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Dynamical R Matrices of Elliptic Quantum Groups and Connection Matrices for the q-KZ Equations*

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Abstract. For any affine Lie algebra $\mathfrak{g}$, we show that any finite dimensional representation of the universal dynamical R matrix $\mathcal{R}(\lambda)$ of the elliptic quantum group $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ coincides with a corresponding connection matrix for the solutions of the q-KZ equation associated with $U_q(\mathfrak{g})$. This provides a general connection between $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ and the elliptic face (IRF or SOS) models. In particular, we construct vector representations of $\mathcal{R}(\lambda)$ for $\mathfrak{g} = A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and show that they coincide with the face weights derived by Jimbo, Miwa and Okado. We hence confirm the conjecture by Frenkel and Reshetikhin.

Key words: elliptic quantum group; quasi-Hopf algebra

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1 Introduction

The quantum group $U_q(\mathfrak{g})$ is one of the fundamental structures appearing in the wide class of trigonometric quantum integrable systems. Among others, we remark the following two facts.

1) For $\mathfrak{g}$ being affine Lie algebra, finite dimensional representations of $U_q(\mathfrak{g})$ allow a systematic derivation of trigonometric solutions of the Yang–Baxter equation (YBE) [1, 2].
2) A combined use of finite and infinite dimensional representations allows us to formulate trigonometric vertex models and calculate correlation functions [3].

To extend this success to elliptic systems is our basic aim. In this paper, we consider a problem analogous to 1). As for developments in the direction 2), we refer the reader to the papers [4, 5, 6, 7, 8]. We are especially interested in the two dimensional exactly solvable lattice models. There are two types of elliptic solvable lattice models. The vertex type and the face (IRF or SOS) type. The vertex type elliptic solutions to the YBE were found by Baxter [9] and Belavin [10]. These are classified as the elliptic R matrices of the type $A_n^{(1)}$. The face type elliptic Boltzmann weights associated with $A_1^{(1)}$ were first constructed by Andrews–Baxter–Forrester [11], and extended to $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ by Jimbo–Miwa–Okado [12, 13], to $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$ by Kuniba [14], and to $G_2^{(1)}$ by Kuniba–Suzuki [15].

Concerning the elliptic face weights, Frenkel and Reshetikhin made an interesting observation [16] that the connection matrices for the solution of the q-KZ equation associated with $U_q(\mathfrak{g})$ ($\mathfrak{g}$: affine Lie algebra) provide elliptic solutions to the face type YBE. They also conjectured

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that the connection matrices in the vector representation are equal to Jimbo–Miwa–Okado's face weights for $\mathfrak{g}=A_n^{(1)},B_n^{(1)},C_n^{(1)},D_n^{(1)}$. In order to confirm this conjecture, one needs to solve the q-KZ equation of general level and find connection matrices. Within our knowledge, no one has yet confirmed it. Instead of doing this, Date, Jimbo and Okado [17] considered the face models defined by taking the connection matrices as Boltzmann weights. They showed that the one-point function of such models is given by the branching function associated with $\mathfrak{g}$. The same property of the one-point function had been discovered in Jimbo–Miwa–Okado's $A_n^{(1)}$, $B_n^{(1)}$, $D_n^{(1)}$ face models.

An attempt to formulate elliptic algebras was first made by Sklyanin [18]. He considered an algebra defined by the RLL-relation associated with Baxter's elliptic R-matrix. It was extended to the elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ by Foda et al. [19], based on a central extension of Sklyanin's RLL-relation. In the same year, Felder proposed a face type elliptic algebra $E_{\tau,\eta}(\mathfrak{g})$ associated with the dynamical RLL-relation [20]. Jimbo–Miwa–Okado's elliptic solutions to the face type YBE were interpreted there as the dynamical R matrices. We classify the former elliptic algebra the vertex-type and the latter the face-type. Another formulation of the face type elliptic algebra was discovered by the author [4]. It is based on an elliptic deformation of the Drinfel'd currents.

A coalgebra structure of these elliptic algebras was clarified in the works by Frønsdal [21], Enriquez–Felder [22] and Jimbo–Konno–Odake–Shiraishi [23]. It is based on an idea of quasi-Hopf deformation [24] by using the twistor operators satisfying the shifted cocycle condition [25]. In this formulation, we regard the coalgebra structures of the vertex and the face type elliptic algebras as two different quasi-Hopf deformation of the corresponding affine quantum group $U_q(\mathfrak{g})$. We call the resultant quasi-Hopf algebras the elliptic quantum groups of the vertex type $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ and the face type $\mathcal{B}_{q,\lambda}(\mathfrak{g})$. A detailed description for the face type case is reviewed in Section 2. For the face type, a different coalgebra structure as a $\mathfrak{h}$-Hopf algebroid was developed by Felder, Etingof and Varchenko [26, 27, 28], Koelink–van Norden–Rosengren [29].

One of the advantages of the quasi-Hopf formulation is that it allows a natural derivation of the universal dynamical R matrix from one of $U_q(\mathfrak{g})$ as a twist. However, a disadvantage is that there are no a priori reasons for the resultant universal R matrix to yield elliptic R matrices. One needs to check this point in all representations. We have done this for the vector representations of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ and $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$, which led to Baxter's elliptic R matrix and Andrews–Baxter–Forrester's elliptic face weights, respectively [23]. The same checks for the face weights were also done in the cases $\mathfrak{g}=A_n^{(1)},A_2^{(2)}$ [6, 7].

The aim of this paper is to overcome this disadvantage by clarifying the following point concerning the face type.

i) Any representations of the universal dynamical R matrix of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ are equivalent to the corresponding connection matrices for the q-KZ equation of $U_q(\mathfrak{g})$.

The connection matrices are known to be elliptic. See Theorem 3.5. In addition, we show

ii) For $\mathfrak{g}=A_n^{(1)},B_n^{(1)},C_n^{(1)},D_n^{(1)}$, the vector representation of the the universal dynamical R matrix of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ is equivalent to Jimbo–Miwa–Okado's elliptic face weight up to a gauge transformation.

Combining i) and ii), we confirm Frenkel–Reshetikhin's conjecture on the equivalence between the connection matrices and Jimbo–Miwa–Okado's face weights. For the purpose of showing i), we follow the idea by Etingof and Varchenko [28], and give an exact relation between the face type twistors and the highest to highest expectation values of the composed vertex operators (fusion matrices) of $U_q(\mathfrak{g})$. To show ii), we solve the difference equation for the face type twistor, which is equivalent to the q-KZ equation of general level.

This paper is organized as follows. In the next section, we summarize some basic facts on the affine quantum groups $U_q(\mathfrak{g})$ and the face type elliptic quantum groups $\mathcal{B}_{q,\lambda}(\mathfrak{g})$. In Section 3, we introduce the vertex operators of $U_q(\mathfrak{g})$ and fusion matrices. We discuss equivalence between the face type twistors and the fusion matrices. Then in Section 4, we show equivalence between the dynamical R matrices of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ and the connection matrices for the q-KZ equation of $U_q(\mathfrak{g})$ in general finite dimensional representation. Section 5 is devoted to a discussion on an equivalence between the vector representations of the the universal dynamical R matrix of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ and Jimbo–Miwa–Okado's elliptic face weights for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$.

2 Affine quantum groups $U_q(\mathfrak{g})$ and elliptic quantum groups $\mathcal{B}_{q,\lambda}(\mathfrak{g})$

2.1 Affine quantum groups $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be an affine Lie algebra associated with a generalized Cartan matrix $A = (a_{ij})$, $i, j \in I = \{0, 1, \ldots, n\}$. We fix an invariant inner product $(\cdot|\cdot)$ on the Cartan subalgebra $\mathfrak{h}$ and identify $\mathfrak{h}^*$ with $\mathfrak{h}$ through $(\cdot|\cdot)$. We follow the notations and conventions in [30] except for $A_{2n}^{(2)}$, which we define in such a way that the order of the vertices of the Dynkin diagram is reversed from the one in [30]. We hence have $a_0 = 1$ for all $\mathfrak{g}$. Let $\{\alpha_i\}_{i\in I}$ be a set of simple roots and set $h_i = \alpha_i^\vee$. We have $a_{ij} = \langle \alpha_j, h_i\rangle = \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}$ and $d_i a_{ij} = a_{ij} d_j$ with $d_i = \frac{1}{2}(\alpha_i|\alpha_i)$. We denote the canonical central element by $c = \sum_{i\in I} a_i^\vee h_i$ and the null root by $\delta = \sum_{i\in I} a_i \alpha_i$. We set

$$
\begin{aligned}
&Q = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_n, \qquad Q_+ = \mathbb{Z}_{\geq 0}\alpha_0 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_n,\\
&P = \mathbb{Z}\Lambda_0 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\delta, \qquad P^* = \mathbb{Z}h_0 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d
\end{aligned}
$$

and impose the pairings

$$
\langle \alpha_i, d\rangle = a_0^\vee \delta_{i,0}, \qquad \langle \Lambda_j, h_i\rangle = \frac{1}{a_0^\vee}\delta_{ij}, \qquad \langle \Lambda_j, d\rangle = 0, \qquad (i, j \in I).
$$

The Λ_j are the fundamental weights. We also use $P_{cl} = P/\mathbb{Z}\delta$, $(P_{cl})^* = \oplus_{i=0}^n \mathbb{Z}h_i \subset P^*$. Let $cl : P \to P_{cl}$ denote the canonical map and define $af : P_{cl} \to P$ by $af(cl(\alpha_i)) = \alpha_i$ $(i \neq 0)$ and $af(cl(\Lambda_0)) = \Lambda_0$ so that $cl \circ af = \mathrm{id}$ and $af(cl(\alpha_0)) = \alpha_0 - \delta$.

Definition 2.1. *The quantum affine algebra $U_q = U_q(\mathfrak{g})$ is an associative algebra over $\mathbb{C}(q^{1/2})$ with 1 generated by the elements e_i, f_i $(i \in I)$ and q^h $(h \in P^*)$ satisfying the following relations*

$$
\begin{aligned}
&q^0 = 1,\ q^h q^{h'} = q^{h+h'} \qquad (h, h' \in P^*),\\
&q^h e_i q^{-h} = q^{\langle \alpha_i, h\rangle} e_i, \qquad q^h f_i q^{-h} = q^{-\langle \alpha_i, h\rangle} f_i,\\
&e_i f_j - f_j e_i = \delta_{ij}\frac{t_i - t_i^{-1}}{q_i - q_i^{-1}},\\
&\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} e_i^{1-a_{ij}-m} e_j e_i^m = 0 \qquad (i \neq j),\\
&\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} f_i^{1-a_{ij}-m} f_j f_i^m = 0 \qquad (i \neq j).
\end{aligned}
$$

Here $q_i = q^{d_i}$, $t_i = q_i^{h_i}$, and

$$
[n]_x = \frac{x^n - x^{-n}}{x - x^{-1}}, \qquad [n]_x! = [n]_x [n-1]_x \cdots [1]_x, \qquad \begin{bmatrix} n \\ m \end{bmatrix}_x = \frac{[n]_x!}{[m]_x! [n-m]_x!}.
$$

The algebra U_q has a Hopf algebra structure with comultiplication Δ, counit ϵ and antipode S defined by

$$\begin{aligned}
&\Delta(q^h) = q^h \otimes q^h,\\
&\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i,\\
&\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i,\\
&\epsilon(q^h) = 1, \qquad \epsilon(e_i) = \epsilon(f_i) = 0,\\
&S(q^h) = q^{-h}, \qquad S(e_i) = -t_i^{-1}e_i, \qquad S(f_i) = -f_i t_i.
\end{aligned} \tag{2.1}$$

U_q is a quasi-triangular Hopf algebra with the universal R matrix $\mathcal{R}$ satisfying

$$\begin{aligned}
&\Delta^{op}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \qquad \forall\, x \in U_q,\\
&(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}, \qquad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}.
\end{aligned}$$

Here Δ^{op} denotes the opposite comultiplication, $\Delta^{op} = \sigma \circ \Delta$ with σ being the flip of the tensor components; $\sigma(a \otimes b) = b \otimes a$.

Proposition 2.2.

$$\begin{aligned}
&\mathcal{R}^{(12)}\mathcal{R}^{(13)}\mathcal{R}^{(23)} = \mathcal{R}^{(23)}\mathcal{R}^{(13)}\mathcal{R}^{(12)}, \qquad (2.2)\\
&(\epsilon \otimes \mathrm{id})\mathcal{R} = (\mathrm{id} \otimes \epsilon)\mathcal{R} = 1,\\
&(S \otimes \mathrm{id})\mathcal{R} = (\mathrm{id} \otimes S^{-1})\mathcal{R} = \mathcal{R}^{-1},\\
&(S \otimes S)\mathcal{R} = \mathcal{R}.
\end{aligned}$$

Let $\{h_l\}$ be a basis of $\mathfrak{h}$ and $\{h^l\}$ be its dual basis. We denote by U^+ (resp. U^-) the subalgebra of U_q generated by e_i (resp. f_i) $i \in I$ and set

$$\begin{aligned}
&U^+_\beta = \{x \in U^+ |\ q^h x q^{-h} = q^{\langle \beta, h\rangle} x\ (h \in \mathfrak{h})\},\\
&U^-_{-\beta} = \{x \in U^- |\ q^h x q^{-h} = q^{-\langle \beta, h\rangle} x\ (h \in \mathfrak{h})\}
\end{aligned}$$

for $\beta \in Q_+$. The universal R matrix has the form [31]

$$\begin{aligned}
&\mathcal{R} = q^{-T}\mathcal{R}_0, \qquad T = \sum_l h_l \otimes h^l,\\
&\mathcal{R}_0 = \sum_{\beta \in Q_+} q^{(\beta,\beta)}\big(q^{-\beta} \otimes q^{\beta}\big)(\mathcal{R}_0)_\beta = 1 - \sum_{i \in I}(q_i - q_i^{-1})e_i t_i^{-1} \otimes t_i f_i + \cdots,\\
&(\mathcal{R}_0)_\beta = \sum_j u_{\beta,j} \otimes u^j_{-\beta} \in U^+_\beta \otimes U^-_{-\beta},
\end{aligned} \tag{2.3}$$

where $\{u_{\beta,j}\}$ and $\{u^j_{-\beta}\}$ are bases of U^+_β and $U^-_{-\beta}$, respectively. Note that T is the canonical element of $\mathfrak{h} \otimes \mathfrak{h}$ w.r.t $(\cdot|\cdot)$ and $(\mathcal{R}_0)_\beta$ is the canonical element of $U^+_\beta \otimes U^-_{-\beta}$ w.r.t a certain Hopf paring.

We write $U'_q = U'_q(\mathfrak{g})$ for the subalgebra of U_q generated by e_i, f_i $(i \in I)$ and $h \in (P_{cl})^*$. Let (π_V, V) be a finite dimensional module over U'_q. We have the evaluation representation $(\pi_{V,z}, V_z)$ of U_q by $V_z = \mathbb{C}(q^{1/2})[z, z^{-1}] \otimes_{\mathbb{C}(q^{1/2})} V$ and

$$\begin{aligned}
&\pi_{V,z}(e_i)(z^n \otimes v) = z^{\delta_{i0}+n} \otimes \pi(e_i)v, \qquad \pi_{V,z}(f_i)(z^n \otimes v) = z^{-\delta_{i0}+n} \otimes \pi(f_i)v,\\
&\pi_{V,z}(t_i)(z^n \otimes v) = z^n \otimes \pi(t_i)v, \qquad \pi_{V,z}(q^d)(z^n \otimes v) = (qz)^n \otimes v,\\
&\mathrm{wt}(z^n \otimes v) = n\delta + af(\mathrm{wt}(v)),
\end{aligned}$$

where $n \in \mathbb{Z}$, and $v \in V$ denotes a weight vector whose weight is $\mathrm{wt}(v)$. We write $vz^n = z^n \otimes v$ ($n \in \mathbb{Z}$).

For generic $\lambda \in \mathfrak{h}^*$, let M_λ denote the irreducible Verma module with the highest weight λ. We have the weight space decomposition $M_\lambda = \bigoplus_{\nu \in \lambda - Q_+} (M_\lambda)_\nu$. We write $\mathrm{wt}(u) = \nu$ for $u \in (M_\lambda)_\nu$.

2.2 Elliptic quantum groups $\mathcal{B}_{q,\lambda}(\mathfrak{g})$

Let $\rho \in \mathfrak{h}$ be an element satisfying $(\rho|\alpha_i) = d_i$ for all $i \in I$. For generic $\lambda \in \mathfrak{h}$, let us consider an automorphism of U_q given by

$$\varphi_\lambda = \mathrm{Ad}(q^{-2\theta(\lambda)}), \qquad \theta(\lambda) = -\lambda + \rho - \frac{1}{2}\sum_l h_l h^l,$$

where $\mathrm{Ad}(x)y = xyx^{-1}$. We define the face type twistor $F(\lambda) \in U_q \widehat{\otimes} U_q$ as follows.

Definition 2.3 (Face type twistor).

$$\begin{aligned} F(\lambda) &= \cdots \big((\varphi_\lambda)^2 \otimes \mathrm{id}\big)\mathcal{R}_0^{-1}(\varphi_\lambda \otimes \mathrm{id})\mathcal{R}_0^{-1} \\ &= \overset{\curvearrowleft}{\prod_{k\geq 1}}\Big((\varphi_\lambda)^k \otimes \mathrm{id}\Big)\mathcal{R}_0^{-1}, \end{aligned} \tag{2.4}$$

where $\overset{\curvearrowleft}{\prod_{k\geq 1}} A_k = \cdots A_3 A_2 A_1$.

Note that the k-th factor in the product (2.4) is a formal power series in $x_i^k = q^{2k\langle\alpha_i,\lambda\rangle}$ ($i \in I$) with leading term 1.

Theorem 2.4 ([23]). *The twistor $F(\lambda)$ satisfies the shifted cocycle condition and the normalization condition given by*

$$1) \quad F^{(12)}(\lambda)(\Delta \otimes \mathrm{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta)F(\lambda), \tag{2.5}$$

$$2) \quad (\epsilon \otimes \mathrm{id})\, F(\lambda) = (\mathrm{id} \otimes \epsilon)\, F(\lambda) = 1. \tag{2.6}$$

In (2.5), λ and $h^{(1)}$ means $\lambda = \sum_l \lambda_l h^l$ and $h^{(1)} = \sum_l h_l^{(1)} h^l$, $h_l^{(1)} = h_l \otimes 1 \otimes 1$, respectively. Note that from (2.3), one has

$$[h \otimes 1 + 1 \otimes h, F(\lambda)] = 0 \qquad \forall\, h \in \mathfrak{h}.$$

Now let us define Δ_λ, $\mathcal{R}(\lambda)$, $\Phi(\lambda)$ and α_λ, β_λ by

$$\Delta_\lambda(a) = F^{(12)}(\lambda)\,\Delta(a)\,F^{(12)}(\lambda)^{-1}, \tag{2.7}$$

$$\mathcal{R}(\lambda) = F^{(21)}(\lambda)\,\mathcal{R}\,F^{(12)}(\lambda)^{-1}, \tag{2.8}$$

$$\Phi(\lambda) = F^{(23)}(\lambda)F^{(23)}(\lambda + h^{(1)})^{-1}, \tag{2.9}$$

$$\alpha_\lambda = \sum_i S(d_i)l_i, \qquad \beta_\lambda = \sum_i m_i S(g_i) \tag{2.10}$$

for $\sum_i k_i \otimes l_i = F(\lambda)^{-1}$, $\sum_i m_i \otimes n_i = F(\lambda)$.

Definition 2.5 (Face type elliptic quantum group). *With S and ϵ defined by (2.1), the set $(U_q(\mathfrak{g}), \Delta_\lambda, S, \varepsilon, \alpha_\lambda, \beta_\lambda, \Phi(\lambda), \mathcal{R}(\lambda))$ forms a quasi-Hopf algebra [23]. We call it the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(\mathfrak{g})$.*

From (2.2), (2.5) and (2.8), one can show that $\mathcal{R}(\lambda)$ satisfies the dynamical YBE.

Theorem 2.6 (Dynamical Yang–Baxter equation).

$$\mathcal{R}^{(12)}(\lambda+h^{(3)})\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda+h^{(1)}) = \mathcal{R}^{(23)}(\lambda)\mathcal{R}^{(13)}(\lambda+h^{(2)})\mathcal{R}^{(12)}(\lambda). \tag{2.11}$$

We hence call $\mathcal{R}(\lambda)$ the universal dynamical R matrix.

Now let us parametrize λ in the following way.

$$\lambda = \left(r + \frac{h^\vee}{a_0^\vee}\right) d + sc + \bar{\lambda} \qquad (r, s \in \mathbb{C}), \tag{2.12}$$

where $\bar{\lambda}$ stands for the classical part of λ, and $h^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$. Note also $\rho = h^\vee\Lambda_0 + \bar{\rho}$ and $d = a_0^\vee\Lambda_0$. Let $\{\bar{h}_j\}$ and $\{\bar{h}^j(=\bar{\Lambda}_j)\}$ denote the classical part of the basis and its dual of $\mathfrak{h}$. We then have

$$\varphi_\lambda = \mathrm{Ad}(p^d q^{2cd} q^{-2\bar{\theta}(\lambda)}), \qquad \bar{\theta}(\lambda) = -\bar{\lambda} + \bar{\rho} - \frac{1}{2}\sum \bar{h}_j\bar{h}^j. \tag{2.13}$$

Here we set $p = q^{2r}$. Set further

$$\mathcal{R}(z) = \mathrm{Ad}(z^d \otimes 1)(\mathcal{R}), \tag{2.14}$$

$$F(z,\lambda) = \mathrm{Ad}(z^d \otimes 1)(F(\lambda)), \tag{2.15}$$

$$\mathcal{R}(z,\lambda) = \mathrm{Ad}(z^d \otimes 1)(\mathcal{R}(\lambda)) = \sigma(F(z^{-1},\lambda))\mathcal{R}(z)F(z,\lambda)^{-1}. \tag{2.16}$$

Then $\mathcal{R}(z)$ and $F(z,\lambda)$ are formal power series in z, whereas $\mathcal{R}(z,\lambda)$ contains both positive and negative powers of z.

From the definition (2.4) of $F(\lambda)$, one can easily derive the following difference equation for the twistor.

Theorem 2.7 (Difference equation [23]).

$$F(pq^{2c^{(1)}}z,\lambda) = \mathrm{Ad}(q^{2\bar{\theta}(\lambda)} \otimes \mathrm{id})\big(F(z,\lambda)\big) \cdot q^T\mathcal{R}(pq^{2c^{(1)}}z). \tag{2.17}$$

Furthermore, noting $\mathrm{Ad}(z^d)(e_i) = z^{\delta_{i,0}}e_i$, one can drop all the e_0 dependent terms in $\mathcal{R}(z)$ and $F(z,\lambda)$ by taking the limit $z \to 0$. We thus obtain

$$\lim_{z\to 0} q^{c\otimes d + d\otimes c}\mathcal{R}(z) = \mathcal{R}_{\bar{\mathfrak{g}}}, \tag{2.18}$$

$$\lim_{z\to 0} F(z,\lambda) = F_{\bar{\mathfrak{g}}}(\bar{\lambda}), \tag{2.19}$$

where $\mathcal{R}_{\bar{\mathfrak{g}}}$ and $F_{\bar{\mathfrak{g}}}(\bar{\lambda})$ are the universal R matrix and the twistor of $U_q(\bar{\mathfrak{g}})$. Then from (2.17), we obtain the following equation for $F_{\bar{\mathfrak{g}}}(\bar{\lambda})$.

Lemma 2.8.

$$F_{\bar{\mathfrak{g}}}(\lambda) = \mathrm{Ad}(q^{2\bar{\theta}(\lambda)} \otimes \mathrm{id})\big(F_{\bar{\mathfrak{g}}}(\bar{\lambda})\big) \cdot q^{\bar{T}}\mathcal{R}_{\bar{\mathfrak{g}}}, \tag{2.20}$$

where $\bar{T} = \sum_{i=1}^n \bar{h}_i \otimes \bar{h}^i$.

Remark. (2.20) corresponds to (18) in [32], where the comultiplication and the universal R matrix are our Δ^{op} and $\mathcal{R}_{\bar{\mathfrak{g}}}^{-1}$, respectively.

Lemma 2.9 ([32])**.** *The equation* (2.20) *has the unique solution* $F_{\bar{\mathfrak{g}}}(\bar{\lambda}) \in U_q(\bar{\mathfrak{b}}_+)\widehat{\otimes}U_q(\bar{\mathfrak{b}}_-)$ *in the form* $F_{\bar{\mathfrak{g}}}(\bar{\lambda}) = 1 + \cdots$. *Here* $U_q(\bar{\mathfrak{b}}_+)$ *(resp.* $U_q(\bar{\mathfrak{b}}_-)$*) is the subalgebra of* $U_q(\bar{\mathfrak{g}})$ *generated by* e_i, t_i $(i = 1, 2, \dots, n)$ *(resp.* f_i, t_i $(i = 1, 2, \dots, n)$*).*

Theorem 2.10. *For* $\lambda \in \mathfrak{h}$ *given by* (2.12)*, the difference equation* (2.17) *has a unique solution.*

Proof. Let us set $\bar{\varphi}_\lambda = \mathrm{Ad}(q^{-2\bar{\theta}(\lambda)})$. Iterating (2.17), N times, we obtain

$$F(z,\lambda) = \big(\bar{\varphi}_\lambda^N \otimes \mathrm{id}\big) F((pq^{2c^{(1)}})^N z, \lambda) \prod_{N \geq k \geq 1}^{\curvearrowleft} \Big((\bar{\varphi}_\lambda)^k \otimes \mathrm{id}\Big) \mathcal{R}_0((pq^{2c^{(1)}})^k z)^{-1}.$$

Taking the limit $N \to \infty$, one obtains

$$F(z,\lambda) = A \prod_{k \geq 1}^{\curvearrowleft} ((\bar{\varphi}_\lambda)^k \otimes \mathrm{id}) \mathcal{R}_0((pq^{2c^{(1)}})^k z)^{-1},$$

where we set

$$\begin{aligned} A &= \lim_{N\to\infty} \big(\bar{\varphi}_\lambda^N \otimes \mathrm{id}\big) F((pq^{2c^{(1)}})^N z, \lambda) \\ &= \lim_{N\to\infty} \big(\bar{\varphi}_\lambda^N \otimes \mathrm{id}\big) F_{\bar{\mathfrak{g}}}(\bar{\lambda}). \end{aligned}$$

Then the statement follows from Lemma 2.9. ∎

3 Vertex operators and fusion matrices

3.1 The vertex operators of $U_q(\mathfrak{g})$

Let V and W be finite dimensional irreducible modules of U'_q. Let $\lambda, \mu \in \mathfrak{h}^*$ be level-k generic elements such that $\langle c, \lambda \rangle = \langle c, \mu \rangle = k$. We denote by M_λ and M_μ the two irreducible Verma modules with highest weights λ and μ, respectively.

Definition 3.1 (Vertex operator). *Writing* $\triangle_\lambda = \frac{(\lambda|\lambda+2\rho)}{2(k+h^\vee)}$[1]*, let us consider the formal series given by*

$$\Psi_\lambda^\mu(z) = z^{\triangle_\mu - \triangle_\lambda} \widetilde{\Psi}_\lambda^\mu(z), \qquad \widetilde{\Psi}_\lambda^\mu(z) = \sum_j \sum_{n\in\mathbb{Z}} v_j z^{-n} \otimes (\widetilde{\Psi}_\lambda^\mu)_{j,n}. \tag{3.1}$$

Here $\{v_j\}$ *denotes a weight basis of* V. *The coefficients* $(\widetilde{\Psi}_\lambda^\mu)_{j,n}$ *are the maps*

$$(\widetilde{\Psi}_\lambda^\mu)_{j,n} : (M_\lambda)_\xi \to (M_\mu)_{\xi - \mathrm{wt}(v_j) + n\delta}, \tag{3.2}$$

such that $\widetilde{\Psi}_\lambda^\mu(z)$ *is the* U_q*-module intertwiners*

$$\begin{aligned} &\widetilde{\Psi}_\lambda^\mu(z) \ : \ M_\lambda \to V_z \widehat{\otimes}\, M_\mu, \\ &\widetilde{\Psi}_\lambda^\mu(z)\ x = \Delta(x)\ \widetilde{\Psi}_\lambda^\mu(z) \qquad \forall\, x \in U_q. \end{aligned} \tag{3.3}$$

Here $\widehat{\otimes}$ *denotes a formal completion*

$$M \widehat{\otimes}\, N = \bigoplus_\nu \prod_\xi M_\xi \otimes N_{\xi-\nu}.$$

We call $\Psi_\lambda^\mu(z)$ *the vertex operator (VO) of* U_q.

[1] Hopefully, there is no confusion of $\triangle_\lambda$ with Δ_λ in (2.7).

Remark. $\Psi_\lambda^\mu(z)$ is the type II VO in the terminology of [3].

We also define U_q'-module intertwiners $\Psi_\lambda^\mu : M_\lambda \to V \otimes \widehat{M}_\mu$ by

$$\Psi_\lambda^\mu = \sum_j v_j \otimes \left(\sum_{n\in\mathbb{Z}} (\widetilde{\Psi}_\lambda^\mu)_{j,n} \right). \tag{3.4}$$

Here $\widehat{M}_\mu = \prod_\nu (M_\mu)_\nu$. Note that there is a bijective correspondence between $\Psi_\lambda^\mu(z)$ and Ψ_λ^μ.

Let u_λ and u_μ denote the highest weight vectors of M_λ and M_μ, respectively. Let us write the image of u_λ by the VO as

$$\Psi_\lambda^\mu\, u_\lambda = v \otimes u_\mu + \sum_{i'} v_{i'} \otimes u_{i'}, \tag{3.5}$$

where $u_{i'} \in M_\mu$, $\mathrm{wt}(u_{i'}) < \mu$ and $v, v_{i'} \in V$. We call the vector v the leading term of Ψ_λ^μ. Note that from (3.2), $cl(\lambda) = \mathrm{wt}(v) + cl(\mu) = \mathrm{wt}(v_{i'}) + cl(\mathrm{wt}(u_{i'}))$. We set

$$V_\lambda^\mu = \{ v \in V \mid \mathrm{wt}(v) = cl(\lambda - \mu) \}.$$

Theorem 3.1 ([16, 17]). *The map* $\langle\ \rangle : \Psi_\lambda^\mu \mapsto \langle \mathrm{id} \otimes u_\mu^*, \Psi_\lambda^\mu u_\lambda \rangle$ *gives a* $\mathbb{C}(q^{1/2})$*-linear isomorphism*

$$\mathrm{Hom}_{U_q'(\mathfrak{g})}(M_\lambda, V \otimes \widehat{M}_\mu) \xrightarrow{\sim} V_\lambda^\mu$$

This theorem tells that Ψ_λ^μ is determined by its leading term. Namely, for given $v_0 \in V_\lambda^\mu$, there exists a unique VO satisfying

$$\langle \Psi_\lambda^\mu \rangle = v_0.$$

We denote such VO by Ψ_λ^{μ,v_0} and corresponding U_q-intertwiner by $\Psi_\lambda^{\mu,v_0}(z)$.

Proposition 3.2 ([17]). *Let* $\{v_j\}$ *be a basis of* V_λ^μ*. The set of VOs* $\{\Psi_\lambda^{\mu,v_j}\}$ *forms a basis of* $\mathrm{Hom}_{U_q'(\mathfrak{g})}(M_\lambda, V \otimes \widehat{M}_\mu)$*.*

3.2 The q-KZ equation and connection matrices

Let $\lambda, \mu, \nu \in \mathfrak{h}^*$ be level-k elements. Let $\{v_i\}$ and $\{w_j\}$ be weight bases of V_λ^μ and W_μ^ν, respectively. Consider the VOs Ψ_λ^{μ,v_i} and Ψ_μ^{ν,w_j} given by

$$\begin{aligned} \Psi_\lambda^{\mu,w_i}(z_1) &: M_\lambda \to W_{z_1} \widehat{\otimes} M_\mu, \\ \Psi_\mu^{\nu,v_j}(z_2) &: M_\mu \to V_{z_2} \widehat{\otimes} M_\nu, \end{aligned}$$

and their composition

$$\left(\mathrm{id} \otimes \Psi_\mu^{\nu,v_i}(z_2)\right) \Psi_\lambda^{\mu,w_j}(z_1) \ : \ M_\lambda \to W_{z_1} \otimes V_{z_2} \widehat{\otimes} M_\nu.$$

Setting

$$\Psi^{(\nu,\mu,\lambda)}(z_1, z_2) = \left\langle \mathrm{id} \otimes \mathrm{id} \otimes u_\nu^*, \left(\mathrm{id} \otimes \Psi_\mu^{\nu,v_i}(z_2)\right) \Psi_\lambda^{\mu,w_j}(z_1) u_\lambda \right\rangle,$$

we call $\Psi^{(\nu,\mu,\lambda)}(z_1, z_2)$ the two-point function.

Theorem 3.3 (q-KZ equation [16, 33]). *The two-point function* $\Psi^{(\nu,\mu,\lambda)}(z_1, z_2)$ *satisfies the* q*-KZ equation*

$$\Psi^{(\nu,\mu,\lambda)}(q^{2(k+h^\vee)} z_1, z_2) = (q^{-\pi_W(\bar\nu + \bar\lambda + 2\bar\rho)} \otimes \mathrm{id}) R_{WV}(z) \Psi^{(\nu,\mu,\lambda)}(z_1, z_2), \tag{3.6}$$

where $R_{WV}(z) = (\pi_W \otimes \pi_V)\mathcal{R}(z)$*.*

Proof. See Appendix. ■

Remark [16]**.** A solution of the q-KZ equation (3.6) is a function of $z = z_1/z_2$ and has a form $G(z)f(z_1, z_2)$. Here $G(z)$ is a meromorphic function multiplied by a fractional power of z determined from the normalization function of the R matrix $R_{WV}(z)$, while $f(z_1, z_2)$ is an analytic function in $|z_1| > |z_2|$ and can be continued meromorphically to $(\mathbb{C}^\times)^2$. Hence a solution of the q-KZ equation is uniquely determined, if one fixes the normalization. It also follows that the composition of the VOs is well defined in the region $|z_1| > |z_2|$ and can be continued meromorphically to $(\mathbb{C}^\times)^2$ apart from an overall fractional power of z.

The following commutation relation holds in the sense of analytic continuation.

Theorem 3.4 (Connection formula [16, 17]).

$$
\begin{aligned}
&(PR_{WV}(z_1/z_2)\otimes \mathrm{id})\left(\mathrm{id}\otimes \Psi_{\mu}^{\nu v_i}(z_2)\right)\Psi_{\lambda}^{\mu w_j}(z_1)\\
&\quad = \sum_{i',j',\mu'}\left(\mathrm{id}\otimes \Psi_{\mu'}^{\nu w_{j'}}(z_1)\right)\Psi_{\lambda}^{\mu' v_{i'}}(z_2) C_{WV}\left(\left.\begin{matrix}\lambda & w_j & \mu\\ v_{i'} & & v_i\\ \mu' & w_{j'} & \nu\end{matrix}\right|\frac{z_1}{z_2}\right),
\end{aligned}
$$

where $v_{i'}$ and $w_{j'}$ are base vectors of $V_\lambda^{\mu'}$ and $W_{\mu'}^{\nu}$, respectively.

The matrix C_{WV} is called the connection matrix. The following theorem states basic properties of the connection matrix.

Theorem 3.5 ([16, 17]). 1) *The matrix elements of C_{WV} are given by a ratio of elliptic theta functions.*

2) *The matrix C_{WV} satisfies i) the face type YBE and ii) the unitarity condition (the first inversion relation):*

$$
\begin{aligned}
i)\quad &\sum_{v_l,w_i,u_j,\mu''} C_{VU}\left(\left.\begin{matrix}\lambda & u_j & \mu''\\ w_{i'} & & w_i\\ \omega & u_{j'} & \nu\end{matrix}\right|\frac{z_2}{z_3}\right) C_{WU}\left(\left.\begin{matrix}\lambda' & u_{j''} & \mu'\\ v_{l'} & & v_l\\ \lambda & u_j & \mu''\end{matrix}\right|\frac{z_1}{z_3}\right)\\
&\quad\times C_{WV}\left(\left.\begin{matrix}\mu' & w_{i''} & \mu\\ v_l & & v_{l''}\\ \mu'' & w_i & \nu\end{matrix}\right|\frac{z_1}{z_2}\right)\\
&= \sum_{v_l,w_i,u_j,\mu''} C_{WV}\left(\left.\begin{matrix}\lambda' & w_i & \mu''\\ v_{l'} & & v_l\\ \lambda & w_{i'} & \omega\end{matrix}\right|\frac{z_1}{z_2}\right) C_{WU}\left(\left.\begin{matrix}\mu'' & u_j & \mu\\ v_l & & v_{l''}\\ \omega & u_{j'} & \nu\end{matrix}\right|\frac{z_1}{z_3}\right)\\
&\quad\times C_{VU}\left(\left.\begin{matrix}\lambda' & u_{j''} & \mu'\\ w_i & & w_{i''}\\ \mu'' & u_j & \mu\end{matrix}\right|\frac{z_2}{z_3}\right),\\
ii)\quad &\sum_{v_{i'},w_{j'},\mu'} C_{WV}\left(\left.\begin{matrix}\lambda & w_j & \mu\\ v_{i'} & & v_i\\ \mu' & w_{j'} & \nu\end{matrix}\right| z\right) C_{VW}\left(\left.\begin{matrix}\lambda & v_{i'} & \mu'\\ w_{j''} & & w_{j'}\\ \mu'' & v_{i''} & \nu\end{matrix}\right| z^{-1}\right)\\
&= \delta_{w_j,w_{j''}}\delta_{v_i,v_{i''}}\delta_{\mu,\mu''}.
\end{aligned}
$$

3) *In the case $V = W$, C_{VV} satisfies the second inversion relation*

$$
\sum_{v_{i'},w_{j'},\lambda}\frac{G_\lambda G_\nu}{G_\mu G_{\mu'}} C_{VV}\left(\left.\begin{matrix}\lambda & w_j & \mu\\ v_{i'} & & v_i\\ \mu' & w_{j'} & \nu\end{matrix}\right| z^{-1}\right) C_{VV}\left(\left.\begin{matrix}\lambda & v_{i'} & \mu'\\ w_j & & w_{j''}\\ \mu & v_{i''} & \nu'\end{matrix}\right| \xi^{-2} z\right)
$$

$$= \alpha_{VV}(z)\delta_{v_i,v_{i''}}\delta_{w_j,w_{j''}}\delta_{\nu,\nu'}.$$

Here $\xi = q^{th^\vee}$ *(*$t = (\text{long root})^2/2$*), and* G_λ *and* $\alpha_{VV}(z)$ *are given by* (5.16) *and* (5.13) *in* [17]*, respectively.*

Furthermore, if V_z is self dual i.e. there exists an isomorphism of U_q-modules $Q : V_{\xi^{-1}z} \simeq V_z^{*a}$, we have the following crossing symmetry.

Theorem 3.6 **([17]).**

$$\beta_{VV}(z^{-1})C_{VV}\left(\begin{matrix}\mu & w_j & \nu \\ v_i & & w_{j'} \\ \lambda & v_{i'} & \mu'\end{matrix}\middle|\, \xi^{-1}z^{-1}\right) = \sum_{\tilde{i},\tilde{j}} \gamma_\lambda^{\mu i\tilde{i}} g_{\mu'}^{\nu\tilde{j}j'} C_{VV}\left(\begin{matrix}\lambda & v_{\tilde{i}} & \mu \\ v_{i'} & & w_j \\ \mu' & w_{\tilde{j}} & \nu\end{matrix}\middle|\, z\right), \tag{3.7}$$

where $g_\mu^{\nu jj'}$ *denotes a certain matrix element appearing in the inversion relation of the VO's, and* γ_μ^ν *is its inverse matrix such that* $\sum_{\tilde{j}} g_\mu^{\nu j\tilde{j}}\gamma_\mu^{\nu\tilde{j}j'} = \delta_{jj'}$.

In Section 5, we will discuss the evaluation molude V_z with V being the vector representation for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$. There V_z is self dual except for $A_n^{(1)}$ $(n > 1)$, and is multiplicity free. Therefore, $\dim V_\lambda^\mu = 1$ etc. Hence the matrices $g_\mu^\nu, \gamma_\mu^\nu$ are scalars satisfying $\gamma_\mu^\nu = 1/g_\mu^\nu$. In this case, let us consider the gauge transformation

$$C_{VV}\left(\begin{matrix}\mu & \nu \\ \lambda & \mu'\end{matrix}\middle|\, z\right) = f(z)\frac{F(\mu,\nu)F(\nu,\mu')}{F(\mu,\lambda)F(\lambda,\mu')}\tilde{C}_{VV}\left(\begin{matrix}\mu & \nu \\ \lambda & \mu'\end{matrix}\middle|\, z\right)$$

with the choice

$$f(z)f(z^{-1}) = 1, \qquad f(\xi^{-1}z^{-1}) = \beta_{VV}(\xi z)f(z),$$
$$F(\nu,\mu)F(\mu,\nu) = g_\mu^\nu\sqrt{\frac{G_\mu}{G_\nu}}.$$

Then we can change the crossing symmetry relation (3.7) to

$$\tilde{C}_{VV}\left(\begin{matrix}\mu & \nu \\ \lambda & \mu'\end{matrix}\middle|\, \xi^{-1}z^{-1}\right) = \sqrt{\frac{G_\nu G_\lambda}{G_\mu G_{\mu'}}}\tilde{C}_{VV}\left(\begin{matrix}\lambda & \mu \\ \mu' & \nu\end{matrix}\middle|\, z\right).$$

The same gauge transformation makes the face type YBE i), the unitarity condition ii) and the second inversion relation 3) in Theorem 3.5 unchanged except for the factor $\alpha_{VV}(z)$ in the RHS of 3), which is changed to 1.

3.3 Fusion matrices

We here follows the idea by Etingof and Varchenko [28], where the cases $U_q(\bar{\mathfrak{g}})$ with $\bar{\mathfrak{g}}$ being simple Lie algebras are discussed. We extend their results to the cases $U_q(\mathfrak{g})$ with $\mathfrak{g}$ being affine Lie algebras.

Definition 3.2 (Fusion matrix). *Fix* $\lambda \in \mathfrak{h}^*$*. The fusion matrix is defined to be a* $\mathfrak{h}$*-linear map* $J_{WV}(\lambda) : W \otimes V \to W \otimes V$ *satisfying*

$$J_{VW}(\lambda) = \bigoplus_\nu J_{VW}(\lambda)_\nu,$$
$$J_{WV}(\lambda)_\nu(w_j \otimes v_i) = \left\langle \mathrm{id}\otimes\mathrm{id}\otimes u_\nu^*, \left(\mathrm{id}\otimes\Psi_\mu^{\nu,v_i}\right)\Psi_\lambda^{\mu,w_j}u_\lambda\right\rangle \ \in (W\otimes V)_{cl(\lambda-\nu)},$$

for $v_i \in V_\mu^\nu$*,* $w_j \in W_\lambda^\mu$.

Note that from (3.2),

$$[h \otimes 1 + 1 \otimes h, J_{WV}(\lambda)] = 0 \qquad \forall\, h \in \mathfrak{h}. \tag{3.8}$$

Noting (3.5) and the intertwining property of the vertex operators, we have

$$J_{WV}(\lambda)_\nu(w_j \otimes v_i) = w_j \otimes v_i + \sum_l C_l(\lambda) w_l \otimes v_l, \tag{3.9}$$

where $\mathrm{wt}(v_l) < \mathrm{wt}(v_i)$, and $C_l(\lambda)$ is a function of λ. Hence $J_{VW}(\lambda)$ is an upper triangular matrix with all the diagonal components being 1. Therefore

Theorem 3.7. *The fusion matrix $J_{WV}(\lambda)$ is invertible.*

The definition of $J_{WV}(\lambda)$ indicates that the leading term of the intertwiner $(\mathrm{id} \otimes \Psi_\mu^{\nu,v_i})\, \Psi_\lambda^{\mu,w_j}$ is $J_{WV}(\lambda)(w_j \otimes v_i)$. Therefore we write

$$\Psi_\lambda^{\nu, J_{WV}(\lambda)(w_j \otimes v_i)} = \left(\mathrm{id} \otimes \Psi_\mu^{\nu,v_i}\right) \Psi_\lambda^{\mu,w_j}.$$

Let us define $J_{U,W\otimes V}(\omega)$ and $J_{U\otimes W,V}(\omega) : U \otimes W \otimes V \to U \otimes W \otimes V$ for $\omega \in \mathfrak{h}^*$ by

$$\begin{aligned} J_{U,W\otimes V}(\omega)(u_l \otimes w_j \otimes v_i) &= \left\langle \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes u_\nu^*, \left(\mathrm{id} \otimes \Psi_\lambda^{\nu, J_{WV}(\lambda)(w_j \otimes v_i)}\right) \Psi_\omega^{\lambda,u_l} u_\omega \right\rangle, \\ J_{U\otimes W,V}(\omega)(u_l \otimes w_j \otimes v_i) &= \left\langle \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes u_\nu^*, (\mathrm{id} \otimes \mathrm{id} \otimes \Psi_\mu^{\nu,v_i}) \Psi_\omega^{\mu, J_{WU}(\omega)(w_j \otimes u_l)} u_\omega \right\rangle, \end{aligned}$$

where $u_l \in U_\omega^\lambda$.

Theorem 3.8. *The fusion matrix satisfies the shifted cocycle condition*

$$J_{U\otimes W,V}(\omega)(J_{UW}(\omega) \otimes \mathrm{id}) = J_{U,W\otimes V}(\omega)(\mathrm{id} \otimes J_{WV}(\omega - h^{(1)})) \qquad \text{on} \quad U \otimes W \otimes V, \tag{3.10}$$

where $hv_j = \mathrm{wt}(v_j) v_j$ $(v_j \in V)$ etc.

Proof. Consider the composition

$$\begin{aligned} &(\mathrm{id} \otimes \mathrm{id} \otimes \Psi_\mu^{\nu,v_i})(\mathrm{id} \otimes \Psi_\lambda^{\mu,w_j}) \Psi_\omega^{\lambda,u_l} : \\ &M_\omega \xrightarrow{\Psi_\omega^{\lambda,u_l}} U \otimes \widehat{M}_\lambda \xrightarrow{\mathrm{id}\otimes\Psi_\lambda^{\mu,w_j}} U \otimes W \otimes \widehat{M}_\mu \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Psi_\mu^{\nu,v_i}} U \otimes W \otimes V \otimes \widehat{M}_\nu \end{aligned}$$

and express the highest to highest expectation value of it in two ways, and use $cl(\lambda) = cl(\omega) - \mathrm{wt}(u_l)$. ■

Remark. Regarding $(\Delta \otimes \mathrm{id})J(\omega) = J_{U\otimes W,V}(\omega)$, $(\mathrm{id} \otimes \Delta)J(\omega) = J_{U,W\otimes V}(\omega)$, $J^{(12)}(\omega) = J_{UW}(\omega) \otimes \mathrm{id}$ and $J^{(23)}(\omega) = \mathrm{id} \otimes J_{WV}(\omega)$, one obtains (2.5) from (3.10) by replacing $J(\omega)$ by $F^{-1}(-\omega)$.

Now let $\lambda \in \mathfrak{h}^*$ be a level-k element. By using the U_q-module VOs (3.1), we define a $\mathfrak{h}$-linear map $J_{WV}(z_1, z_2; \lambda) : W_{z_1} \otimes V_{z_2} \to W_{z_1} \otimes V_{z_2}$ by

$$\begin{aligned} J_{WV}(z_1, z_2; \lambda) &= \bigoplus_\nu J_{WV}(z_1, z_2; \lambda)_\nu, \\ J_{WV}(z_1, z_2; \lambda)_\nu&(w_j \otimes v_i) \\ &= \left\langle \mathrm{id} \otimes \mathrm{id} \otimes u_\nu^*, \left(\mathrm{id} \otimes \widetilde{\Psi}_\mu^{\nu,v_i}(z_2)\right) \widetilde{\Psi}_\lambda^{\mu,w_j}(z_1) u_\lambda \right\rangle \in (W_{z_1} \otimes V_{z_2})_{cl(\lambda-\nu)}, \end{aligned}$$

for $v_j \in V_\mu^\nu$, $w_j \in W_\lambda^\mu$. Then from the q-KZ equation (3.6), one can derive the following difference equation for $J_{WV}(z_1, z_2; \lambda)$.

Lemma 3.9.

$$\begin{aligned}&J_{WV}(q^{2(k+h^\vee)}z_1,z_2;\lambda)(q^{-2\pi_W(\bar{\theta}(-\lambda))}\otimes \mathrm{id})\\&\qquad=(q^{-2\pi_W(\bar{\theta}(-\lambda))}\otimes \mathrm{id})q^{\pi_{W\otimes V}(\bar{T})}R_{WV}(z_1/z_2)J_{WV}(z_1,z_2,\lambda).\end{aligned}\tag{3.11}$$

Proof. See Appendix. ∎

From the remark below Theorem 3.3, $J_{WV}(z_1,z_2;\lambda)$ is a function of the ratio $z=z_1/z_2$. Let us parameterize a level-k $\lambda\in\mathfrak{h}^*$ as (2.12). Comparing Theorem 2.7 and Lemma 3.9, we find that the difference equation for $F_{WV}(z,-\lambda)^{-1}=(\pi_W\otimes\pi_V)F(z,-\lambda)^{-1}$ coincides with the q-KZ equation (3.11) for $J_{WV}(z_1,z_2;\lambda)$ on $w_j\otimes v_i$ under the identification $r=-(k+h^\vee)$. Hence the uniqueness of the solution to the q-KZ equation yields the following theorem.

Theorem 3.10. *For a level-k $\lambda\in\mathfrak{h}^*$ in the parameterization* (2.12),

$$\left\langle \mathrm{id}\otimes\mathrm{id}\otimes u_\nu^*, \left(\mathrm{id}\otimes\widetilde{\Psi}_\mu^{\nu,v_i}(z_2)\right)\widetilde{\Psi}_\lambda^{\mu,w_j}(z_1)u_\lambda\right\rangle=F_{WV}\left(z_1/z_2,-\lambda\right)^{-1}(w_j\otimes v_i).$$

Remark. Relation between the twistors and the fusion matrices was first discussed by Etingof and Varchenko for the case $\mathfrak{g}$ being simple Lie algebra (Appendix 9 in [28]). Their coproduct and the universal R matrix correspond to our Δ^{op} and $\mathcal{R}_{\bar{\mathfrak{g}}}^{-1}$, respectively, and the twistor is identified with the two point function of the $U_q(\bar{\mathfrak{g}})$-analogue of the type I VOs.

4 Dynamical R matrices and connection matrices

Let (π_V,V), (π_W,W) be finite dimensional representations of $U_q'(\mathfrak{g})$, and $\{v_i\}$, $\{w_j\}$ be their weight bases, respectively. Now we consider the dynamical R matrices given as the images of the universal R matrix $\mathcal{R}(\lambda)$

$$\begin{aligned}R_{WV}(z,\lambda)&=(\pi_W\otimes\pi_V)\mathcal{R}(z,\lambda)\\&=F_{VW}^{(21)}(z^{-1},\lambda)R_{WV}(z)F_{WV}(z,\lambda)^{-1}.\end{aligned}\tag{4.1}$$

Note that $F_{VW}^{(21)}(z^{-1},\lambda)=PF_{VW}(z^{-1},\lambda)P$.

By using Theorems 3.10 and 3.4, we show that the dynamical R matrix $R_{WV}(z,\lambda)$ coincides with the corresponding connection matrix for the q-KZ equation of $U_q(\mathfrak{g})$ associated with the representations (π_V,V) and (π_W,W).

Theorem 4.1. *For level-k $\lambda\in\mathfrak{h}^*$ in the form* (2.12) *with $r=-(k+h^\vee)$, we have*

$$R_{WV}(z,-\lambda)(w_j\otimes v_i)=\sum_{i',j',\mu'}C_{WV}\left(\begin{matrix}\lambda & w_j & \mu\\ v_{i'} & & v_i\\ \mu' & w_{j'} & \nu\end{matrix}\middle| z\right)(w_{j'}\otimes v_{i'}),$$

where $v_i\in V_\mu^\nu$, $w_j\in W_\lambda^\mu$, $v_{i'}\in V_\lambda^{\mu'}$ and $w_{j'}\in W_{\mu'}^\nu$.

Proof. From Theorem 3.10 and (4.1), we have

$$\begin{aligned}&R_{WV}(z_1/z_2,-\lambda)(w_j\otimes v_i)\\&\qquad=F_{VW}^{(21)}(z^{-1},-\lambda)R_{WV}(z)\left\langle \mathrm{id}\otimes\mathrm{id}\otimes u_\nu^*, \left(\mathrm{id}\otimes\widetilde{\Psi}_\mu^{\nu,v_i}(z_2)\right)\widetilde{\Psi}_\lambda^{\mu,w_j}(z_1)u_\lambda\right\rangle,\end{aligned}$$

where we set $z=z_1/z_2$. Then from Theorem 3.4, we obtain

$$R_{WV}(z_1/z_2,-\lambda)(w_j\otimes v_i)=F_{VW}^{(21)}(z^{-1},-\lambda)$$

$$\times \sum_{i',j',\mu'} P\left\langle \mathrm{id}\otimes\mathrm{id}\otimes u_\nu^*, \left(\mathrm{id}\otimes\widetilde{\Psi}_{\mu'}^{\nu,w_{j'}}(z_1)\right)\widetilde{\Psi}_\lambda^{\mu,v_{i'}}(z_2)u_\lambda\right\rangle C_{WV}\left(\begin{matrix}\lambda & w_j & \mu\\ v_{i'} & & v_i\\ \mu' & w_{j'} & \nu\end{matrix}\middle| z\right)$$

$$= F_{VW}^{(21)}(z^{-1},-\lambda)\sum_{i',j',\mu'} PF(z_2/z_1,-\lambda)^{-1}(v_{i'}\otimes w_{j'})C_{WV}\left(\begin{matrix}\lambda & w_j & \mu\\ v_{i'} & & v_i\\ \mu' & w_{j'} & \nu\end{matrix}\middle| z\right)$$

$$= \sum_{i',j',\mu'} C_{WV}\left(\begin{matrix}\lambda & w_j & \mu\\ v_{i'} & & v_i\\ \mu' & w_{j'} & \nu\end{matrix}\middle| z\right)(w_{j'}\otimes v_{i'}). \qquad \blacksquare$$

Note that in view of Theorem 3.5, this theorem indicates that the dynamical R matrices of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ are elliptic.

5 Vector representations

In this section, we consider the vector representation of the universal R matrix $\mathcal{R}(\lambda)$ of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$, and show that they coincide with the corresponding face weights obtained by Jimbo, Miwa and Okado [13].

5.1 Jimbo–Miwa–Okado's solutions

Let us summarize Jimbo–Miwa–Okado's elliptic solutions to the face type YBE.

Let (π_V, V) be the vector representation of $U_q'(\mathfrak{g})$. We set $\dim V = N$. Then $N = n+1$, $2n+1$, $2n$, $2n$ for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$, respectively. Let us define an index set J by

$$\begin{aligned} J &= \{1,2,\dots,n+1\} &&\text{for } A_n^{(1)}\\ &= \{0,\pm1,\dots,\pm n\} &&\text{for } B_n^{(1)}\\ &= \{\pm1,\dots,\pm n\} &&\text{for } C_n^{(1)}, D_n^{(1)}\end{aligned}$$

and introduce a linear order $\prec$ in J by

$$1\prec 2\prec\cdots\prec n\ (\prec 0)\ \prec -n\prec\cdots\prec -2\prec -1.$$

We also use the usual order $<$ in J.

Let $\bar\Lambda_j$ $(1\le j\le n)$ be the fundamental weights of $\bar{\mathfrak{g}}$. Following Bourbaki [34] we introduce orthonormal vectors $\{\varepsilon_1,\dots,\varepsilon_n\}$ with the bilinear form $(\varepsilon_i|\varepsilon_j) = \delta_{i,j}$. Then one has the following expression of $\bar\Lambda_j$ as well as the set $\mathcal{A}$ of weights belonging to the vector representation of $\bar{\mathfrak{g}}$.

$$\begin{aligned}
A_n:\quad & \mathcal{A} = \{\varepsilon_1-\varepsilon,\dots,\varepsilon_{n+1}-\varepsilon\},\\
& \bar\Lambda_j = \varepsilon_1+\cdots+\varepsilon_j - j\varepsilon \quad (1\le j\le n), \qquad \varepsilon = \frac{1}{n+1}\sum_{j=1}^{n+1}\varepsilon_j,\\
B_n:\quad & \mathcal{A} = \{\pm\varepsilon_1,\dots,\pm\varepsilon_n, 0\},\\
& \bar\Lambda_j = \varepsilon_1+\cdots+\varepsilon_j \qquad (1\le j\le n-1),\\
& = \frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_n) \qquad (j=n),\\
C_n:\quad & \mathcal{A} = \{\pm\varepsilon_1,\dots,\pm\varepsilon_n\},\\
& \bar\Lambda_j = \varepsilon_1+\cdots+\varepsilon_j \qquad (1\le j\le n),\\
D_n:\quad & \mathcal{A} = \{\pm\varepsilon_1,\dots,\pm\varepsilon_n\},
\end{aligned}$$

$$
\begin{aligned}
\bar{\Lambda}_j &= \varepsilon_1+\cdots+\varepsilon_j \qquad (1\le j\le n-2),\\
&= \frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{n-2}+\varepsilon_{n-1}-\varepsilon_n) \qquad (j=n-1),\\
&= \frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{n-2}+\varepsilon_{n-1}+\varepsilon_n) \qquad (j=n),
\end{aligned}
$$

Now for $\mu\in J$, let us define $\hat{\mu}\in\mathcal{A}$ by

$$
\begin{aligned}
\hat{\mu} &= \varepsilon_\mu-\varepsilon \qquad (1\le \mu\le n+1) \quad \text{for } A_n,\\
&= \pm\varepsilon_j \text{ or } 0 \qquad (\mu=\pm j\ (1\le j\le n),\ \text{or } \mu=0) \quad \text{for } B_n,\\
&= \pm\varepsilon_j \qquad (\mu=\pm j\ (1\le j\le n)) \quad \text{for } C_n, D_n.
\end{aligned}
$$

We then define a dynamical variable $a\in\mathfrak{h}^*$ of the face model of type $\mathfrak{g}$ as follows.

$$
\begin{aligned}
a_\mu &= (a+\rho|\hat{\mu}) \qquad (\mu\neq 0),\\
&= -\frac{1}{2} \qquad (\mu=0).
\end{aligned}
$$

We also set $a_{\mu\nu}=a_\mu-a_\nu$.

Proposition 5.1. *If one parameterizes $a\in\mathfrak{h}^*$ such that $a+\rho=\sum_{i=0}^{n}s_i\Lambda_i$, one has*

$$
\begin{aligned}
a_\mu &= \frac{1}{n+1}\left(-\sum_{j=1}^{\mu-1} js_j+\sum_{j=\mu}^{n}(n+1-j)s_j\right) \qquad (1\le\mu\le n+1) \quad \text{for } A_n^{(1)},\\
&= \pm\left(\sum_{j=i}^{n-1}s_j+\frac{s_n}{2}\right) \ \text{or}\ -\frac{1}{2} \qquad (\mu=\pm i\ (1\le i\le n),\ \text{or } 0) \quad \text{for } B_n^{(1)},\\
&= \pm\sum_{j=i}^{n}s_j \qquad (\mu=\pm i\ (1\le i\le n)) \quad \text{for } C_n^{(1)},\\
&= \pm\left(\sum_{j=i}^{n-1}s_j+\frac{s_n-s_{n-1}}{2}\right) \qquad (\mu=\pm i\ (1\le i\le n)) \quad \text{for } D_n^{(1)}.
\end{aligned}
$$

Note that $a_{\mu\nu}=\sum_{j=\mu}^{\nu-1}s_j$ for $A_n^{(1)}$ and $a_{-\mu}=-a_\mu$ for $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$.

Then Jimbo–Miwa–Okado's solutions to the face type YBE are given as follows.

$$
W\left(\begin{array}{cc|c} a & b & \\ c & d & u\end{array}\right)=\kappa(u)\overline{W}\left(\begin{array}{cc|c} a & b & \\ c & d & u\end{array}\right), \tag{5.1}
$$

$$
\begin{aligned}
(I)\quad \overline{W}\left(\begin{array}{cc|c} a & a+\hat{\mu} & \\ a+\hat{\mu} & a+2\hat{\mu} & u\end{array}\right) &= 1 \quad (\mu\neq 0),\\
\overline{W}\left(\begin{array}{cc|c} a & a+\hat{\mu} & \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} & u\end{array}\right) &= \frac{[1][a_{\mu\nu}-u]}{[1+u][a_{\mu\nu}]} \qquad (\mu\neq\pm\nu),\\
\overline{W}\left(\begin{array}{cc|c} a & a+\hat{\nu} & \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} & u\end{array}\right) &= \frac{[u]\sqrt{[a_{\mu\nu}+1][a_{\mu\nu}-1]}}{[1+u][a_{\mu\nu}]} \quad (\mu\neq\pm\nu),
\end{aligned}
$$

for $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$,

$$(II)\ \overline{W}\left(\begin{array}{cc} a & a+\hat{\nu} \\ a+\hat{\mu} & a \end{array}\middle| u\right) = \frac{[u][1][a_{\mu,-\nu}+1+\eta-u]}{[\eta-u][1+u][a_{\mu,-\nu}+1]}\sqrt{G_{a\mu}G_{a\nu}} \quad (\mu\neq\nu),$$

$$\overline{W}\left(\begin{array}{cc} a & a+\hat{\mu} \\ a+\hat{\mu} & a \end{array}\middle| u\right) = \frac{[\eta+u][1][a_{\mu,-\mu}+1+2\eta-u]}{[\eta-u][1+u][a_{\mu,-\mu}+1+2\eta]}$$
$$-\frac{[u][1][a_{\mu,-\mu}+1+\eta-u]}{[\eta-u][1+u][a_{\mu,-\mu}+1+2\eta]}\sum_{\kappa\neq\mu}\frac{[a_{\mu,-\kappa}+1+2\eta]}{[a_{\mu,-\kappa}+1]}G_{a\mu}, \quad \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)},$$

where $\eta = -th^\vee/2$ ($t = (\text{long root})^2/2$) is the crossing parameter, and the symbol $[u]$ denotes the Jacobi elliptic theta function

$$[u] = q^{r/4}e^{\pi i/4}\left(-\frac{2\pi i}{\log p}\right)^{-1/2} q^{\frac{u^2}{r}-u}\Theta_p(q^{2u}), \qquad p = q^{2r}, \tag{5.2}$$

$$\Theta_p(z) = (z;p)_\infty(p/z;p)_\infty(p;p)_\infty, \qquad (z;p)_\infty = \prod_{n=0}^{\infty}(1-zp^n).$$

The $\kappa(u)$ denotes a function satisfying the following relations

$$\kappa(u)\kappa(-u) = 1 \qquad \text{for } A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)},$$
$$\kappa(\eta-u)\kappa(\eta+u) = \frac{[1+\eta+u][1+\eta-u]}{[\eta+u][\eta-u]} \qquad \text{for } A_n^{(1)},$$
$$\kappa(u)\kappa(\eta+u) = \frac{[-u][1+\eta+u]}{[1-u][\eta+u]} \qquad \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)}.$$

The $G_{a\mu} = \frac{G_{a+\hat{\mu}}}{G_a}$ denotes a ratio of the principally specialized character G_a for the dual affine Lie algebra $\mathfrak{g}^\vee$ [13]

$$G_a = \prod_{1\le i<j\le n+1}[a_i-a_j] \qquad \text{for } A_n^{(1)},$$
$$= \varepsilon(a)\prod_{i=1}^{n}h(a_i)\prod_{1\le i<j\le n}[a_i-a_j][a_i+a_j] \qquad \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)}.$$

Here $\varepsilon(a)$ denotes a sign factor such that $\varepsilon(a+\hat{\mu})/\varepsilon(a) = s$. s and $h(a)$ are listed in the following Table

	$A_n^{(1)}$	$B_n^{(1)}$	$C_n^{(1)}$	$D_n^{(1)}$
$h^\vee$	$n+1$	$2n-1$	$n+1$	$2n-2$
t	1	1	2	1
s	1	1	-1	1
$h(a)$	1	$[a]$	$[2a]$	1

Remark. Our normalization of the weights W and some notations are different from those in [13]. Their relations are given as follows

$$W\left(\begin{array}{cc} a & b \\ c & d \end{array}\middle| u\right) = \frac{[1]}{[1+u]}\kappa(u)W_{JMO}\left(\begin{array}{cc} a & b \\ c & d \end{array}\middle| u\right) \qquad \text{for } A_n^{(1)},$$
$$= \frac{[1][\eta]}{[1+u][\eta-u]}\kappa(u)W_{JMO}\left(\begin{array}{cc} a & b \\ c & d \end{array}\middle| u\right) \qquad \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)},$$

$$r = L_{JMO}, \qquad \log p = \frac{4\pi^2}{\log p_{JMO}}.$$

Here the symbols with subindex JMO denote the ones in [13]. Note $[u] = [u]_{JMO}$.

The following theorem states basic properties of the face weights.

Theorem 5.2 ([13]). *The face weight W satisfies i) the face type Yang–Baxter equation, ii) the first and iii) the second inversion relations*

$$i)\quad \sum_g W\left(\begin{array}{cc} f & g \\ e & d \end{array}\middle| u\right) W\left(\begin{array}{cc} a & b \\ f & g \end{array}\middle| u+v\right) W\left(\begin{array}{cc} b & c \\ g & d \end{array}\middle| v\right)$$
$$= \sum_g W\left(\begin{array}{cc} a & b \\ g & c \end{array}\middle| u\right) W\left(\begin{array}{cc} g & c \\ e & d \end{array}\middle| u+v\right) W\left(\begin{array}{cc} a & g \\ f & e \end{array}\middle| v\right),$$
$$ii)\quad \sum_g W\left(\begin{array}{cc} a & g \\ d & c \end{array}\middle| u\right) W\left(\begin{array}{cc} a & b \\ g & c \end{array}\middle| -u\right) = \delta_{bd},$$
$$iii)\quad \sum_g \frac{G_a G_g}{G_b G_d} W\left(\begin{array}{cc} a & b \\ d & g \end{array}\middle| -u\right) W\left(\begin{array}{cc} c & d \\ b & g \end{array}\middle| 2\eta+u\right) = \delta_{ac}.$$

In addition, we have the crossing symmetry except for $\mathfrak{g} = A_n^{(1)}$ $(n > 1)$

$$iv)\quad W\left(\begin{array}{cc} a & b \\ c & d \end{array}\middle| u\right) = \sqrt{\frac{G_b G_c}{G_a G_d}} W\left(\begin{array}{cc} c & a \\ d & b \end{array}\middle| \eta-u\right).$$

The following theorem is communicated by Jimbo and Okado and is not written explicitly in [13].

Theorem 5.3. *For $\mathfrak{g} = B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$, the weights listed in the part (II) of (5.1) is determined uniquely from those in (I) by requiring the face type Yang–Baxter equation and the crossing symmetry relations.*

Sketch of proof. It is easy to see that the first type of weights in (II) is determined by those in (I) by using the crossing symmetry relation. Then the second type of weights in (II) is determined by solving the system of two linear equations (YBE) shown in Fig. 1. Here unknowns are the weights $W\begin{pmatrix} a & a+\mu \\ a+\mu & a \end{pmatrix}$ and $W\begin{pmatrix} a+\nu & a+\mu+\nu \\ a+\mu+\nu & a+\nu \end{pmatrix}$, and the other weights are in (I). ■

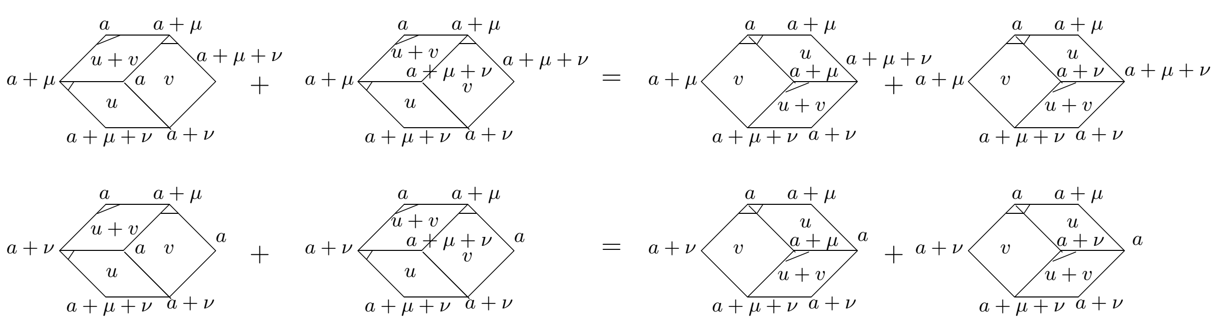

Figure 1. Two relevant equations $(\mu \neq \pm\nu)$.

5.2 The difference equations for the twistor

We here solve the difference equation for the twistor. Then using Theorem 4.1, we derive the dynamical R matrix $R_{VV}(z, \lambda)$ as the connection matrix in the vector representation (π_V, V), and argue that it coincides with Jimbo–Miwa–Okado's solution up to a gauge transformation.

Let us consider the difference equation (2.17) in the vector representation.

$$F(pz,\lambda) = (q^{2\pi_V(\bar\theta(\lambda))}\otimes \mathrm{id})F(z,\lambda)(q^{-2\pi_V(\bar\theta(\lambda))}\otimes \mathrm{id})q^{\pi_{V\otimes V}(T)}R(pz), \tag{5.3}$$

where λ is parameterized as (2.12), $\bar\theta(\lambda)$ is given by (2.13), $T = c\otimes\Lambda_0 + \Lambda_0\otimes c + \sum_{i=1}^{n}\bar h_i\otimes\bar h^i$, $F(z,\lambda) = (\pi_V\otimes\pi_V)F(z,\lambda)$, and $R(z) = (\pi_V\otimes\pi_V)\mathcal{R}(z)$.

Let $\{v_j \,|j\in J\}$ be a basis of V and $E_{i,j}$ be the matrix unit defined by $E_{i,j}v_k = \delta_{j,k}v_i$. The action of the generators on V is given by [35, 36, 37] (for $C_n^{(1)}$, the conventions used here are slightly different from [37])

$$\begin{aligned}
\pi_V(e_0) &= E_{n+1,1} \qquad \text{for } A_n^{(1)},\\
&= (-)^n(E_{-1,2}-E_{-2,1}) \qquad \text{for } B_n^{(1)},\\
&= (-)^{n-1}(E_{-1,2}-E_{-2,1}) \qquad \text{for } D_n^{(1)},\\
&= E_{-1,1} \qquad \text{for } C_n^{(1)},\\
\pi_V(e_i) &= E_{i,i+1} \qquad (1\le i\le n) \quad \text{for } A_n^{(1)},\\
&= E_{i,i+1}-E_{-i-1,-i} \qquad (1\le i\le n-1) \quad \text{for } B_n^{(1)}, D_n^{(1)}, C_n^{(1)},\\
\pi_V(e_n) &= \sqrt{[2]_{q_n}}(E_{n,0}-E_{0,-n}) \qquad \text{for } B_n^{(1)},\\
&= E_{n,-n} \qquad \text{for } C_n^{(1)},\\
&= E_{n-1,-n}-E_{n,-n+1} \qquad \text{for } D_n^{(1)},\\
\pi_V(t_0) &= \sum_{j\in J} q^{-\delta_{j,1}+\delta_{j,n+1}}E_{j,j} \qquad \text{for } A_n^{(1)},\\
&= \sum_{j\in J} q^{-\delta_{j,1}-\delta_{j,2}+\delta_{j,-1}+\delta_{j,-2}}E_{j,j} \qquad \text{for } B_n^{(1)}, D_n^{(1)}\\
&= \sum_{j\in J} q^{-2\delta_{j,1}+2\delta_{j,-1}}E_{j,j} \qquad \text{for } C_n^{(1)},\\
\pi_V(t_i) &= \sum_{j\in J} q^{\delta_{j,i}-\delta_{j,i+1}}E_{j,j} \qquad (1\le i\le n) \quad \text{for } A_n^{(1)}\\
&= \sum_{j\in J} q^{\delta_{j,i}-\delta_{j,i+1}+\delta_{j,-i-1}-\delta_{j,-i}}E_{j,j} \qquad (1\le i\le n-1) \quad \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)},\\
\pi_V(t_n) &= \sum_{j\in J} q^{\delta_{j,n}-\delta_{j,-n}}E_{j,j} \qquad \text{for } B_n^{(1)},\\
&= \sum_{j\in J} q^{2\delta_{j,n}-2\delta_{j,-n}}E_{j,j} \qquad \text{for } C_n^{(1)},\\
&= \sum_{j\in J} q^{\delta_{j,n-1}+\delta_{j,n}-\delta_{j,-n}-\delta_{j,-n+1}}E_{j,j} \qquad \text{for } D_n^{(1)},
\end{aligned}$$

and $\pi_V(f_i) = \pi_V(e_i)^t$.

A basis $\{\bar h_i\}$ of $\bar{\mathfrak{h}}$ and its dual basis $\{\bar h^i\}$ w.r.t $(\cdot|\cdot)$ are given as follows

$$\begin{aligned}
A_n:\quad &\pi_V(\bar h_i) = E_{i,i}-E_{i+1,i+1} \qquad (1\le i\le n),\\
&\pi_V(\bar h^i) = \frac{1}{n+1}\left((n-i+1)\sum_{j=1}^{i}E_{j,j} - i\sum_{j=i+1}^{n+1}E_{j,j}\right) \qquad (1\le i\le n),\\
B_n:\quad &\pi_V(\bar h_i) = E_{i,i}-E_{i+1,i+1}+E_{-i-1,-i-1}-E_{-i,-i} \qquad (1\le i\le n-1),
\end{aligned}$$

$$\pi_V(\bar{h}_n) = 2(E_{n,n} - E_{-n,-n}),$$
$$\pi_V(\bar{h}^i) = \sum_{j=1}^{i}(E_{j,j} - E_{-j,-j}) \qquad (1 \le i \le n-1),$$
$$\pi_V(\bar{h}^n) = \frac{1}{2}\sum_{j=1}^{n}(E_{j,j} - E_{-j,-j}),$$
$$C_n:\quad \pi_V(\bar{h}_i) = E_{i,i} - E_{i+1,i+1} + E_{-i-1,-i-1} - E_{-i,-i} \qquad (1 \le i \le n-1),$$
$$\pi_V(\bar{h}_n) = E_{n,n} - E_{-n,-n},$$
$$\pi_V(\bar{h}^i) = \sum_{j=1}^{i}(E_{j,j} - E_{-j,-j}) \qquad (1 \le i \le n),$$
$$D_n:\quad \pi_V(\bar{h}_i) = E_{i,i} - E_{i+1,i+1} + E_{-i-1,-i-1} - E_{-i,-i} \qquad (1 \le i \le n-1),$$
$$\pi_V(\bar{h}_n) = E_{n-1,n-1} + E_{n,n} - E_{-n,-n} - E_{-n+1,-n+1},$$
$$\pi_V(\bar{h}^i) = \sum_{j=1}^{i}(E_{j,j} - E_{-j,-j}) \qquad (1 \le i \le n-2),$$
$$\pi_V(\bar{h}^{n-1}) = \frac{1}{2}\sum_{j=1}^{n-1}(E_{j,j} - E_{-j,-j}) - \frac{1}{2}(E_{n,n} - E_{-n,-n}),$$
$$\pi_V(\bar{h}^n) = \frac{1}{2}\sum_{j=1}^{n-1}(E_{j,j} - E_{-j,-j}) + \frac{1}{2}(E_{n,n} - E_{-n,-n}).$$

Then one can easily verify the following.

Proposition 5.4.

$$q^{\pi_{V\otimes V}(T)} = q^{-\frac{1}{n+1}} \sum_{i,j\in J} q^{\delta_{i,j}} E_{i,i} \otimes E_{j,j} \qquad \text{for } A_n^{(1)},$$
$$= \sum_{i,j\in J} q^{\delta_{i,j}-\delta_{i,-j}} E_{i,i} \otimes E_{j,j} \qquad \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)}.$$

Proposition 5.5. *If we parameterize* $\bar{\lambda}$ *such that* $\bar{\lambda} = \sum_{i=1}^{n}(s_i+1)\bar{h}^i$, *we have*

$$q^{-2\pi_V(\bar{\theta}(\lambda))} = q^{\frac{n}{n+1}} \sum_{j\in J} q^{2a_j} E_{j,j} \qquad \text{for } A_n^{(1)},$$
$$= \sum_{j\in J} q^{2a_j+1} E_{j,j} \qquad \text{for } B_n^{(1)}, C_n^{(1)}, D_n^{(1)},$$

where a_j $(j \in J)$ *is given by Proposition* 5.1.

The R matrix $R(z)$ of $U_q(\mathfrak{g})$ in the vector representation is well known [2, 35, 36, 37] (for $C_n^{(1)}$, we modified the R matrix in [37] according to the convention used here)

$$R(z) = \rho(z)\Bigg\{\sum_{\substack{i\in J\\ i\neq 0}} E_{i,i}\otimes E_{i,i} + b(z)\sum_{\substack{i,j\\ i\neq \pm j}} E_{i,i}\otimes E_{j,j}$$
$$+ \sum_{\substack{i\prec j\\ i\neq -j}} \Big(c(z)E_{i,j}\otimes E_{j,i} + zc(z)E_{j,i}\otimes E_{i,j}\Big)$$

$$+\frac{1}{(1-q^2z)(1-\xi z)}\sum_{i,j}a_{ij}(z)E_{i,j}\otimes E_{-i,-j}\Bigg\}, \tag{5.4}$$

$$b(z)=\frac{q(1-z)}{1-q^2z},\quad c(z)=\frac{1-q^2}{1-q^2z},$$

$$\rho(z)=q^{-\frac{n}{n+1}}\frac{(q^2z;\xi^2)_\infty(q^{-2}\xi^2z;\xi^2)_\infty}{(z;\xi^2)_\infty(\xi^2z;\xi^2)_\infty}\qquad \text{for } A_n^{(1)},$$

$$=q^{-1}\frac{(q^2z;\xi^2)_\infty(\xi z;\xi^2)_\infty^2(q^{-2}\xi^2z;\xi^2)_\infty}{(z;\xi^2)_\infty(q^{-2}\xi z;\xi^2)_\infty(q^2\xi z;\xi^2)_\infty(\xi^2z;\xi^2)_\infty}\qquad \text{for } B_n^{(1)},C_n^{(1)},D_n^{(1)},$$

$$a_{ij}(z)=0\qquad \text{for } A_n^{(1)},$$

$$=\begin{cases}(q^2-\xi z)(1-z)+\delta_{i,0}(1-q)(q+z)(1-\xi z) & (i=j),\\ (1-q^2)[\epsilon_i\epsilon_j q^{\bar{j}-\bar{i}}(z-1)+\delta_{i,-j}(1-\xi z)] & (i\prec j),\\ (1-q^2)z[\xi\epsilon_i\epsilon_j q^{\bar{j}-\bar{i}}(z-1)+\delta_{i,-j}(1-\xi z)] & (i\succ j),\end{cases}\qquad \text{for } B_n^{(1)},C_n^{(1)},D_n^{(1)}.$$

Here $\xi=q^{th^\vee}$, and $\epsilon_j=1$ $(j>0)$, -1 $(j<0)$ for $\mathfrak{g}=C_n^{(1)}$ and $\epsilon_j=1$ $(j\in J)$ for the other cases. The symbol $\bar{j}$ is defined by

$$\bar{j}=\begin{cases}j-\epsilon_j & (j=1,\ldots,n),\\ n-\epsilon_j & (j=0),\\ j+N-\epsilon_j & (j=-n,\ldots,-1).\end{cases}$$

Then due to the formula (2.4), we make the following ansatz for the twistor $F(z,\lambda)$ in the vector representation.

$$F(z,\lambda)=f(z)\Bigg\{\sum_{\substack{i\in J\\ i\neq 0}}E_{i,i}\otimes E_{i,i}+\sum_{\substack{i,j\\ i\neq\pm j}}X_{ij}^{ij}(z)E_{i,i}\otimes E_{j,j} \tag{5.5}$$
$$+\sum_{\substack{i\prec j\\ i\neq -j}}\Big(X_{ij}^{ji}(z)E_{i,j}\otimes E_{j,i}+X_{ji}^{ij}(z)E_{j,i}\otimes E_{i,j}\Big)+\sum_{i,j}X_{i,-i}^{j,-j}(z)E_{i,j}\otimes E_{-i,-j}\Bigg\},$$

where X_{ij}^{kl} denote unknown functions to be determined.

From the from of $R(z)$ and $F(z,\lambda)$ in (5.4) and (5.5), one finds that the difference equation (5.3) consists of 1×1, 2×2 and $N\times N$ blocks. The numbers of blocks of each size contained in the equation are listed as follows

	1×1	2×2	$N\times N$
$A_n^{(1)}$	$n+1$	$\frac{n(n+1)}{2}$	0
$B_n^{(1)}$	$2n$	$2n^2$	1
$C_n^{(1)}$	$2n$	$2n(n-1)$	1
$D_n^{(1)}$	$2n$	$2n(n-1)$	1

By using Propositions 5.4, 5.5 and (5.4), (5.5), we obtain the following equations.

1×1 blocks:

$$f(pz)=q^{\frac{n}{n+1}}\rho(pz)f(z)\qquad \text{for } A_n,$$
$$=q\rho(pz)f(z)\qquad \text{for } B_n,C_n,D_n.$$

2×2 blocks:

$$\begin{pmatrix}X_{ij}^{ij}(pz) & X_{ij}^{ji}(pz)\\ X_{ji}^{ij}(pz) & X_{ji}^{ji}(pz)\end{pmatrix}=q^{-1}\begin{pmatrix}X_{ij}^{ij}(z) & w_{ij}^{-1}X_{ij}^{ji}(z)\\ w_{ij}X_{ji}^{ij}(z) & X_{ji}^{ji}(z)\end{pmatrix}\begin{pmatrix}b(pz) & c(pz)\\ pzc(pz) & b(pz)\end{pmatrix},$$

$$(i,j\in J,\ i\prec j,\ i\neq -j)$$

where we set $w_{ij}=q^{2(a_i-a_j)}$.

$N\times N$ block:

$$X^{j,-j}_{i,-i}(pz)=\frac{q^{-2}}{(1-pq^2z)(1-p\xi z)}\sum_{k\in J}q^{-2(a_i-a_k)}a_{kj}(pz)X^{k,-k}_{i,-i}(z)\qquad (i,j\in J).$$

Here we dropped a scalar factor in the 2×2 and $N\times N$ blocks by using the equation in the 1×1 block.

Note that the difference equations in the 2×2 blocks have the same structure as the one in the case $\mathfrak{g}=\widehat{\mathfrak{sl}}_2$, which was analyzed completely in [23]. Let us summarize the essence of it. The 2×2 block equation consists of two 2nd order q-difference equations of the type

$$(q^c-q^{a+b+1}z)u(q^2z)-\{(q+q^c)-(q^a+q^b)qz\}u(qz)+q(1-z)u(z)=0.$$

This equation has two independent solutions of the form $z^\alpha\sum\limits_{n=0}^{\infty}a_nz^n$ around $z=0$, which are given by the basic hypergeometric series

$${}_2\phi_1\left(\begin{matrix}q^a & q^b\\ & q^c\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{(q^a;q)_n(q^b;q)_n}{(q^c;q)_n(q;q)_n}z^n,$$

and

$$z^{1-c}{}_2\phi_1\left(\begin{matrix}q^{a-c+1} & q^{b-c+1}\\ & q^{2-c}\end{matrix};q,z\right),$$

where $(x;q)_n=\prod\limits_{j=0}^{n-1}(1-xq^j)$, $(x;q)_0=1$. The connection formula for these solutions is well known:

$$\begin{aligned}{}_2\phi_1\left(\begin{matrix}q^a & q^b\\ & q^c\end{matrix};q,1/z\right)&=\frac{\Gamma_q(c)\Gamma_q(b-a)\Theta_q(q^{1-a}z)}{\Gamma_q(b)\Gamma_q(c-a)\Theta_q(qz)}{}_2\phi_1\left(\begin{matrix}q^a & q^{a-c+1}\\ & q^{a-b+1}\end{matrix};q,q^{c-a-b+1}z\right)\\&\quad+\frac{\Gamma_q(c)\Gamma_q(a-b)\Theta_q(q^{1-b}z)}{\Gamma_q(a)\Gamma_q(c-b)\Theta_q(qz)}{}_2\phi_1\left(\begin{matrix}q^b & q^{b-c+1}\\ & q^{b-a+1}\end{matrix};q,q^{c-a-b+1}z\right),\end{aligned}\tag{5.6}$$

where

$$\Gamma_q(z)=\frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}.$$

By using this, one can derive the connection matrices for the 2×2 block parts.

In our case, the initial condition (2.19) leads to

$$f(0)=1,$$
$$\begin{pmatrix}X^{ij}_{ij}(0) & X^{ji}_{ij}(0)\\ X^{ij}_{ji}(0) & X^{ji}_{ji}(0)\end{pmatrix}=\begin{pmatrix}1 & \frac{(q-q^{-1})w_{ij}}{1-w_{ij}}\\ 0 & 1\end{pmatrix}.$$

The solutions to the 1×1 and 2×2 blocks are given as follows.

1×1 block:

$$f(z)=\frac{\{pz\}\{p\xi^2z\}}{\{pq^2z\}\{pq^{-2}\xi^2z\}}\qquad\text{for }A_n,$$

$$= \frac{\{pz\}\{pq^{-2}\xi z\}\{pq^2\xi z\}\{p\xi^2 z\}}{\{pq^2 z\}\{p\xi z\}^2\{pq^{-2}\xi^2 z\}} \qquad \text{for } B_n, C_n, D_n,$$

where

$$\{z\} = \prod_{n,m=0}^{\infty}(1 - z\xi^{2n}p^m).$$

2×2 block:

$$\begin{aligned}
X_{ij}^{ij}(z) &= {}_2\phi_1\left(\begin{matrix} w_{ij}q^2 & q^2 \\ & w_{ij}\end{matrix}; p, pq^{-2}z\right),\\
X_{ij}^{ji}(z) &= \frac{(q-q^{-1})w_{ij}}{1-w_{ij}}\,{}_2\phi_1\left(\begin{matrix} w_{ij}q^2 & pq^2 \\ & pw_{ij}\end{matrix}; p, pq^{-2}z\right),\\
X_{ji}^{ij}(z) &= \frac{(q-q^{-1})pw_{ij}^{-1}}{1-pw_{ij}^{-1}z}\,{}_2\phi_1\left(\begin{matrix} pw_{ij}^{-1}q^2 & pq^2 \\ & p^2w_{ij}^{-1}\end{matrix}; p, pq^{-2}z\right),\\
X_{ji}^{ji}(z) &= {}_2\phi_1\left(\begin{matrix} pw_{ij}^{-1}q^2 & q^2 \\ & pw_{ij}^{-1}\end{matrix}; p, pq^{-2}z\right).
\end{aligned}$$

Then due to the formulae (5.6) and (4.1) or Theorem 4.1, we determine the 1×1 and 2×2 blocks of the dynamical R matrix

$$\begin{aligned}
&(\pi_V\otimes\pi_V)\mathcal{R}(z,\lambda)\\
&\quad = \rho_{ell}(z)\left\{\sum_{\substack{i\in J\\ i\neq 0}} E_{i,i}\otimes E_{i,i} + \sum_{\substack{i\prec j\\ i\neq\pm j}}\Big(R_{ij}^{ij}(z,w_{ij})E_{i,i}\otimes E_{j,j} + R_{ji}^{ji}(z,w_{ij})E_{j,j}\otimes E_{i,i}\Big)\right.\\
&\qquad + \sum_{\substack{i\prec j\\ i\neq -j}}\Big(R_{ij}^{ji}(z,w_{ij})E_{i,j}\otimes E_{j,i} + R_{ji}^{ij}(z,w_{ij})E_{j,i}\otimes E_{i,j}\Big)\\
&\qquad \left. + \sum_{i,j} R_{i-i}^{j-j}(z,w_{ij})E_{i,j}\otimes E_{-i,-j}\right\}
\end{aligned}$$

as follows.

1×1 block:

$$\begin{aligned}
\rho_{ell}(z) &= f(z^{-1})\rho(z)f(z)^{-1}\\
&= q^{-\frac{n}{n+1}}\frac{\{q^2z\}\{q^{-2}\xi^2 z\}\{p/z\}\{p\xi^2/z\}}{\{z\}\{\xi^2 z\}\{pq^2/z\}\{pq^{-2}\xi^2/z\}} \qquad \text{for } A_n^{(1)},\\
&= q^{-1}\frac{\{q^2z\}\{\xi z\}^2\{q^{-2}\xi^2 z\}\{p/z\}\{pq^{-2}\xi/z\}\{pq^2\xi/z\}\{p\xi^2/z\}}{\{z\}\{q^{-2}\xi z\}\{q^2\xi z\}\{\xi^2 z\}\{pq^2/z\}\{p\xi/z\}^2\{pq^{-2}\xi^2/z\}}\\
&\qquad \text{for } B_n^{(1)},\ C_n^{(1)},\ D_n^{(1)}.
\end{aligned}$$

2×2 blocks: for $i\prec j$, $i\neq -j$,

$$\begin{aligned}
R_{ij}^{ij}(z,w_{ij}) &= q\frac{(pw_{ij}^{-1}q^2;p)_\infty(pw_{ij}^{-1}q^{-2};p)_\infty}{(pw_{ij}^{-1};p)_\infty^2}\frac{\Theta_p(z)}{\Theta_p(q^2z)},\\
R_{ji}^{ji}(z,w_{ij}) &= q\frac{(w_{ij}q^2;p)_\infty(w_{ij}q^{-2};p)_\infty}{(w_{ij};p)_\infty^2}\frac{\Theta_p(z)}{\Theta_p(q^2z)},
\end{aligned}$$

$$R_{ij}^{ji}(z,w_{ij}) = \frac{\Theta_p(q^2)}{\Theta_p(w_{ij})}\frac{\Theta_p(w_{ij}z)}{\Theta_p(q^2z)},$$
$$R_{ji}^{ij}(z,w_{ij}) = z\frac{\Theta_p(q^2)}{\Theta_p(pw_{ij}^{-1})}\frac{\Theta_p(pw_{ij}^{-1}z)}{\Theta_p(q^2z)}.$$

By setting $z = q^{2u}$ and using (5.2), we can reexpress these matrix elements in terms of the theta functions with some extra factors including q with fractional power and infinite products. Then making an appropriate gauge transformation, we can sweep away all the extra factors and find that the 1×1 and 2×2 block parts coincide with the part (I) of Jimbo–Miwa–Okado's solution, i.e.

$$R_{ij}^{kl}(z,w_{ij}) \Leftrightarrow W\left(\begin{array}{cc} a & a+\hat{k} \\ a+\hat{\imath} & a+\hat{\imath}+\hat{\jmath}\end{array}\middle| z\right) \qquad \text{in } (I).$$

From 2) of Theorem 3.5, (3.2) and Theorem 5.3, the remaining part (II) is determined uniquely from the part (I). We hence obtain the following theorem.

Theorem 5.6. *For $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$, the vector representation of the universal dynamical R matrix $\mathcal{R}(\lambda)$ coincides with Jimbo–Miwa–Okado's elliptic solutions to the face type YBE.*

To solve the difference equation in the $N\times N$ block directly is an open problem.

For the cases $\mathfrak{g}$ being the twisted affine Lie algebras $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$, Kuniba derived elliptic solutions to the face type YBE. His construction is based on a common structure of the R matrices of the twisted $U_q(\mathfrak{g})$ to those of the $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ types. In fact, the resultant face weights have the common 2×2 block part, as a function of a_μ, to the cases $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$. The simplest $A_2^{(2)}$ case was investigated in [7]. In view of these facts, we expect the same statement as Theorem 5.3 is valid in the twisted cases, too.

Conjecture 5.7. *Similar statement to Theorem 5.6 is true for Kuniba's solution of $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$ types and for Kuniba–Suzuki's solution of $G_2^{(1)}$ type.*

A Proof of Lemma 3.9

We here give a direct proof of Lemma 3.9 and leave a derivation of the q-KZ Equation(3.6) from it as an exercise.

Let $\mathcal{R}$ be the universal R matrix of $U_q(\mathfrak{g})$ and write

$$\mathcal{R} = \sum_j a_j \otimes b_j, \tag{A.1}$$

and set $\mathcal{U} = \sum_j S(b_j)a_j = \sum_j b_j S^{-1}(a_j)$ and $\mathcal{Z} = q^{2\rho}\mathcal{U}$.

Lemma A.1 ([38])**.**

(1) $\mathcal{U}x\mathcal{U}^{-1} = S^2(x) \qquad \forall\, x \in U_q,$

(2) $\mathcal{Z}x\mathcal{Z}^{-1} = x,$

(3) $\mathcal{Z}|_{V(\lambda)} = q^{(\lambda|\lambda+2\rho)}\mathrm{id}_{V(\lambda)}.$

Lemma A.2. *For* (A.1),

$$\sum_j a_j \otimes \Delta(b_j) = \sum_{i,j} a_i a_j \otimes b_j \otimes b_i.$$

Proof. The statement follows $(\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}^{(13)}\mathcal{R}^{(12)}$. ■

Lemma A.3. *Let $\Psi(z)$ denote a vertex operator. Then we have*

$$(\mathrm{id}\otimes a)\Psi(z) = \sum (S(a_{(1)})\otimes 1)\Psi(z)a_{(2)},$$

where we write $\Delta(a) = \sum a_{(1)}\otimes a_{(2)}$.

Proof.

$$\begin{aligned} RHS &= \sum (S(a_{(1)})\otimes 1)\Delta(a_{(2)})\Psi(z) \\ &= \sum (S(a_{(1)})a'_{(2)}\otimes a''_{(2)})\Psi(z) \\ &= \sum (S(a'_{(1)})a''_{(1)}\otimes a_{(2)})\Psi(z) \\ &= \sum (1\otimes \epsilon(a_{(1)})a_{(2)})\Psi(z) \\ &= LHS. \end{aligned}$$

Here we wrote $\Delta(a_{(2)}) = \sum a'_{(2)}\otimes a''_{(2)}$ etc. and used $(\Delta\otimes\mathrm{id})\Delta(a) = (\mathrm{id}\otimes\Delta)\Delta(a)$ in the 3rd line and $m(S\otimes\mathrm{id})\Delta(a) = \epsilon(a)$ in the 4th line. ■

Proof of Lemma 3.9. Let $\lambda,\mu,\nu\in\mathfrak{h}^*$ be level-k elements. Let us set $\tilde{p} = q^{2(k+h^\vee)}$ and consider

$$\widetilde{\Psi}(z_1,z_2) = \big\langle \mathrm{id}\otimes\mathrm{id}\otimes u_\nu^*, (\mathrm{id}\otimes\widetilde{\Psi}_\mu^\nu(z_2)\mathcal{U})\widetilde{\Psi}_\lambda^\mu(\tilde{p}z_1)u_\lambda\big\rangle,$$

where we abbreviate $\widetilde{\Psi}_\mu^{\nu,v_i}(z_2)$ and $\widetilde{\Psi}_\lambda^{\mu,w_j}(z_1)$ as $\widetilde{\Psi}_\mu^\nu(z_2)$ and $\widetilde{\Psi}_\lambda^\mu(z_1)$, respectively. We regard $\mathcal{U}$ and its expression in terms of a_j, b_j as certain images of appropriate representations of $U_q(\mathfrak{g})$ in the following processes. We evaluate $\widetilde{\Psi}(z_1,z_2)$ in the following two ways.

1) Substituting $\mathcal{U} = q^{-2\rho}\mathcal{Z}$ and using the intertwining property (3.3) and Lemma A.1 (3), we have

$$\begin{aligned} \widetilde{\Psi}(z_1,z_2) &= (\mathrm{id}\otimes q^{-2\rho})q^{-(\nu|2\rho)+(\mu|\mu+2\rho)}J_{WV}(\tilde{p}z_1,z_2;\lambda)(w_j\otimes v_i) \\ &= (\mathrm{id}\otimes q^{-2\rho})q^{-(\nu|2\rho)+(\mu|\mu+2\rho)+2(\mathrm{wt}(w_j)|\rho+\lambda)-(\mathrm{wt}(w_j)|\mathrm{wt}(w_j))} \\ &\quad\times J_{WV}(\tilde{p}z_1,z_2;\lambda)(q^{-2\pi_W(\bar{\theta}(\lambda))}\otimes\mathrm{id})(w_j\otimes v_i). \end{aligned}$$

In the last line, we used $(q^{-2\pi_W(\bar{\theta}(\lambda))}\otimes\mathrm{id})|_{w_j\otimes v_i} = q^{-2(\mathrm{wt}(w_j)|\rho+\lambda)+(\mathrm{wt}(w_j)|\mathrm{wt}(w_j))}$.

2) Using $\mathcal{U} = \sum_j b_j S^{-1}(a_j)$ and (A.2), we have

$$\begin{aligned} (\mathrm{id}\otimes\widetilde{\Psi}_\mu^\nu(z_2)\mathcal{U})\widetilde{\Psi}_\lambda^\mu(z_1) &= \sum_j (\mathrm{id}\otimes\Delta(b_j)\widetilde{\Psi}_\mu^\nu(z_2)S^{-1}(a_j))\widetilde{\Psi}_\lambda^\mu(z_1) \\ \mathrm{id}\otimes\widetilde{\Psi}_\mu^\nu(z_2)\mathcal{U})\widetilde{\Psi}_\lambda^\mu(z_1) &= \sum_{i,j} (\mathrm{id}\otimes(b_i\otimes b_j)\widetilde{\Psi}_\mu^\nu(z_2)S^{-1}(a_ja_i))\widetilde{\Psi}_\lambda^\mu(z_1) \\ \mathrm{id}\otimes\widetilde{\Psi}_\mu^\nu(z_2)\mathcal{U})\widetilde{\Psi}_\lambda^\mu(z_1) &= \sum_{i,j} (\mathrm{id}\otimes(S(b_i)\otimes S(b_j))\widetilde{\Psi}_\mu^\nu(z_2)a_ia_j)\widetilde{\Psi}_\lambda^\mu(z_1) \\ \mathrm{id}\otimes\widetilde{\Psi}_\mu^\nu(z_2)\mathcal{U})\widetilde{\Psi}_\lambda^\mu(z_1) &= \sum_{i,j} (\mathrm{id}\otimes(S(b_i)\otimes S(b_j))\widetilde{\Psi}_\mu^\nu(z_2))(1\otimes a_i)(1\otimes a_j)\widetilde{\Psi}_\lambda^\mu(z_1) \end{aligned}$$

In the 3rd line we used $(\mathrm{id}\otimes S)\mathcal{R}=(S^{-1}\otimes\mathrm{id})\mathcal{R}$. Then apply Lemma A.3 and Lemma A.2 twice each, we have

$$(\mathrm{id}\otimes\widetilde{\Psi}^{\nu}_{\mu}(z_2)\mathcal{U})\widetilde{\Psi}^{\mu}_{\lambda}(z_1)=\sum_{i,j,k,l}(S(a_i)S(a_j)\otimes(S(b_ib_k)\otimes S(b_jb_l))\widetilde{\Psi}^{\nu}_{\mu}(z_2))\widetilde{\Psi}^{\mu}_{\lambda}(z_1)a_ka_l.$$

Take the expectation value $\langle\mathrm{id}\otimes\mathrm{id}\otimes u^*_\nu,u_\lambda\rangle$, and use

$$\left\langle u^*_\nu,\sum_l S(b_l)a_lu_\lambda\right\rangle=q^{(\lambda|\nu)},$$
$$\sum_k S(b_k)\otimes a_ku_\lambda=q^{k\Lambda_0+\bar\lambda}\otimes u_\lambda,$$
$$\sum_j S(a_j)\otimes u^*_\nu S(b_j)=q^{-k\Lambda_0-\bar\nu}\otimes u^*_\nu.$$

Noting further that (3.2) implies $\widetilde{\Psi}^{\mu}_{\lambda}(\tilde pz)=(\tilde p^{\Lambda_0}\otimes\mathrm{id})\widetilde{\Psi}^{\mu}_{\lambda}(z)$, we obtain

$$\begin{aligned}\widetilde{\Psi}(z_1,z_2)&=(\tilde p^{\Lambda_0}\otimes\mathrm{id})q^{(\lambda|\nu)}(1\otimes q^{k\Lambda_0+\bar\lambda})(q^{-k\Lambda_0-\bar\nu}\otimes 1)\\&\quad\times\sum_i(S(a_i)\otimes S(b_i))J_{WV}(z_1,z_2)(w_j\otimes v_i)\\&=q^{(\lambda|\nu)+(\lambda+2\rho|\mathrm{wt}(w_j)+\mathrm{wt}(v_i))}(q^{-2\bar\theta(\lambda)}\otimes 1)q^{\pi_{W\otimes V}(\bar T)}\\&\quad\times R_{WV}(z_1/z_2)J_{WV}(z_1,z_2)(w_j\otimes v_i).\end{aligned}$$

Combining 1) and 2), we obtain (3.11). ■

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