{2k+1} = 2v C_{2k+1}$$

$$= -4^{k+1} v \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_{n-k-1} \left[v_x - v^2 \right]$$

$$= \frac{4^{k+1}}{2} \frac{\partial}{\partial x} \left\{ \mathcal{L}_{n-k} \left[v_x - v^2 \right] - \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_{n-k-1} \left[v_x - v^2 \right] \right\},$$

because

$$\left(\frac{\partial}{\partial x} - 2v \right) \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_{n-k-1} \left[v_x - v^2 \right] = \frac{\partial}{\partial x} \mathcal{L}_{n-k} \left[v_x - v^2 \right].$$

This proves (14a). We then prove all the others, in the given order, in an analogous way. In particular, we proved that A_{2k} is a constant. At the next step, we assume this constant to be zero, compute B_{2k-1} and C_{2k-1} using (12b) and (12c), and start again. This concludes the proof of Proposition 1. $\qquad \square$

3. The Natural Isomonodromic Problem for the $\mathrm{P_{II}}$ Hierarchy.

The $\mathrm{P_{II}}$ hierarchy is given by

$$(15) \quad \mathrm{P_{II}^{(n)}}: \quad \left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w \right) \widehat{\mathcal{L}}_n \left\{ w_z - w^2 \right\} = zw + \alpha_n, \qquad n \geq 1$$

where α_n are constants and $\widehat{\mathcal{L}}_n$ is the operator defined by equation (6) with x replaced by z. For $n = 1$, equation (15) is P$_{\text{II}}$. This hierarchy arises as the following symmetry reduction of the mKdV (see [**6**] for details)

$$v(x, t_{n+1}) = \frac{w(z)}{[(2n+1)t_{n+1}]^{1/(2n+1)}}, \qquad z = \frac{x}{[(2n+1)t_{n+1}]^{1/(2n+1)}}.$$

On the Lax pair we perform the following symmetry reduction

$$\Phi(x, t_{n+1}, \zeta) = \Psi(z, \lambda),$$

$$z = \frac{x}{[(2n+1)t_{n+1}]^{1/(2n+1)}},$$

$$\lambda = [(2n+1)t_{n+1}]^{1/(2n+1)}\,\zeta.$$

By (9) we obtain

$$\frac{\partial \Psi}{\partial z} = [(2n+1)t_{n+1}]^{1/(2n+1)}\,\frac{\partial \Phi}{\partial x}$$

$$= [(2n+1)t_{n+1}]^{1/(2n+1)}\,\mathcal{L}\Psi \equiv \widehat{\mathcal{L}}\Psi$$

$$\lambda\frac{\partial \Psi}{\partial \lambda} = z\frac{\partial \Psi}{\partial z} + (2n+1)t_{n+1}\frac{\partial \Phi}{\partial t_{n+1}}$$

$$= \left\{ z\left[(2n+1)t_{n+1}\right]^{1/(2n+1)}\mathcal{L} + (2n+1)t_{n+1}\mathcal{M} \right\}\Psi \equiv \left[z\widehat{\mathcal{L}} + \widehat{\mathcal{M}} \right]\Psi,$$

where

$$\widehat{\mathcal{L}} \equiv [(2n+1)t_{n+1}]^{1/(2n+1)}\,\mathcal{L}, \qquad \widehat{\mathcal{M}} \equiv (2n+1)t_{n+1}\mathcal{M}$$

that gives

$$(16a) \quad \frac{\partial \Psi}{\partial z} = \begin{pmatrix} -i\lambda & w \\ w & i\lambda \end{pmatrix}\Psi$$

$$(16b) \quad \lambda\frac{\partial \Psi}{\partial \lambda} = \left\{ z\begin{pmatrix} -i\lambda & w \\ w & i\lambda \end{pmatrix} + \begin{pmatrix} \displaystyle\sum_{j=0}^{2n+1}\widehat{A}_j(i\lambda)^j & \displaystyle\sum_{j=0}^{2n}\widehat{B}_j(i\lambda)^j \\ \displaystyle\sum_{j=0}^{2n}\widehat{C}_j(i\lambda)^j & -\displaystyle\sum_{j=0}^{2n+1}\widehat{A}_j(i\lambda)^j \end{pmatrix} \right\}\Psi$$

where, from (10)

(17a) $\widehat{A}_{2n+1} = 4^n, \qquad \widehat{A}_{2k} = 0, \quad \forall\, k = 0, \ldots, n,$

(17b) $\widehat{A}_{2k+1} = \dfrac{4^{k+1}}{2} \left\{ \widehat{\mathcal{L}}_{n-k} \left[w' - w^2 \right] - \dfrac{\mathrm{d}}{\mathrm{d}z} \left(\dfrac{\mathrm{d}}{\mathrm{d}z} + 2w \right) \widehat{\mathcal{L}}_{n-k-1} \left[w' - w^2 \right] \right\},$
$$k = 0, \ldots, n-1,$$

(17c) $\widehat{B}_{2k+1} = \dfrac{4^{k+1}}{2} \dfrac{\mathrm{d}}{\mathrm{d}z} \left(\dfrac{\mathrm{d}}{\mathrm{d}z} + 2w \right) \widehat{\mathcal{L}}_{n-k-1} \left[w' - w^2 \right], \quad k = 0, \ldots, n-1,$

(17d) $\widehat{B}_{2k} = -4^k \left(\dfrac{\mathrm{d}}{\mathrm{d}z} + 2w \right) \widehat{\mathcal{L}}_{n-k} \left[w' - w^2 \right], \quad k = 1, \ldots, n,$

(17e) $\widehat{B}_0 = -zw - \alpha_n,$

(17f) $\widehat{C}_{2k+1} = -\widehat{B}_{2k+1}, \quad k = 0, \ldots, n-1,$

(17g) $\widehat{C}_{2k} = \widehat{B}_{2k}, \quad k = 0, \ldots, n,$

since

$$\mathcal{L}_k \left[v_x - v^2 \right] = \frac{1}{\left[(2n+1)t_{n+1} \right]^{2k/(2n+1)}} \widehat{\mathcal{L}}_k [w' - w^2],$$

$$\left(\frac{\partial}{\partial x} + 2v \right) \mathcal{L}_k \left[v_x - v^2 \right] = \frac{1}{\left[(2n+1)t_{n+1} \right]^{(2k+1)/(2n+1)}} \left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w \right) \widehat{\mathcal{L}}_k [w' - w^2].$$

4. Conclusions

In this paper we have given an algorithm for deriving a hierarchy of integrable partial differential equations and simultaneously obtaining the associated Lax pair. A consequence of this procedure is an algorithm for deriving a hierarchy of integrable ordinary differential equations and the associated monodromy problem. An important point here is that our procedure is algorithmic, whereas previous methods have relied on making appropriate ansatzes or relied on other knowledge about the differential equations under consideration, for example the associated recursion operator or the Hamiltonian structure.

A second Painlevé hierarchy was suggested in the paper by Ablowitz and Segur [**3**] which demonstrated a relationship between the soliton equations solvable by inverse scattering and the Painlevé equations. Subsequently Airault [**5**] and Flaschka and Newell [**7**] derived properties of the second Painlevé hierarchy. Flaschka and Newell [**7**] implicitly derived the Lax pairs for the mKdV and second Painlevé hierarchies

using a Hamiltonian approach, rather than the AKNS expansion technique which we use in this paper.

Kudryashov [13] also uses an AKNS type approach to generate a hierarchy of equations which he claims is a generalization of the PII hierarchy. However Kudryashov starts with the linear system

$$\Psi_z = \mathcal{M}\Psi, \qquad \lambda^2 \Psi_\lambda = \mathcal{N}\Psi$$

where λ is the monodromy parameter, and makes assumptions on the matrices $\mathcal{M}$ and $\mathcal{N}$. The expressions are very similar to what we have here. However, in addition to making an ansatz rather than presenting an algorithm, no connection is made linking the hierarchy of ordinary differential equations with a hierarchy of partial differential equations. In another paper, Kudryashov [9] derives some fourth order ordinary differential equations by seeing what arises as the compatibility condition of certain isomonodromy problems. Again no algorithm is presented nor is any connection made linking the ordinary differential equations with integrable partial differential equations.

In a series of papers with various co-authors [17, 20, 21, 22], Zeng discusses the derivation of the KdV and mKdV hierarchies and their associated Lax pairs. The approach taken is through an adjoint representation, rather than the usual AKNS expansion. We expect these should be equivalent to those obtained in §2 above since the expressions obtained seem quite similar, though we feel that our approach is simpler and "more natural". Further there is no mention of reductions to Painlevé hierarchies in any of these papers.

Appendix A
The Lax pair from the KdV hierarchy

Here we show how to derive a Lax pair for the mKdV hierarchy from the one of the KdV hierarchy.

Proposition 2. — [14] *The Lax pair for the n-th equation of the KdV hierarchy is*

$$(18) \quad \begin{cases} \phi_{xx} + [\zeta + u(x, t_{n+1})]\phi = 0 \\ \phi_{t_{n+1}} = \left[\dfrac{\partial}{\partial x}\left(\sum_{k=0}^{n}(-4\zeta)^{n-k}\right)\mathcal{L}_k[u] + a_n \right]\phi - 2\sum_{k=0}^{n}(-4\zeta)^{n-k}\mathcal{L}_k[u]\cdot\phi_x \end{cases}$$

with $a_n = (-4)^n a_0$, a_0 a constant.

Proof. — In fact, the compatibility condition of

$$\begin{cases} \phi_{xx} + [\zeta + u(x, t)]\phi = 0 \\ \phi_{t_{n+1}} = A[u, \zeta]\phi - B[u, \zeta]\phi_x, \end{cases}$$

where $A = A_0[u]\zeta^n + \cdots + A_n[u]$, $A_0[u] = a_n$ is constant, $B = B_0[u]\zeta^n + B_1[u]\zeta^{n-1} + \cdots + B_n[u]$ and $B_0[u] = (-4)^n$, implies

$$-\frac{\partial u}{\partial t_{n+1}}\phi - (\zeta + u)(A\phi - B\phi_x) = A_{xx}\phi + 2A_x\phi_x + A(-\zeta - u)\phi$$
$$- B_{xx}\phi_x + 2B_x(\zeta + u)\phi + B[\zeta\phi_x + u_x\phi + u\phi_x],$$

Setting coefficients of ϕ and ϕ_x to zero, we get

(19) $\phi_x:\quad A = \frac{1}{2}B_x + a_0$

(20) $\phi:\quad u_{t_{n+1}} + \frac{1}{2}B_{xxx} + 2(\zeta + u)B_x + u_xB = 0$

that gives

(21) $$u_{t_{n+1}} + \frac{1}{2}\left(\frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2u_x\right)B + 2\zeta\frac{\partial}{\partial x}B = 0.$$

One can show by straightforward computations that, at each order in ζ, (21) holds identically and, at the order $\mathcal{O}(1)$, one has

$$u_{t_{n+1}} + \frac{1}{2}\left(\frac{\partial^3}{\partial x^3} + 4u\frac{\partial}{\partial x} + 2u_x\right)B_n = 0,$$

namely,

$$u_{t_{n+1}} + \frac{\partial}{\partial x}\mathcal{L}_{n+1}[u] = 0$$

that is the n-th equation of the KdV hierarchy. $\square$

One can obtain the Lax pair of the mKdV hierarchy simply by substituting $u = v_x - v^2$ in (18). To obtain the P_{II} hierarchy one imposes the symmetry reduction

$\Phi(x, t_{n+1}, \zeta) = \psi(z, \lambda),$

$z = \dfrac{x}{[(2n + 1)t_{n+1}]^{1/(2n+1)}},$

$\lambda = \zeta\left[(2n + 1)t_{n+1}\right]^{2/(2n+1)},$

$\Phi_{xx} = \dfrac{\psi_{zz}}{[(2n + 1)t_{n+1}]^{2/(2n+1)}},$

$u = \dfrac{w' - w^2}{[(2n + 1)t_{n+1}]^{2/(2n+1)}},$

$\dfrac{\partial}{\partial t_{n+1}} = \dfrac{-x}{[(2n + 1)t_{n+1}]^{2(n+1)/(2n+1)}}\dfrac{\partial}{\partial z} + \dfrac{2\zeta}{[(2n + 1)t_{n+1}]^{(2n+1)/(2n+1)}}\dfrac{\partial}{\partial \lambda}$

$\qquad = \dfrac{1}{(2n + 1)t_{n+1}}\left(-z\dfrac{\partial}{\partial z} + 2\lambda\dfrac{\partial}{\partial \lambda}\right).$

Now, since

$$\mathcal{L}_k[u] = \frac{1}{\left[(2n+1)t_{n+1}\right]^{2k/(2n+1)}} \, \widehat{\mathcal{L}}_k[w' - w^2],$$

we obtain

$$(-4\zeta)^{n-k}\mathcal{L}_k[u] = \frac{(-4\lambda)^{n-k}}{\left[(2n+1)t_{n+1}\right]^{2n/(2n+1)}} \, \widehat{\mathcal{L}}_k[w' - w^2]$$

that gives the isomonodromic problem

$$\begin{cases} \psi_{zz} + (s + w' - w^2)\psi = 0 \\[2mm] 2s\psi_s - z\psi_z = \\ \quad \left[\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\sum_{k=0}^{n} \widehat{\mathcal{L}}_k\left[w' - w^2\right](-4s)^{n-k}\right) + b_n\right]\psi - 2\sum_{k=0}^{n} \widehat{\mathcal{L}}_k\left[w' - w^2\right](-4s)^{n-k} \cdot \psi_z \end{cases}$$

where $b_n = a_n(2n+1)t_{n+1}$.

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