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On a Negative Flow of the AKNS Hierarchy and Its Relation to a Two-Component Camassa–Holm Equation*

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Abstract. Different gauge copies of the Ablowitz–Kaup–Newell–Segur (AKNS) model labeled by an angle θ are constructed and then reduced to the two-component Camassa–Holm model. Only three different independent classes of reductions are encountered corresponding to the angle θ being 0, $\pi/2$ or taking any value in the interval $0 < \theta < \pi/2$. This construction induces Bäcklund transformations between solutions of the two-component Camassa–Holm model associated with different classes of reduction.

Key words: integrable hierarchies; Camassa–Holm equation; Bäcklund transformation

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1 Introduction

It is widely known that the standard integrable hierarchies can be supplemented by a set of commuting flows of a negative order in a spectral parameter [1]. A standard example is provided by the modified KdV-hierarchy, which can be embedded in a new extended hierarchy. This extended hierarchy contains in addition to the original modified KdV equation also the differential equation of the sine-Gordon model realized as the first negative flow [2, 3, 4, 5, 6, 7].

Quite often the negative flows can only be realized in a form of non-local integral differential equations. The cases where the negative flow can be cast in form of local differential equation which has physical application are therefore of special interest. Recently in [11], a negative flow of the extended AKNS hierarchy [8] was identified with a two-component generalization of the Camassa–Holm equation. The standard Camassa–Holm equation [9, 10]

$$u_t - u_{txx} = -3uu_x + 2u_x u_{xx} + uu_{xxx} - \kappa u_x, \qquad \kappa = \text{const} \tag{1.1}$$

enjoys a long history of serving as a model of long waves in shallow water. The two-component extension [11, 13] differs from equation (1.1) by presence on the right hand side of a new term $\rho\rho_x$, with the new variable ρ obeying the continuity equation $\rho_t + (u\rho)_x = 0$. Such generalization was first encountered in a study of deformations of the bihamiltonian structure of hydrodynamic type [12]. Various multi-component generalizations of the Camassa–Holm model have been subject of intense investigations in recent literature [14, 15, 16, 17, 18].

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A particular connection between extended AKNS model and a two-component generalization of the Camassa–Holm equation was found in [11] and in [13]. It was pointed out in [19] that the second order spectral equation for a two-component Camassa–Holm model can be cast in form of the first order spectral equation which after appropriate gauge transformations fits into an $sl(2)$ setup of linear spectral problem and associated zero-curvature equations.

The goal of this article is to formulate a general scheme for connecting an extended AKNS model to a two-component Camassa–Holm model which would encompass all known ways of connecting the solution f of the latter model to variables r and q of the former model. Our approach is built on making gauge copies of an extended AKNS model labeled by angle θ belonging to an interval $0 \le \theta \le \pi/2$ and then by elimination of one of two components of the $sl(2)$ wave function reach a second order non-linear partial differential equation which governs the two-component Camassa–Holm model. We found that the construction naturally decomposes into three different classes depending on whether angle θ belongs to an interior of interval $0 \le \theta \le \pi/2$ or is equal to one of two boundary values unifying therefore the results of [11] and [20]. The map between these three cases induces a Bäcklund like transformations between different solutions f of the two-component Camassa–Holm equation.

2 A simple derivation of a relation between AKNS and two-component Camassa–Holm models

Our starting point is a standard first-order linear spectral problem of the AKNS model:

$$\Psi_y = (\lambda\sigma_3 + \mathcal{A}_0)\,\Psi = \lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Psi + \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix} \Psi, \tag{2.1}$$

where λ is a spectral parameter, y a space variable and Ψ a two-component object:

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \tag{2.2}$$

In addition, the system is augmented by a negative flow defined in terms of a matrix, which is inverse proportional to λ:

$$\Psi_s = D^{(-1)}\Psi = \frac{1}{\lambda} \begin{bmatrix} A & B \\ C & -A \end{bmatrix} \Psi. \tag{2.3}$$

The compatibility condition arising from equations (2.1) and (2.3):

$$(\mathcal{A}_0)_s - D^{(-1)}_y + \left[\lambda\sigma_3 + \mathcal{A}_0, D^{(-1)}\right] = 0. \tag{2.4}$$

has a general solution:

$$D^{(-1)} = \frac{1}{4\beta\lambda} M_0 \sigma_3 M_0^{-1}, \qquad \mathcal{A}_0 = M_{0\,y} M_0^{-1}, \tag{2.5}$$

in terms of the zero-grade group element, M_0, of SL(2). Note that the solution, $D^{(-1)}$, of the compatibility condition is connected to $(1/\lambda)\sigma_3$-matrix by a similarity transformation.

The factor $1/4\beta$ in (2.5) is a general proportionality factor which implies a determinant formula:

$$A^2 + BC = \frac{1}{16\beta^2} \tag{2.6}$$

for the matrix elements of $D^{(-1)}$.

From (2.4) we find that

$$\begin{aligned}\left(\operatorname{Tr}(\mathcal{A}_0^2)\right)_s &= 2\operatorname{Tr}(\mathcal{A}_0\mathcal{A}_{0\,s}) = -2\operatorname{Tr}\left(\mathcal{A}_0\left[\lambda\sigma_3, D^{(-1)}\right]\right) = 2\operatorname{Tr}\left(\lambda\sigma_3\left[\mathcal{A}_0, D^{(-1)}\right]\right)\\ &= 2\operatorname{Tr}\left(\lambda\sigma_3 D_y^{(-1)}\right) = 4A_y\end{aligned}$$

or

$$A_y = \frac{1}{2}(rq)_s. \tag{2.7}$$

When projected on the zero and the first powers of λ the compatibility condition (2.4) yields

$$q_s = -2B, \qquad r_s = 2C, \tag{2.8}$$

and

$$A_y = qC - rB, \qquad B_y = -2Aq, \qquad C_y = 2Ar, \tag{2.9}$$

respectively. Note that the first of equations (2.9) together with equations (2.8) reproduces formula (2.7).

Combining the above equations we find that

$$A = -\frac{B_y}{2q} = \frac{q_{sy}}{4q} = \frac{C_y}{2r} = \frac{r_{sy}}{4r}. \tag{2.10}$$

The spectral equation (2.1) reads in components:

$$\psi_{1\,y} = \lambda\psi_1 + q\psi_2, \qquad \psi_{2\,y} = -\lambda\psi_2 + r\psi_1. \tag{2.11}$$

Now we eliminate the wave-function component ψ_2 by substituting

$$\psi_2 = \frac{1}{q}\left(\psi_{1y} - \lambda\psi_1\right)$$

into the remaining second equation of (2.11). In this way we obtain for ψ_1

$$\psi_{1yy} - \frac{q_y}{q}\psi_{1y} + \frac{\lambda q_y}{q}\psi_1 - \lambda^2\psi_1 - rq\psi_1 = 0.$$

Introducing

$$\psi = e^{-\int p\,\mathrm{d}y}\psi_1 \tag{2.12}$$

with the integrating factor

$$p(y) = \frac{1}{2}\left(\ln q\right)_y$$

allows to eliminate the term with ψ_{1y} and obtain

$$\psi_{yy} = \Big(\lambda^2 - \lambda\left(\ln q\right)_y - Q\Big)\,\psi \tag{2.13}$$

with

$$Q = \frac{1}{2}\left(\ln q\right)_{yy} - \frac{1}{4}\left(\ln q\right)_y^2 - rq = \frac{q_{yy}}{2q} - \frac{3}{4}\left(\frac{q_y}{q}\right)^2 - rq \tag{2.14}$$

as in [20].

Eliminating ψ_2 from equation (2.3) yields for ψ the following equation:

$$\psi_s = \frac{1}{4\lambda}\left(\frac{q_s}{q}\right)_y \psi - \frac{1}{2\lambda}\frac{q_s}{q}\psi_y. \tag{2.15}$$

Compatibility equation $\psi_{yys} - \psi_{syy} = 0$ yields

$$\left(\frac{q_{sy}}{4q}\right)_y = \frac{1}{2}(rq)_s \tag{2.16}$$

in total agreement with (2.7). To eliminate r from (2.16) we use that

$$r = \frac{-A_y + qC}{B} \tag{2.17}$$

as follows from the first equation from (2.9). Replacing C by $1/(B16\beta^2) - A^2/B$ as follows from the determinant relation (2.6) and recalling that $B = -q_s/2$ according to equation (2.8) we obtain after substituting r from (2.17) into (2.16):

$$\left(\frac{q_{sy}}{q}\right)_y = \left(\frac{q_{syy}}{q_s} - \frac{q_{sy}q_y}{qq_s} + \frac{1}{2\beta^2}\frac{q^2}{q_s^2} - \frac{q_{sy}^2}{2q_s^2}\right)_s. \tag{2.18}$$

Note that alternatively we could have eliminated q from equation

$$\left(\frac{r_{sy}}{4r}\right)_y = \frac{1}{2}(rq)_s$$

and obtained an equation for r only. It turns out that the equation for r follows from equation (2.18) by simply substituting r for q.

For brevity we introduce, as in [20], $f = \ln q$. Then expression (2.18) becomes:

$$(f_s f_y)_y = -\left(\frac{f_y^2}{2} + \frac{f_{sy}^2}{2f_s^2} - \frac{1}{2\beta^2 f_s^2} - \frac{f_{syy}}{f_s}\right)_s. \tag{2.19}$$

The above relation can be cast in an equivalent form:

$$\frac{f_{ss}}{2\beta^2 f_s^3} + f_{sy}f_y + \frac{1}{2}f_s f_{yy} - \frac{f_{ssyy}}{2f_s} + \frac{f_{ssy}f_{sy}}{2f_s^2} + \frac{f_{ss}f_{syy}}{2f_s^2} - \frac{f_{ss}f_{sy}^2}{2f_s^3} = 0, \tag{2.20}$$

which first appeared in [11]. The relation (2.20) is also equivalent to the following condition

$$\left(\frac{1}{f_s}\right)_s = \beta^2\left(f_s^2 f_y - f_{ssy} + \frac{f_{ss}f_{sy}}{f_s}\right)_y. \tag{2.21}$$

For a quantity u defined as:

$$u = \beta^2\left(f_s^2 f_y - f_{ssy} + \frac{f_{ss}f_{sy}}{f_s}\right) - \frac{1}{2}\kappa, \tag{2.22}$$

with κ being an integration constant, it holds from relation (2.21) that

$$u_y = \left(\frac{1}{f_s}\right)_s. \tag{2.23}$$

Next, as in [21], we define a quantity m as $\beta^2 f_s^2 f_y$ and derive from relations (2.22) and (2.23) that

$$m = \beta^2 f_s^2 f_y = u + \beta^2 \left(f_{ssy} - \frac{f_{ss} f_{sy}}{f_s} \right) + \frac{1}{2}\kappa = u - \beta^2 f_s \left(f_s \left(\frac{1}{f_s} \right)_s \right)_y + \frac{1}{2}\kappa$$
$$= u - \beta^2 f_s (f_s u_y)_y + \frac{1}{2}\kappa. \tag{2.24}$$

Taking a derivative of m with respect to s yields

$$m_s = \beta^2 \left(2 f_y f_s f_{ss} + f_s^2 f_{sy} \right) = 2m \frac{f_{ss}}{f_s} + \beta^2 f_s^2 f_{sy} = -2m f_s \left(\frac{1}{f_s} \right)_s + \beta^2 f_s^2 f_{sy}$$
$$= -2m f_s u_y + \beta^2 f_s^2 f_{sy}. \tag{2.25}$$

In terms of quantities u and $\rho = f_s$ equations (2.23) and (2.25) take the following form

$$\rho_s = -\rho^2 u_y, \tag{2.26}$$
$$m_s = -2m\rho u_y + \beta^2 \rho^2 \rho_y, \tag{2.27}$$

for m given by

$$m = u - \beta^2 \rho (\rho u_y)_y + \frac{1}{2}\kappa. \tag{2.28}$$

An inverse reciprocal transformation $(y, s) \mapsto (x, t)$ is defined by relations:

$$F_x = \rho F_y, \qquad F_t = F_s - \rho u F_y \tag{2.29}$$

for an arbitrary function F. Equations (2.26), (2.27) and (2.28) take a form

$$\rho_t = -\left(u\rho \right)_x, \tag{2.30}$$
$$m_t = -2m u_x - m_x u + \beta^2 \rho \rho_x, \tag{2.31}$$
$$m = u - \beta^2 u_{xx} + \frac{1}{2}\kappa \tag{2.32}$$

in terms of the (x, t) variables. Equation (2.30) is called the compatibility condition, while equation (2.31) is the two-component Camassa–Holm equation [11], which agrees with standard Camassa–Holm equation (1.1) for $\rho = 0$.

3 General reduction scheme from AKNS system to the two-component Camassa–Holm equation

Next, we perform the transformation

$$\Psi \;\rightarrow\; \mathcal{U}(\theta, f)\Psi = \begin{bmatrix} \varphi \\ \eta \end{bmatrix} \tag{3.1}$$

on AKNS two-component Ψ function from (2.2). $\mathcal{U}(\theta, f)$ stands for an orthogonal matrix:

$$\mathcal{U}(\theta, f) = \Omega(\theta) \exp\left(-\frac{1}{2} f \sigma_3 \right), \qquad 0 \le \theta \le \frac{\pi}{2}, \tag{3.2}$$

where $\Omega(\theta)$ is given by

$$\Omega(\theta) = \sigma_3 e^{\mathrm{i}\,\theta\sigma_2} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{3.3}$$

and f is a function of y and s, which is going to be determined below for each value of θ.

Note that $\Omega^{-1}(\theta) = \Omega(\theta)$ and $\Omega(0) = \sigma_3$, $\Omega(\pi/2) = \sigma_1$.

Taking a derivative with respect to y and s on both sides of (3.1) one gets

$$\begin{bmatrix} \varphi \\ \eta \end{bmatrix}_y = \left(\mathcal{U}_y \mathcal{U}^{-1} + \mathcal{U} \begin{bmatrix} \lambda & q \\ r & -\lambda \end{bmatrix} \mathcal{U}^{-1} \right) \begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \tag{3.4}$$

$$\begin{bmatrix} \varphi \\ \eta \end{bmatrix}_s = \left(\mathcal{U}_s \mathcal{U}^{-1} + \mathcal{U} D^{(-1)} \mathcal{U}^{-1} \right) \begin{bmatrix} \varphi \\ \eta \end{bmatrix}. \tag{3.5}$$

Thus, the flows of the new two-component function defined in (3.2) are governed by the gauge transformations of the AKNS matrices $\lambda\sigma_3 + \mathcal{A}_0$ and $D^{(-1)}$, respectively. This ensures that the original AKNS compatibility condition (2.4) still holds for the rotated system defined by equations (3.4) and (3.5).

From equation (3.4) we derive that:

$$\begin{aligned} \lambda\left(\varphi\cos(2\theta) + \eta\sin(2\theta)\right) = {} & \varphi_y + \frac{1}{2}\varphi\left(f_y\cos(2\theta) - \sin(2\theta)\left(qe^{-f} + re^{f}\right)\right) \\ & + \eta\left(\frac{1}{2}f_y\sin(2\theta) - re^{f}\sin^2(\theta) + qe^{-f}\cos^2(\theta)\right). \end{aligned} \tag{3.6}$$

Repeating derivation with respect to y one more time yields

$$\begin{aligned} \begin{bmatrix} \varphi \\ \eta \end{bmatrix}_{yy} &= \left[\left(\mathcal{U}_y \mathcal{U}^{-1} + \mathcal{U} \begin{bmatrix} \lambda & q \\ r & -\lambda \end{bmatrix} \mathcal{U}^{-1} \right)_y + \left(\mathcal{U}_y \mathcal{U}^{-1} + \mathcal{U} \begin{bmatrix} \lambda & q \\ r & -\lambda \end{bmatrix} \mathcal{U}^{-1} \right)^2 \right] \begin{bmatrix} \varphi \\ \eta \end{bmatrix} \\ &= \mathcal{U} \begin{bmatrix} \lambda^2 - \lambda f_y + f_y^2/4 - f_{yy}/2 + qr & q_y - f_y q \\ r_y + f_y r & \lambda^2 - \lambda f_y + f_y^2/4 + f_{yy}/2 + qr \end{bmatrix} \mathcal{U}^{-1} \begin{bmatrix} \varphi \\ \eta \end{bmatrix}. \end{aligned} \tag{3.7}$$

For

$$\begin{bmatrix} \bar\varphi \\ \bar\eta \end{bmatrix} = \Omega(\theta) \begin{bmatrix} \varphi \\ \eta \end{bmatrix}$$

the result is

$$\begin{bmatrix} \bar\varphi \\ \bar\eta \end{bmatrix}_{yy} = \begin{bmatrix} \lambda^2 - \lambda f_y + f_y^2/4 - f_{yy}/2 + qr & (q_y - f_y q)e^{-f} \\ (r_y + f_y r)e^{f} & \lambda^2 - \lambda f_y + f_y^2/4 + f_{yy}/2 + qr \end{bmatrix} \begin{bmatrix} \bar\varphi \\ \bar\eta \end{bmatrix}$$

and shows in a transparent way that the condition for eliminating $\bar\eta$ from the equation for $\bar\varphi_{yy}$ requires $(q_y - f_y q)\exp(-f) = 0$ or $q = \exp(f)$. Similarly, the condition for eliminating $\bar\varphi$ from the equation for $\bar\eta_{yy}$ requires $(r_y + f_y r)\exp(f) = 0$ or $r = \exp(-f)$. Clearly these reductions reproduce results of the previous section.

To obtain a more general result we return to equation (3.7). Projecting on the φ-component in equation (3.7) gives

$$\begin{aligned} \varphi_{yy} = {} & \lambda^2\varphi - \lambda f_y \varphi + \left(\frac{1}{4}f_y^2 + qr\right)\varphi + \left(-\frac{1}{2}f_y\cos(2\theta) + \frac{1}{2}(qe^{-f} + re^{f})\sin(2\theta)\right)_y \varphi \\ & + \left(-\frac{1}{2}f_y\sin(2\theta) - qe^{-f}\cos^2\theta + re^{f}\sin^2\theta\right)_y \eta. \end{aligned} \tag{3.8}$$

Next, we will eliminate η in order to obtain an equation for the one-component variable φ. This is analogous to the calculation made below equation (2.11), where the first order two-component AKNS spectral problem was reduced to second order equation for the one-component function ψ. To accomplish the task we must choose f so that the identity

$$\frac{1}{2} f_y \sin(2\theta) = re^f \sin^2\theta - qe^{-f}\cos^2\theta + c_0 \tag{3.9}$$

holds, where c_0 is an integration constant. The identity (3.9) ensures that terms with η drop out of equation (3.8).

Note, that for $\theta = \pi/4$ and $c_0 = 0$ we recover identity $f_y = r\exp(f) - q\exp(-f)$ from [11, 19]. For $\theta = 0$, $c_0 = 1$ and $\theta = \pi/2$, $c_0 = -1$ we get, respectively, $q = \exp(f)$ and $r = \exp(-f)$ as in [20]. From now on we take $c_0 = 0$ as long as $0 < \theta < \pi/2$.

Let us shift a function f by a constant term, $\ln(\tan\theta)$:

$$f \longrightarrow f_\theta = f + \ln(\tan\theta)\,. \tag{3.10}$$

Then relation (3.9) can be rewritten for $0 < \theta < \pi/2$ as

$$f_{\theta y} = re^{f_\theta} - qe^{-f_\theta} \tag{3.11}$$

which is of the same form as the relation found in reference [11]. It therefore appears that for all values of θ in the the $0 < \theta < \pi/2$ relation between function f and AKNS variables q and r remains invariant up to shift of f by a constant.

Now, we turn our attention back to equation (3.5) rewritten as

$$\begin{bmatrix}\varphi\\ \eta\end{bmatrix}_s = \mathcal{U}\left(-\frac{1}{2}f_s\sigma_3 + \frac{1}{\lambda}\begin{bmatrix}A & B\\ C & -A\end{bmatrix}\right)\mathcal{U}^{-1}\begin{bmatrix}\varphi\\ \eta\end{bmatrix}.$$

For the φ component we find:

$$\begin{aligned}\varphi_s = &-\frac{1}{2}f_s\left(\varphi\cos(2\theta) + \eta\sin(2\theta)\right) + \frac{1}{2\lambda}\varphi\left(2A\cos(2\theta) + Ce^f\sin(2\theta) + Be^{-f}\sin(2\theta)\right)\\ &+ \frac{1}{\lambda}\eta\left(A\sin(2\theta) + Ce^f\sin^2\theta - Be^{-f}\cos^2\theta\right).\end{aligned} \tag{3.12}$$

For $0 < \theta < \pi/2$ we choose

$$B = \left(A - \frac{1}{4\beta}\right)e^{f_\theta}, \qquad C = -\left(A + \frac{1}{4\beta}\right)e^{-f_\theta}, \tag{3.13}$$

which agrees with the determinant formula $A^2 + BC = 1/16\beta^2$ and implies identities:

$$2A - Be^{-f_\theta} + Ce^{f_\theta} = 0, \tag{3.14}$$

$$Be^{-f_\theta} + Ce^{f_\theta} = -\frac{1}{2\beta}. \tag{3.15}$$

The first of these identities, (3.14), ensures that the last three terms containing η on the right hand side of equation (3.12) cancel.

Recall at this point relation (3.6). Simplifying this relation by invoking identity (3.9) and plugging it into equation (3.12) gives

$$\varphi_s = -\frac{f_s}{2\lambda}\varphi_y + \frac{1}{\lambda}\varphi\left(-\frac{1}{4}f_sf_y\cos(2\theta) + \frac{f_s}{4}\sin(2\theta)\left(qe^{-f} + re^f\right)\right.$$

$$+ A\cos(2\theta) + \frac{1}{2}Be^{-f}\sin(2\theta) + \frac{1}{2}Ce^{f}\sin(2\theta)\Big). \tag{3.16}$$

From (3.13) we find

$$re^{f_\theta} = \frac{C_y}{2A}e^{f_\theta} = \frac{1}{2A}\left(f_y\left(A + \frac{1}{4\beta}\right) - A_y\right), \tag{3.17}$$

$$qe^{-f_\theta} = -\frac{B_y}{2A}e^{-f_\theta} = \frac{-1}{2A}\left(f_y\left(A - \frac{1}{4\beta}\right) + A_y\right) \tag{3.18}$$

and therefore

$$qe^{-f_\theta} + re^{f_\theta} = \frac{f_y}{4A\beta} - \frac{A_y}{A}. \tag{3.19}$$

Due to the above relation and identity (3.15) equation (3.16) becomes

$$\varphi_s = -\frac{f_s}{2\lambda}\varphi_y + \frac{1}{\lambda}\varphi\left(-\frac{1}{4\beta}\left(1 - \frac{f_s f_y}{4A}\right) - \frac{A_y}{A}\frac{f_s}{4}\right). \tag{3.20}$$

Taking derivative of (3.9) with respect to s we find

$$\frac{1}{2}f_{sy} = Ce^{f_\theta} + Be^{-f_\theta} + \frac{1}{2}f_s\left(qe^{-f_\theta} + re^{f_\theta}\right) = -\frac{1}{2\beta} + \frac{f_s f_y}{8A\beta} - \frac{f_s A_y}{2A}. \tag{3.21}$$

Thus equation (3.20) becomes

$$\varphi_s = -\frac{f_s}{2\lambda}\varphi_y + \frac{f_{sy}}{4\lambda}\varphi. \tag{3.22}$$

We now turn our attention to equation (3.8). The last term containing η vanishes due to the identity (3.9). In addition it holds that

$$\frac{f_{sy}}{2f_s} + \frac{1}{2\beta f_s} = -\frac{1}{2}f_y\cos(2\theta) + \frac{1}{2}\left(qe^{-f} + re^{f}\right)\sin(2\theta) = \frac{1}{2}\left(qe^{-f_\theta} + re^{f_\theta}\right) \tag{3.23}$$

as follows from relations (3.19) and (3.21). Also, it holds from relations (3.17)–(3.18) that for $0 < \theta < \pi/2$:

$$rq = \left(\frac{f_{sy}}{2f_s} + \frac{1}{2\beta f_s}\right)^2 - \frac{1}{4}f_y^2 = g^2 - f_y^2/4, \tag{3.24}$$

where

$$g = \frac{f_{sy}}{2f_s} + \frac{1}{2\beta f_s}. \tag{3.25}$$

Thus, the remaining constant (the ones which do not contain λ) terms on the right hand side of equation (3.8) are equal to

$$\begin{aligned}&\frac{1}{4}f_y^2 + qr + \left(-\frac{1}{2}f_y\cos(2\theta) + \frac{1}{2}(qe^{-f} + re^{f})\sin(2\theta)\right)_y\\ &\qquad = \frac{1}{4}f_y^2 + qr + \frac{1}{2}\left(qe^{-f_\theta} + re^{f_\theta}\right)_y = g^2 + g_y.\end{aligned} \tag{3.26}$$

Therefore, we can write equation (3.8) as:

$$\varphi_{yy} = \left(\lambda^2 - \lambda f_y - Q\right)\varphi, \qquad Q = -g^2 - g_y \tag{3.27}$$

with g given by (3.25). The above spectral problem together with equation (3.22) ensures via compatibility condition $\varphi_{yys} - \varphi_{syy} = 0$, that

$$Q_s + \frac{1}{2}f_{yy}f_s + f_y f_{sy} = 0 \tag{3.28}$$

holds. The latter is equivalent to the two-component Camassa–Holm equation (2.19).

4 The $\theta = 0$ case and Bäcklund transformation between different solutions

We now consider θ at the boundary of the $0 < \theta < \pi/2$ interval. For illustration we take $\theta = 0$, the remaining case $\theta = \pi/2$ can be analyzed in a similar way. Plugging $\theta = 0$ into relation (3.26) we obtain

$$rq|_{\theta=0} = -\frac{1}{4}f_y^2 + \frac{1}{2}f_{yy} + g^2 + g_y = g^2 - \frac{1}{4}f_y^2 + \left(\frac{1}{2}f_y + g\right)_y .$$

Comparing with relation (3.24) we get

$$rq|_{\theta=0} = rq|_\theta + \left(\frac{1}{2}f_y + g\right)_y \tag{4.1}$$

which describes a relation between the product rq for zero and non-zero values of the angle θ, with $rq|_\theta$ being associated with θ within an interval $0 < \theta < \pi/2$.

Recall that $q = \exp(f)$ for $\theta = 0$. It follows that $A = q_{sy}/4q = (f_{sy} + f_s f_y)/4$ and equation (2.7) is equivalent to

$$(rq|_{\theta=0})_s = \frac{1}{2}(f_{sy} + f_s f_y)_y. \tag{4.2}$$

On the other hand, it follows from (2.17) and $C = 1/(16\beta^2 B) - A^2/B$ that

$$rq|_{\theta=0} = \frac{1}{2}\left(f_{yy} - \frac{1}{2}f_y^2 - \frac{f_{sy}^2}{2f_s^2} + \frac{1}{2\beta^2 f_s^2} + \frac{f_{syy}}{f_s}\right)$$

and accordingly equation (4.2) is equivalent to the two-component Camassa–Holm equation (2.19).

From (3.18) one finds for $0 < \theta < \pi/2$ that:

$$q = \mathcal{P}_-(f_\theta)e^{f_\theta}, \tag{4.3}$$

where

$$\mathcal{P}_\pm(f) = \pm\frac{1}{2}f_y + g = \pm\frac{f_y}{2} + \frac{f_{sy}}{2f_s} + \frac{1}{2\beta f_s}.$$

Obviously $\mathcal{P}_\pm(f_\theta) = \mathcal{P}_\pm(f)$.

We are now ready to show that

$$\bar{f} = f_\theta + \ln\left(\mathcal{P}_-(f_\theta)\right) = f_\theta + \ln\left(-\frac{f_{\theta\, y}}{2} + \frac{f_{\theta\, sy}}{2f_{\theta\, s}} + \frac{1}{2\beta f_{\theta\, s}}\right)$$

satisfies the two-component Camassa–Holm equation (2.19) for any f or f_θ, which satisfies equation (2.19). For $0 < \theta < \pi/2$, it holds that $q = \exp(\bar{f})$ and therefore

$$A = q_{sy}/4q = (\bar{f}_{sy} + \bar{f}_s\bar{f}_y)/4 = (f_{sy} + f_s f_y)/4 + \frac{f_s\mathcal{P}_{-\,y} + \mathcal{P}_{-\,ys} + \mathcal{P}_{-\,s}f_y}{4\mathcal{P}_-}. \tag{4.4}$$

We will now show that

$$(rq|_\theta)_s = (rq|_{\theta=0})_s - \left(\frac{1}{2}f_y + g\right)_{ys} = \frac{1}{2}(f_{sy} + f_s f_y)_y + \left(\frac{f_s\mathcal{P}_{-y} + \mathcal{P}_{-ys} + \mathcal{P}_{-s}f_y}{2\mathcal{P}_-}\right)_y. \tag{4.5}$$

Using equation (4.2) one can easily show that equation (4.5) holds if the following relation

$$-\left(f_y+\mathcal{P}_-\right)_s=\frac{f_s\mathcal{P}_{-\,y}+\mathcal{P}_{-\,ys}+\mathcal{P}_{-\,s}f_y}{2\mathcal{P}_-}$$

is true. We note that the above relation can be rewritten as

$$(\mathcal{P}_-^2)_s+2f_{ys}\mathcal{P}_-+f_s\mathcal{P}_{-\,y}+\mathcal{P}_{-\,sy}+\mathcal{P}_{-\,s}f_y=0\,.$$

The last equation is fully equivalent to the two-component Camassa–Holm equation (3.28) as can be seen by rewriting Q from relation (3.27) as $Q=-(\mathcal{P}_-+f_y/2)^2-(\mathcal{P}_-+f_y/2)_y$. This completes the proof for relation (4.5).

It follows from (2.17) and $C=1/(16\beta^2B)-A^2/B$ that

$$rq|_\theta=\frac{1}{2}\left(\bar f_{yy}-\frac{1}{2}\bar f_y^2-\frac{\bar f_{sy}^2}{2\bar f_s^2}+\frac{1}{2\beta^2\bar f_s^2}+\frac{\bar f_{syy}}{\bar f_s}\right).$$

Thus, due to (4.4) and (4.5) we have proved explicitly that

$$\bar f=f+\ln\left(\tan\theta\left(-\frac{f_y}{2}+\frac{f_{sy}}{2f_s}+\frac{1}{2\beta f_s}\right)\right)=f_\theta+\ln\mathcal{P}_-(f_\theta) \tag{4.6}$$

is a solution of a 2-component version of the Camassa–Holm equation. Thus the transformation

$$f\to\bar f$$

maps a solution f of a 2-component version of the Camassa–Holm equation to a different solution $\bar f$. For example, let us consider, as in [21], the Camassa–Holm function:

$$f(y,s)=\ln\frac{a_1^{(1)}a_2^{(1)}z_1e^{\frac{s}{2z_1}+2yz_1}+a_1^{(2)}a_2^{(2)}z_2e^{\frac{s}{2z_2}+2yz_2}}{(z_2-z_1)a_1^{(2)}a_2^{(1)}}, \tag{4.7}$$

where $a_i^{(j)}$, $i,j=1,2$ and z_1 and z_2 are constants. The function f solves equation (2.19) for $\beta^2=1$. Then, as an explicit calculation verifies, the map $f\to\bar f$ with $\bar f$ given by expression (4.6) yields another solution of equation (2.19) for $\beta^2=1$ and $\theta\neq0$.

For $\theta=\pi/2$ we have $r=\exp(-f)$ and comparing with the result for $0<\theta<\pi/2$:

$$r=\mathcal{P}_+(f_\theta)e^{-f_\theta}, \tag{4.8}$$

we get a Bäcklund transformation

$$f\to f_\theta-\ln\left(\mathcal{P}_+(f_\theta)\right)=f_\theta-\ln\left(\frac{f_{\theta y}}{2}+\frac{f_{\theta sy}}{2f_{\theta s}}+\frac{1}{2\beta f_{\theta s}}\right).$$

Additional Bäcklund transformations can be obtained by comparing expressions for q and r variables in terms of f for the boundary values of θ.

We first turn our attention to the case of $\theta=0$ for which we have $q=\exp(f)$ and

$$r=\frac{1}{2}\left(f_{yy}-\frac{1}{2}f_y^2-\frac{f_{sy}^2}{2f_s^2}+\frac{1}{2\beta^2f_s^2}+\frac{f_{syy}}{f_s}\right)e^{-f}=\left(\mathcal{P}_+^2-\mathcal{P}_+f_y+\mathcal{P}_{+y}\right)e^{-f}. \tag{4.9}$$

From the AKNS equation (2.18) we see immediately that $f=\ln q$ must satisfy the 2-component Camassa–Holm equation (2.19). Note, in addition, that the AKNS equation (2.18) is still valid if we replace q by r and therefore

$$f-\ln\left(\mathcal{P}_+^2-\mathcal{P}_+f_y+\mathcal{P}_{+y}\right)$$

must satisfy the 2-component Camassa–Holm equation (2.19) as well.

Next, for $\theta = \pi/2$ we have $r = \exp(-f)$ and

$$q = \frac{1}{2}\left(-f_{yy} - \frac{1}{2}f_y^2 - \frac{f_{sy}^2}{2f_s^2} + \frac{1}{2\beta^2 f_s^2} + \frac{f_{syy}}{f_s}\right)e^f = \left(\mathcal{P}_-^2 + \mathcal{P}_- f_y + \mathcal{P}_{-y}\right)e^f. \tag{4.10}$$

Comparing expressions for q and r we find find that if f is a solution of the 2-component Camassa–Holm equation (2.19) then so is also

$$f + \ln\left(\mathcal{P}_-^2 + \mathcal{P}_- f_y + \mathcal{P}_{-y}\right).$$

To summarize we found the following Bäcklund maps

$$f \to \begin{cases} f_\theta \pm \ln\left(\mathcal{P}_\mp(f_\theta)\right), & f_\theta = f + \mathrm{const}, \\ f \pm \ln\left(\mathcal{P}_\mp^2 \pm \mathcal{P}_\mp f_y + \mathcal{P}_{\mp y}\right). \end{cases}$$

The top row lists maps between $\theta = 0, \pi/2$ cases and θ within the interval $0 < \theta < \pi/2$ [20]. The bottom row shows new maps derived for the $\theta = 0$ and $\pi/2$ cases only.

5 Conclusions

These notes describe an attempt to construct a general and universal formalism which would realize possible connections between the 2-component Camassa–Holm equation and AKNS hierarchy extended by a negative flow.

Construction yields gauge copies of an extended AKNS model connected by a continuous parameter (angle) θ taking values in an interval $0 \le \theta \le \pi/2$. Eliminating one of two components of the $sl(2)$ wave function gives a second order non-linear partial differential equation for a single function f of the two-component Camassa–Holm model. Functions f corresponding to different values of θ in an interior of interval $0 \le \theta \le \pi/2$ differ only by a trivial constant and fall into a class considered in [11]. Two remaining and separate cases correspond to θ equal to 0 and $\pi/2$ and agree with a structure described in [20].

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