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Global Stability of Dynamic Systems of High Order

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Abstract. This paper deals with global asymptotic stability of prolongations of flows induced by specific vector fields and their prolongations. The method used is based on various estimates of the flows.

Key words: global stability; vector fields; prolongations of flows

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1 Introduction

Global stability of dynamic systems is a vast domain in ordinary differential equations and it is one of its main topics. Many works have been done in this context, we list some of them: [3, 4, 5, 6, 7, 8]. However, little is known in the stability of high order (see [10] and [2]). In this paper, we are concerned with the global asymptotic stability of prolongations of flows generated by some specific vector fields and their perturbations. The method used is based on various estimates of the flows and their prolongations. To justify the study of the dynamic of prolongations of flows, we consider the Lie algebra $\chi(\mathbb{R}^n)$ of vector fields on $\mathbb{R}^n$ endowed with the weak topology, which is the topology of the uniform convergence of vector fields and all their derivatives on a compact sets. The Lie bracket is a fundamental operation not only in differential geometry but in many fields of mathematics, such as dynamic and control theory. The invertibility of this latter is of many uses i.e. given any vector fields X, Z find a vector field Y such that $[X,Y]=Z$. In the case of vector fields X defined in a neighborhood of a point a with $X(a)\neq 0$ we have a positive answer: since in this case the vector field X is locally of the form $\frac{\partial}{\partial x_1}$ and the solution is given by

$$Y(x_1,\dots,x_n)=\int_{-r}^{x_1} Z(t,x_2,\dots,x_n)dt,$$

where $\|x\|=\max\limits_{1\leq i\leq n}|x_i|<r$. In the case of singular vector fields, i.e. $X(a)=0$ little is known. Consider a singular vector field X defined in a neighborhood U of the origin 0 with $X(0)=0$ and let ϕ_t be the flow generated by X. Suppose that X is complete and consider a vector field Y defined on an open set $V\supset\phi_t(U)$ for all $t\in\mathbb{R}$. The transportation of a vector field Y along the flow ϕ_t is defined as

$$(\phi_t)_*Y(x)=(D\phi_t\cdot Y)\circ\phi_{-t}(x)$$

and the derivative with respect to t is given as follows

$$\frac{d}{dt}(\phi_t)_*Y=\left[(\phi_t)_*X,(\phi_t)_*Y\right].$$

Put $Y_t = -\int_0^t (\phi_s)_* Z ds$, then

$$[X,Y_t] = -\frac{d}{dt}\Big|_{t=0} (\phi_t)_* \int_0^t (\phi_s)_* Z ds = -\int_0^t \frac{d}{ds}(\phi_s)_* Z ds = Z - (\phi_t)_* Z.$$

So if $(\phi_t)_* Z$ converges to 0 and the integral $Y = -\int_0^{+\infty} (\phi_s)_* Z ds$ is convergent in the weak topology, then Y is a solution of our equation.

As applications of the right invertibility of the bracket operation on germs of vector fields at a singular point we refer the reader to the papers by the authors [1, 2] (see also [10]).

2 Generalities

First we recall some definitions on global asymptotic stability as introduced in [9]. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$, $K \subset \mathbb{R}^n$ is a compact set and f any smooth function on $\mathbb{R}^n$, we put

$$\|f\|_r^K = \sup_{x\in K} \max_{|\alpha|\le r} \|D^\alpha f(x)\| . \tag{1}$$

Definition 1. A point $a \in \mathbb{R}^n$ is said globally asymptotically stable (in brief $G.A.S.$) of the flow ϕ_t if

i) a is an asymptotically stable (in brief $A.S.$) equilibrium of the flow ϕ_t;

ii) for any compact set $K \subset \mathbb{R}^n$ and any $\varepsilon > 0$ there exists $T_K > 0$ such that for any $t \ge T_K$ we have $\|\phi_t(x) - a\| \le \epsilon$ for all $x \in K$.

Definition 2. The point $a \in \mathbb{R}^n$ is said globally asymptotically stable of order r ($1 \le r \le \infty$) for the flow ϕ_t if

i) a is a $G.A.S.$ point for the flow ϕ_t;

ii) for any compact set $K \subset \mathbb{R}^n$ and

$$\forall\, \epsilon > 0, \quad \exists\, T_K > 0 \text{ such that } \forall\, t \ge T_K \;\Rightarrow\; \|\phi_t - aI\|_r^K \le \epsilon,$$

where I denotes the identity map.

A vector field X will be called semi-complete if the X-flow $\phi_t = \exp(tX)$ is defined for all $t \ge 0$.

First we quote the following proposition which characterizes the uniform asymptotic stability, for a proof see the book of W. Hahn [5].

Let $(\phi)_t$ denote a flow defined on $\mathbb{R}^n$.

Proposition 1. *The origin* 0 *in* $\mathbb{R}^n$ *is* $G.A.S.$ *point for the flow* ϕ_t *if for any ball* $B(0,\rho)$*, centered at* 0 *and of radius* $\rho > 0$*, there exist* $t_0 \ge 0$ *and functions* a*,* b *such that*

$$\|\phi_t(x)\| \le a(\|x\|)b(t) \tag{2}$$

with a *a continuous function on* $B(0,\rho)$ *monotonously increasing such that* $a(0) = 0$ *and* b *is a continuous function defined for any* $t \ge t_0$ *monotonously decreasing such that* $\lim_{t\to+\infty} b(t) = 0$.

3 Estimates of prolongations of flows

We start with some perturbations of linear vector fields.

3.1 Perturbation of linear vector fields

Consider the following linear vector field

$$X_1 = \sum_{i=1}^{n} \alpha_i x_i \frac{\partial}{\partial x_i},$$

where the coefficients $\alpha_i \in [a, b] \subset \mathbb{R}$ and are not all 0.

The X_1-flow, $\psi_t^1 = \exp(tX_1)$ is then

$$\psi_t^1(x) = xe^{\alpha t} = \left(x_1 e^{\alpha_1 t}, \ldots, x_n e^{\alpha_n t}\right) \quad \forall\, t \in \mathbb{R} \tag{3}$$

and its estimates are given by

$$\|x\|\, e^{at} \le \|\psi_t^1(x)\| \le \|x\|\, e^{bt}. \tag{4}$$

Consider now a perturbation of the vector field X_1 of the form $Y_1 = X_1 + Z_1$, where Z_1 is a smooth vector field globally Lipschitzian on $\mathbb{R}^n$. The explicit form of the Y_1-flow is then

$$\psi_t^1(x) = xe^{At} + \int_0^t Z_1\left(\psi_s^1(x)\right) ds, \tag{5}$$

where $A = \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{pmatrix}$.

Lemma 1. *If the perturbation Z_1 fulfills*

$$\|Z_1(x)\| \le c_0 \quad \forall\, x \in \mathbb{R}^n \tag{6}$$

then the vector field Y_1 is complete and the Y_1-flow satisfies the estimates

$$\left(\|x\| - \frac{c_0}{a}\right) e^{bt} + \frac{c_0}{a} \le \|\psi_t^1(x)\| \le \left(\|x\| + \frac{c_0}{b}\right) e^{bt} - \frac{c_0}{b}.$$

Proof. Clearly the Y_1-flow ψ_t^1 is bounded for any $t \in [0, T]$ with $T < +\infty$ and any $x \in \mathbb{R}^n$. The same is true if we replace t by $-t$. Then ψ_t^1 is complete.

Consider now the equation

$$\frac{1}{2}\frac{d}{dt}\left\|\psi_t^1(x)\right\|^2 = \left\langle \psi_t^1(x), \alpha\psi_t^1(x) + Z_1\left(\psi_t^1(x)\right)\right\rangle. \tag{7}$$

Letting $y = \|\psi_t^1(x)\|$, we deduce

$$ay^2 - c_0 y \le \frac{1}{2}\frac{d}{dt}y^2 \le by^2 + c_0 y, \qquad y(0) = \|x\|$$

and by integrating we obtain

$$\left(\|x\| - \frac{c_0}{a}\right) e^{bt} + \frac{c_0}{a} \le y \le \left(\|x\| + \frac{c_0}{b}\right) e^{bt} - \frac{c_0}{b}. \qquad \blacksquare$$

Let $B(0, 1)$ be the open unit ball centered at the origin 0.

Lemma 2. *If the perturbation Z_1 fulfills the estimates*

$$\begin{aligned} &\|Z_1(x)\| \leq c_0' \|x\|^{1+m} \quad \forall\, x \in B(0,1) \ \textit{and any integer } m \geq 1, \\ &\|Z_1(x)\| \leq c_0'' \|x\| \quad \textit{for every } x \in \mathbb{R}^n \setminus B(0,1), \end{aligned} \tag{8}$$

then Y_1 is complete and the Y_1-flow fulfills the following estimates for ant $t \geq 0$

$$\begin{aligned} &\|x\| e^{a_0 t} \leq \left\|\psi_t^1(x)\right\| \leq \|x\| e^{b_0 t}, \\ &\|x\| e^{-b_0 t} \leq \left\|\psi_{-t}^1(x)\right\| \leq \|x\| e^{-a_0 t} \end{aligned} \tag{9}$$

with $c_0 = \max\{c_0', c_0''\}$, $a_0 = a - c_0$ and $b_0 = b + c_0$.

Proof. Taking account of the explicit form of the flow (5) and the estimates (8), we deduce that Y_1 is complete. If $x \in B(0,1)$ then $\|Z_1(x)\| \leq c_0' \|x\|^{1+m} \leq c_0' \|x\|$, letting $c_0 = \max\{c_0', c_0''\}$ then $\|Z_1(x)\| \leq c_0 \|x\|$ for any $x \in \mathbb{R}^n$. If we put $y = \left\|\psi_t^1(x)\right\|$ the equation (7) leads to

$$(a - c_0)y \leq \frac{d}{dt} y \leq (b + c_0)y, \qquad y(0) = \|x\|$$

and putting $b_0 = b + c_0$, $a_0 = a - c_0$, we deduce the following estimates

$$\|x\| e^{a_0 t} \leq y \leq \|x\| e^{b_0 t} \quad \text{for any } t \geq 0.$$

The same is also true in the on $\mathbb{R}^n \setminus B(0,1)$. ∎

Lemma 3. *Suppose that all the coefficients α_i are negative, $a \leq \alpha_i \leq b < 0$.*

If the perturbation Z_1 fulfills the estimates

$$\|Z_1(x)\| \leq c_0 \|x\|^{1+m} \qquad \textit{for any } x \in \mathbb{R}^n \ \textit{and any integer } m \geq 1, \tag{10}$$

then the vector field Y_1 is semi-complete and the Y_1-flow satisfies the estimates for any $t \geq 0$

$$\begin{aligned} &\|x\| e^{at} \left(1 - \frac{c_0}{a} \|x\|^m \left(1 - e^{amt}\right)\right)^{-\frac{1}{m}} \\ &\quad \leq \left\|\psi_t^1(x)\right\| \leq \|x\| e^{bt} \left(1 - \frac{c_0}{b} \|x\|^m \left(1 - e^{bmt}\right)\right)^{-\frac{1}{m}}. \end{aligned} \tag{11}$$

Proof. By the relation (5) and the estimates (10), we deduce that the vector field Y_1 is semi-complete. Letting $y = \left\|\psi_t^1(x)\right\|$ and taking into account the equation (7) and the estimates (10) we deduce that

$$ay - c_0 y^{1+m} \leq \frac{d}{dt} y \leq by + c_0 y^{1+m}, \qquad y(0) = \|x\|$$

and by integration we have

$$\|x\| e^{at} \left(1 - \frac{c_0}{a} \|x\|^m \left(1 - e^{amt}\right)\right)^{-\frac{1}{m}} \leq y \leq \|x\| e^{bt} \left(1 - \frac{c_0}{b} \|x\|^m \left(1 - e^{bmt}\right)\right)^{-\frac{1}{m}}.$$ ∎

Example 1. Let the vector field

$$X_3 = \sum_{i=1}^{n} \left(\alpha_i x_i + \beta_i x_i^{1+m_i}\right) \frac{\partial}{\partial x_i}$$

such that all the coefficients fulfilling

$$a \leq \alpha_i \leq b < 0, \qquad a' \leq \beta_i \leq b' \leq 0$$

and all the exponents m_i are even positive integers with $0 < m_0' \leq m_i \leq m_0$. The associated flow $\phi_t^3 = \exp(tX_3)$ is the solution of the dynamic system

$$\frac{d}{dt}\phi_t(x) = X_3 \circ \phi_t(x), \qquad \phi_0(x) = x$$

or in coordinates

$$\frac{d}{dt}\left(\phi_t(x)\right)_i = \alpha_i \left(\phi_t(x)\right)_i + \beta_i \left(\phi_t(x)\right)_i^{1+m_i}, \qquad \phi_0(x) = x.$$

This latter is a Bernoulli type equation and its solution is given by

$$\left(\phi_t^3(x)\right)_i = x_i e^{\alpha_i t}\left(1 + \frac{\beta_i}{\alpha_i} x_i^{m_i}\left(1 - e^{\alpha_i m_i t}\right)\right)^{\frac{-1}{m_i}}. \tag{12}$$

The X_3-flow $\phi_t^3 = \exp\left(tX_3\right)$ then has the explicit form

$$\phi_t^3(x) = x e^{\alpha t}\left(1 + \frac{\beta}{\alpha} x^m\left(1 - e^{\alpha m t}\right)\right)^{\frac{-1}{m}}$$

and the following estimates are true, $\forall\, t \geq 0$

$$\|x\|\, e^{at} \leq \left\|\phi_t^3(x)\right\| \leq \|x\|\, e^{bt}. \tag{13}$$

3.2 Estimation of the k^{th} prolongation of the Y_1-flow

Denote by $\eta_1^1(t,x,\nu) = D\psi_t^1(x)\nu$, where $\nu \in \mathbb{R}^n$, the first derivative with respect to x of the Y_1-flow, solution of the dynamic system

$$\frac{d}{dt}\eta_1^1(t,x,\nu) = \left(D_y X_1 + D_y Z_1\right)\eta_1^1(t,x,\nu), \qquad \eta_1^1(0,x,\nu) = \nu$$

with $y = \psi_t^1(x)$.

Lemma 4. *If the perturbation Z_1 fulfills the estimate*

$$\|DZ_1(x)\| \leq c_1 \quad \textit{for any } x \in \mathbb{R}^n, \tag{14}$$

then the derivative of the Y_1-flow is complete and has the following estimates, for any $t \geq 0$

$$e^{a_1 t} \leq \left\|D\psi_t^1(x)\right\| \leq e^{b_1 t}, \qquad e^{-b_1 t} \leq \left\|D\psi_{-t}^1(x)\right\| \leq e^{-a_1 t} \tag{15}$$

with $a_1 = a - c_1$ and $b_1 = b + c_1$.

Proof. Consider as in previous lemmas the following equation

$$\frac{1}{2}\frac{d}{dt}\left\|\eta_1^1(t,x,\nu)\right\|^2 = \left\langle \eta_1^1(t,x,\nu), \left(\alpha + DZ_1\right)\eta_1^1(t,x,\nu)\right\rangle \tag{16}$$

and put $z = \left\|\eta_1^1(t,x,\nu)\right\|$, so

$$(a - c_1)z^2 \leq \frac{1}{2}\frac{d}{dt}z^2 \leq (b + c_1)z^2, \qquad z(0) = \|\nu\| \tag{17}$$

and then

$$\|\nu\|\, e^{a_1 t} \leq z \leq \|\nu\|\, e^{b_1 t} \quad \text{for any } \ t \geq 0 \ \text{ and } \ \nu \in \mathbb{R}^n. \qquad \blacksquare$$

Lemma 5. *If the perturbation Z_1 fulfils the estimates*

$$\left\| D^l Z_1(x) \right\| \le c_l' \left\| x \right\|^{1-l+m} \text{ for any } x \in B(0,1) \text{ and all integers } m \ge 1,$$
$$\left\| D^l Z_1(x) \right\| \le c_l'' \left\| x \right\|^{1-l} \quad \forall\, x \in \mathbb{R}^n \setminus B(0,1)$$

with $l = 0, 1$, then the first derivative of the Y_1-flow is complete and is estimated by, for any $t \ge 0$

$$e^{a_1 t} \le \| D\psi_t^1(x) \| \le e^{b_1 t}, \qquad e^{-b_1 t} \le \| D\psi_{-t}^1(x) \| \le e^{-a_1 t} \tag{18}$$

with $c_l = \max\{c_l', c_l''\}$, $a_l = a - c_l$ and $b_l = b + c_l$, $l = 0, 1$.

Proof. For any $x \in B(0,1)$ we have $\|D^l Z_1(x)\| \le c_l' \|x\|^{1-l+m} \le c_l' \|x\|^{1-l}$ and letting $c_l = \max\{c_l', c_l''\}$, we get for any $x \in \mathbb{R}^n$ $\|D^l Z_1(x)\| \le c_l \|x\|^{1-l}$. By the same arguments as in previous lemmas we get the estimates (18). ∎

Lemma 6. *Suppose that all the coefficients α_i are negative, $a \le \alpha_i \le b < 0$.*
If the perturbation Z_1 fulfills the estimates

$$\| Z_1(x) \| \le c_0 \|x\|^{1+m}, \quad \| DZ_1(x) \| \le c_1 \|x\|^m \quad \text{for all } x \in \mathbb{R}^n \text{ and any integers } m \ge 1.$$

Then the estimates of the first derivation of the Y_1-flow are as follows, for any $t \ge 0$

$$e^{at}\left(1 - \frac{c_0}{a}\|x\|^m\left(1 - e^{amt}\right)\right)^{-\frac{c_1}{mc_0}} \le \left\| D\psi_t^1(x) \right\| \le e^{bt}\left(1 - \frac{c_0}{b}\|x\|^m\left(1 - e^{bmt}\right)\right)^{-\frac{c_1}{mc_0}}.$$

Proof. Letting $y = \left\| \psi_t^1(x) \right\|$ and $z = \left\| \eta_1^1(t, x, \nu) \right\|$ in equation (16), we get

$$(a - c_1 y^m) z^2 \le \frac{1}{2}\frac{d}{dt} z^2 \le (b + c_1 y^m) z^2, \qquad z(0) = \|\nu\|$$

and taking into account the estimates given by the relation (11), we obtain

$$\|x\|^m e^{mat}\left(1 - \frac{c_0}{a}\|x\|^m\left(1 - e^{amt}\right)\right)^{-1} \le y^m \le \|x\|^m e^{mbt}\left(1 - \frac{c_0}{b}\|x\|^m\left(1 - e^{bmt}\right)\right)^{-1}$$

consequently

$$\|\nu\| \exp\left(at - c_1 \int_0^t \frac{\|x\|^m e^{mas} ds}{1 - \frac{c_0}{a}\|x\|^m\left(1 - e^{ams}\right)}\right)$$
$$\le z \le \|\nu\| \exp\left(bt + c_1 \int_0^t \frac{\|x\|^m e^{mbs} ds}{1 - \frac{c_0}{b}\|x\|^m\left(1 - e^{bms}\right)}\right)$$

which has the solution

$$\|\nu\| e^{at}\left(1 - \frac{c_0}{a}\|x\|^m\left(1 - e^{amt}\right)\right)^{-\frac{c_1}{mc_0}}$$
$$\le z \le \|\nu\| e^{bt}\left(1 - \frac{c_0}{b}\|x\|^m\left(1 - e^{bmt}\right)\right)^{-\frac{c_1}{mc_0}} \quad \text{for } \nu \in \mathbb{R}^n.$$
∎

Example 2. We consider the same vector field as in Example 1. Denote by $\xi_3^1(t, x, \nu) = D\phi_t^3(x)\nu$, $\forall\, \nu \in \mathbb{R}^n$, the first derivation of the X_3-flow. In coordinates, we have for any $i, j = 1, \ldots, n$,

$$\left(\phi_t^3(x)\right)_i = x_i e^{\alpha_i t}\left(1 + \frac{\beta_i}{\alpha_i} x_i^{m_i}\left(1 - e^{\alpha_i m_i t}\right)\right)^{\frac{-1}{m_i}}$$

so we deduce that

$$\frac{\partial}{\partial x_j}\left(\phi_t^3(x)\right)_i = e^{\alpha_i t}\left(1+\frac{\beta_i}{\alpha_i}x_i^{m_i}\left(1-e^{\alpha_i m_i t}\right)\right)^{-1-\frac{1}{m_i}}\delta_j^i$$

and by the estimates (13) we get

$$e^{at} \leq \left\|D\phi_t^3(x)\right\| \leq e^{bt}.$$

The second derivative is

$$\frac{\partial^2}{\partial x_i^2}\left(\phi_t^3(x)\right)_i = -(1+m_i)\frac{\beta_i}{\alpha_i}x_i^{-1+m_i}e^{\alpha_i t}\left(1-e^{\alpha_i m_i t}\right)\left(1+\frac{\beta_i}{\alpha_i}x_i^{m_i}\left(1-e^{\alpha_i m_i t}\right)\right)^{-2-\frac{1}{m_i}}.$$

Consequently, for $l=1,2$ and any $x\in B(0,\rho)$ with $\rho>0$ arbitrary fixed, there are constants $M_l>0$ such that

$$\left\|D^l\phi_t^3(x)\right\| \leq M_l e^{bt}.$$

3.3 Perturbation of a nonlinear vector field

Consider the nonlinear vector field

$$X_2 = \sum_{i=1}^{n}\beta_i x_i^{1+m_i}\frac{\partial}{\partial x_i} \quad \text{with all } \ m_i>0 \ \text{ and all } \ \beta_i\leq 0.$$

The explicit form of the X_2-flow is then given by

$$\phi_t^2(x) = x(1-m\beta t x^m)^{\frac{-1}{m}} \tag{19}$$

for any $t\geq 0$ in the sense

$$\left(\phi_t^2(x)\right)_i = x_i(1-m_i\beta_i t x_i^{m_i})^{\frac{-1}{m_i}}, \qquad 1\leq i\leq n.$$

Lemma 7. *If the following assumptions are true*
i) all the coefficients β_i are non positive, $-a'\leq\beta_i\leq -b'\leq 0$
ii) all the exponents m_i are even positive integers; $0<m_0\leq m_i\leq m_0'$.
Then the vector field X_2 is semi-complete and the X_2-flow satisfies the estimates

$$\|x\|\left(1+b'm_0 t\|x\|^{m_0}\right)^{\frac{-1}{m_0}} \leq \left\|\phi_t^2(x)\right\| \leq \|x\|\left(1+a'm_0' t\|x\|^{m_0'}\right)^{\frac{-1}{m_0'}} \quad \text{for any} \ \ t\geq 0. \tag{20}$$

Proof. Clearly the flow $\phi_t^2=\exp(tX_2)$ given by (19) is semi-complete i.e. defined for all $t\geq 0$.
Consider the equation

$$\frac{1}{2}\frac{d}{dt}\|\phi_t^2(x)\|^2 = \left\langle \phi_t^2(x), \beta\left(\phi_t^2(x)\right)^{1+m}\right\rangle$$

and put $y=\phi_t^2(x)$, then

$$b'y^{2+m_0} \leq \frac{1}{2}\frac{d}{dt}y^2 \leq a'y^{2+m_0'}, \qquad y(0)=\|x\|$$

and we get the estimates given in (20). ■

3.4 Estimation of the k^{th} order derivation of the X_2-flow

Let $\xi_2^1(t,x,\nu) = D\phi_t^2(x)\nu$, $\forall\, \nu \in \mathbb{R}^n$ be the first derivation of the X_2-flow.

By formula (19), we get in coordinates

$$\frac{\partial}{\partial x_j}\left(\phi_t^2(x)\right)_i = \left(1 - m_i\beta_i t x_i^{m_i}\right)^{-1-\frac{1}{m_i}}\delta_i^j \quad \text{with} \quad \delta_i^j = \begin{cases} 1 & \text{if } \ i = j, \\ 0 & \text{if } \ i \neq j, \end{cases}$$

where $i, j = 1, \ldots, n$.

Consequently

$$\left(1 + b'mt\,\|x\|^{m_0}\right)^{-1-\frac{1}{m_0}} \leq \|D\phi_t^2(x)\| \leq \left(1 + a'm_0't\,\|x\|^{m_0'}\right)^{-1-\frac{1}{m_0'}}. \tag{21}$$

To get the estimates of the second derivative, we put

$$w_i = 1 - m_i\beta_i t x_i^{m_i},$$

so

$$\frac{d}{dx_i}w_i = m_i(w_i - 1)x_i^{-1} \qquad \text{and} \qquad \frac{\partial}{\partial x_i}\left(\phi_t^2(x)\right)_i = {w_i}^{-1-\frac{1}{m_i}}.$$

Consequently

$$\frac{\partial^2}{\partial x_i^2}\left(\phi_t^2(x)\right)_i = (1+m_i)x_i^{-1}{w_i}^{-\frac{1}{m_i}}\left(w_i^{-2} - w_i^{-1}\right) = x_i^{-1}{w_i}^{-\frac{1}{m_i}}\left(\frac{a_1^2}{w_i} + \frac{a_2^2}{w_i^2}\right),$$

where a_1^2 and a_2^2 are real constants. Let $\rho > 0$ be any arbitrary and fixed real number, then for any $x \in B(0,\rho)$ and any $t \geq t_0 > 0$ and $l = 1, 2$ there is $M_l > 0$ such that

$$\left\|D^l\phi_t^2(x)\right\| \leq M_l t^{-1-\frac{1}{m_0'}}.$$

Suppose that for $l = 1, \ldots, k-1$, with fixed k, there exist constants a_j^l and $M_l > 0$ such that

$$\frac{\partial^l}{\partial x_i^l}\left(\phi_t^2(x)\right)_i = x_i^{1-l}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{l}\frac{a_j^l}{w_i^j},$$

where a_j^l are real constants and

$$\left\|D^l\phi_t^2(x)\right\| \leq M_l t^{-1-\frac{1}{m_0'}} \quad \forall\, t > 0.$$

For the estimates of the k^{th} derivative, we compute

$$\frac{\partial^k}{\partial x_i^k}\left(\phi_t^2(x)\right)_i = x_i^{1-k}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{k}\frac{a_j^k}{w_i^j},$$

$$\frac{\partial^k}{\partial x_i^k}\left(\phi_t^2(x)\right)_i = \frac{d}{dx_i}x_i^{2-k}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{k-1}\frac{a_j^{k-1}}{w_i^j}$$

$$= x_i^{1-k}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{k-1}\left(\frac{a_j^{k-1}}{w_i^j}(1-k-jm_i) + \frac{a_j^{k-1}}{w_i^{j+1}}(1+jm_i)\right) = x_i^{1-k}{w_i}^{-\frac{1}{m_i}}\sum_{j=1}^{k}\frac{a_j^k}{w_i^j},$$

where a_j^k are real constants.

So we resume

Proposition 2. *Suppose that*

i) all the coefficients satisfy $\beta_i \leq 0$, $-a' \leq \beta_i \leq -b'$,

ii) the exponents m_i *are even natural numbers such that* $0 < m_0 \leq m_i \leq m_0'$.

Let $\rho > 0$ *be any arbitrary fixed real number. For any* $x \in B(0,\rho)$, *for any* $t \geq t_0 > 0$ *and* $\forall\ k \geq 1$ *there exist a constant* $M_k > 0$ *such that*

$$\left\| D^k \phi_t^2(x) \right\| \leq M_k t^{-1-\frac{1}{m_0'}}. \tag{22}$$

3.5 Estimates of the Y_2-flow

Let

$$Y_2 = \sum_{i=1}^{n} \left(\beta_i x_i^{1+m_i} + Z_{2i}(x) \right) \frac{\partial}{\partial x_i}$$

the perturbation of the nonlinear vector field X_2 and denote by $\psi_t^2 = \exp(tY_2)$ the solution of the dynamic system

$$\frac{d}{dt} \psi_t^2(x) = Y_2 \circ \psi_t^2(x), \qquad \psi_0^2(x) = x.$$

In coordinates we have, $i = 1, \ldots, n$,

$$\frac{\partial}{\partial t} \psi_{2,i}(t,x) = \beta_i \psi_{2,i}^{1+m_i}(t,x) + Z_{2i}\left(\psi_t^2(x) \right), \qquad \psi_{2,i}(0,x) = x_i.$$

Putting

$$y_i(t) = \psi_{2,i}^{-m_i}(t,x)$$

and

$$\psi_t^2(x) = y^{\frac{-1}{m}}(t) = \left(y_1^{\frac{-1}{m_1}}(t), \ldots, y_n^{\frac{-1}{m_n}}(t) \right)$$

we get

$$y_i'(t) = -m_i \psi_{2,i}^{-1-m_i}(t,x) \frac{\partial}{\partial t} \psi_{2,i}(t,x).$$

The Cauchy problem reads as

$$y_i'(t) = -m_i \beta_i - m_i \left(y_i(t) \right)^{1+\frac{1}{m_i}} Z_{2i}(y^{\frac{-1}{m}}(t)), \qquad y_i(0) = x_i^{-m_i}$$

and has the following solution

$$y_i(t) = x_i^{-m_i} - m_i \beta_i t - m_i \int_0^t y_i(s)^{1+\frac{1}{m_i}} Z_{2i}(y^{\frac{-1}{m}}(s) ds,$$

i.e.

$$\psi_{2,i}(t,x) = x_i \left(1 - m_i \beta_i t x_i^{m_i} - m_i x_i^{m_i} \int_0^t \psi_i(s,x)^{-1-m_i} Z_{2i}(\psi_s^2(x)) ds \right)^{-\frac{1}{m_i}},$$

so we have the explicit form of the Y_2-flow

$$\psi_t^2(x) = x \left(1 - m\beta t x^m - m x^m \int_0^t \psi_s^2(x)^{-1-m} Z_2(\psi_s^2(x)) ds \right)^{-\frac{1}{m}}. \tag{23}$$

Now we will estimate the Y_2-flow.

Lemma 8. *Suppose that*

i) all the coefficients satisfy $\beta_i \leq 0$, $-a' \leq \beta_i \leq -b'$;

ii) the exponents m_i *are even natural numbers with* $0 < m_0 \leq m_i \leq m_0'$;

iii)

$$\|Z_{2i}(x)\| \leq c_0' |x_i|^{2+m_i} \quad \text{if } \ x \in B(0,1),$$
$$\|Z_{2i}(x)\| \leq c_0'' |x_i|^{1+m_i} \quad \text{if } \ x \in \mathbb{R}^n \setminus B(0,1)$$

with $c_0 = \max\{c_0', c_0''\}$, $b_0 = b' - c_0 > 0$, $a_0 = a' + c_0$.

Then

1) *the vector field* Y_2 *is semi-complete;*

2) *the* Y_2*-flow has the estimates*

$$\|x\| \left(1 + a_0 m_0 t \|x\|^{m_0}\right)^{\frac{-1}{m_0}} \leq \|\psi_t^2(x)\| \leq \|x\| \left(1 + b_0 m_0' t \|x\|^{m_0'}\right)^{\frac{-1}{m_0'}}; \tag{24}$$

3) *let* $\rho > 0$ *and* $t_0 > 0$ *be fixed, then for any* $x \in B(0,\rho)$ *and any* $t \geq t_0 > 0$ *there is a constant* $M_0 > 0$ *such that*

$$\|\psi_t^2(x)\| \leq M_0 \|x\| t^{-\frac{1}{m_0'}}. \tag{25}$$

Proof. Let $x \in B(0,1)$, by assumption we have $\|Z_{2i}(x)\| \leq c_0' |x_i|^{2+m_i} \leq c_0' |x_i|^{1+m_i}$, put $c_0 = \max\{c_0', c_0''\}$ then for any $x \in \mathbb{R}^n$ we deduce $\|Z_{2i}(x)\| \leq c_0 |x_i|^{1+m_i}$. Now taking account of the relation (23) we deduce that for any $t \in [0,T]$

$$\left\|\psi_t^2(x)\right\| \leq \|x\| \left(1 + mt \|x\|^m (b' - c_0)\right)^{-\frac{1}{m}} \leq \|x\|$$

hence the vector Y_2 is semi-complete, i.e. defined for all $t \geq 0$.

Consider the equation

$$\frac{1}{2}\frac{d}{dt}\| \left(\psi_t^2(x)\right)_i \|^2 = \left\langle \left(\psi_t^2(x)\right)_i, \beta_i \left(\psi_t^2(x)\right)_i^{1+m_i} + Z_{2i}\left(\psi_t^2(x)\right) \right\rangle$$

we get $y_i = \| \left(\psi_t^2(x)\right)_i \|$ and $y_i(0) = |x_i|$, so we deduce

$$\frac{1}{2}\frac{d}{dt} y_i^2 \leq (\beta_i + c_0) y_i^{2+m_i} \leq -(b' - c_0) y_i^{2+m_i}$$

and

$$\frac{1}{2}\frac{d}{dt} y_i^2 \geq (\beta_i - c_0) y_i^{2+m_i} \geq -(a' + c_0) y_i^{2+m_i}.$$

We put $b_0 = b' - c_0$ and $a_0 = a' + c_0$, the solutions are estimated as

$$\left(|x_i|^{-m_i} + a_0 m_i t\right)^{-\frac{1}{m_i}} \leq \| \left(\psi_t^2(x)\right)_i \| \leq \left(|x_i|^{-m_i} + b_0 m_i t\right)^{-\frac{1}{m_i}}. \tag{26}$$

Hence, we have the estimate (25). ■

Now, we estimate the first derivation of the Y_2-flow. Let $\eta_2^1(t,x,\nu) = D\psi_t^2(x)\nu$, $\forall\ \nu \in \mathbb{R}^n$ the solution of the dynamic system

$$\frac{d}{dt}\eta_2^1(t,x,\nu) = \left(D_y X_2 + D_y Z_2\right)\eta_2^1(t,x,\nu), \qquad \eta_2^1(0,x,\nu) = \nu$$

with $y = \psi_t^2(x)$.

Lemma 9. *Suppose that*

i) the coefficients are such that $\beta_i \leq 0$, $-a' \leq \beta_i \leq -b'$;

ii) the coefficients m_i *are even natural numbers,* $0 < m_0 \leq m_i \leq m'_0$;

iii)

$$\|D^l Z_{2i}(x)\| \leq c'_l \, |x_i|^{2-l+m_i} \quad \text{if } x \in B\,(0,1)\,,$$
$$\|D^l Z_{2i}(x)\| \leq c''_l \, |x_i|^{1-l+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B\,(0,1)$$

with $l = 0, 1$;

iv)

$$a_0 = a' + c_0, \qquad b_0 = b' - c_0 > 0$$

and

$$a_1 = a'(1 + m_0) + c_1, \qquad b_1 = b'(1 + m_0) - c_1 > 0$$

with $c_l = \max\{c'_l, c''_l\}$.

Then the first derivation of the Y_2*-flow has the following estimates, for any* $t > 0$

$$\left(1 + b_0 m_0 t \, \|x\|^{m_0}\right)^{-\frac{a_1}{b_0 m_0}} \leq \|D\psi_t^2(x)\| \leq \left(1 + a_0 m'_0 t \, \|x\|^{m'_0}\right)^{-\frac{b_1}{a_0 m'_0}}. \tag{27}$$

Let $\rho > 0$ *be arbitrary and fixed for any* $x \in B(0, \rho)$, *and any* $t \geq t_0 > 0$ *there is a constant* $M_1 > 0$ *such that*

$$\|D\psi_t^2(x)\| \leq M_1 t^{-\frac{b_1}{a_0 m'_0}}. \tag{28}$$

Proof. Let $x \in B\,(0,1)$, for $l = 0, 1$ we have

$$\|D^l Z_{2i}(x)\| \leq c'_l \, |x_i|^{2-l+m_i} \leq c'_l \, |x_i|^{1-l+m_i}\,.$$

Let $c_l = \max\{c'_l, c''_l\}$ then for $x \in \mathbb{R}^n$ one has

$$\|D^l Z_{2i}(x)\| \leq c_l \, |x_i|^{1-l+m_i}\,.$$

Consider the equation

$$\frac{1}{2}\frac{d}{dt}\|\eta_2^1(t,x,\nu)\|^2 = \left\langle \eta_2^1(t,x,\nu), (D_y X_2 + D_y Z_2)\, \eta_2^1(t,x,\nu) \right\rangle$$

and put $z(t) = \|\eta_2^1(t,x,\nu)\|$ with $z(0) = \|\nu\|$, then

$$\frac{1}{2}\frac{d}{dt} z^2 \leq \sup_{i=1,\ldots,n} \left(((1+m_i)\beta_i + c_1)\| \left(\psi_t^2(x)\right)_i \|^{m_i}\right) z^2 \leq z^2 \sup_{i=1,\ldots,n} \left(-b_1 \| \left(\psi_t^2(x)\right)_i \|^{m_i}\right)$$

and

$$\frac{1}{2}\frac{d}{dt} z^2 \geq \inf_{i=1,\ldots,n} \left(((1+m_i)\beta_i - c_1)\| \left(\psi_t^2(x)\right)_i \|^{m_i}\right) z^2 \geq z^2 \inf_{i=1,\ldots,n} \left(-a_1 \| \left(\psi_t^2(x)\right)_i \|^{m_i}\right).$$

The solutions fulfill the following estimates

$$\|\nu\| \exp \inf_{i=1,\ldots,n} \left(-a_1 \int_0^t \| \left(\psi_s^2(x)\right)_i \|^{m_i} ds\right)$$
$$\leq z(t) \leq \|\nu\| \exp \sup_{i=1,\ldots,n} \left(-b_1 \int_0^t \| \left(\psi_s^2(x)\right)_i \|^{m_i} ds\right)$$

with, by (26)

$$\frac{|x_i|^{m_i}}{1+a_0 m_i t\,|x_i|^{m_i}} \leq \| \left(\psi_t^2(x)\right)_i \|^{m_i} \leq \frac{|x_i|^{m_i}}{1+b_0 m_i t\,|x_i|^{m_i}}.$$

So we deduce

$$\|\nu\| \exp \inf_{i=1,\dots,n} \left(-a_1 \int_0^t \frac{|x_i|^{m_i}}{1+b_0 m_i s\,|x_i|^{m_i}} ds\right) \\ \leq z(t) \leq \|\nu\| \exp \sup_{i=1,\dots,n} \left(-b_1 \int_0^t \frac{|x_i|^{m_i}}{1+a_0 m_i s\,|x_i|^{m_i}} ds\right).$$

Consequently the solutions satisfy

$$\|\nu\| \inf_{i=1,\dots,n} (1+b_0 m_i t\,|x_i|^{m_i})^{-\frac{a_1}{b_0 m_i}} \leq z(t) \leq \|\nu\| \sup_{i=1,\dots,n} (1+a_0 m_i t\,|x_i|^{m_i})^{-\frac{b_1}{a_0 m_i}}.$$

Then there are constants $m_0 > 0$ and $m_0' > 0$ such that

$$\|\nu\| \left(1+b_0 m_0 t\,\|x\|^{m_0}\right)^{-\frac{a_1}{b_0 m_0}} \leq \|D\psi_t^2(x)\nu\| \\ \leq \|\nu\| \left(1+a_0 m_0' t\,\|x\|^{m_0'}\right)^{-\frac{b_1}{a_0 m_0'}} \quad \forall\, \nu \in \mathbb{R}^n \text{ and for any } t>0.$$

Hence, we have the estimate (28). ■

3.6 Perturbation of binomial vector fields

Let

$$Y_3 = \sum_{i=1}^n \left(\alpha_i x_i + \beta_i x_i^{1+m_i} + Z_{3i}(x)\right) \frac{\partial}{\partial x_i}$$

with $a \leq \alpha_i \leq b < 0$, $a' \leq \beta_i \leq b' \leq 0$ and $0 < m_0 \leq m_i \leq m_0'$, be the perturbation of the binomial vector field X_3 and let $\psi_t^3 = \exp(tY_3)$ be the Y_3-flow which is the solution of the dynamic system

$$\frac{d}{dt}\psi_t(x) = Y_3 \circ \psi_t(x), \qquad \psi_0(x) = x$$

and in coordinates, we get

$$\frac{\partial}{\partial t}\psi_{3,i}(t,x) = \alpha_i \psi_{3,i}(t,x) + \beta_i \psi_{3,i}^{1+m_i}(t,x) + Z_{3,i}\left(\psi_t^3(x)\right), \qquad \psi_i(0,x) = x_i$$

which is a Bernoulli type equation and by the same method as in the proof of previous lemmas and with putting

$$y_i(t) = \psi_{3,i}^{-m_i}(t,x)$$

and

$$\psi_t^3(x) = y^{\frac{-1}{m}}(t) = \left(y_1^{\frac{-1}{m_1}}(t), \dots, y_n^{\frac{-1}{m_n}}(t)\right)$$

we get the solution

$$\psi_{3,i}(t,x) = x_i e^{\alpha_i t}\left(1 + \frac{\beta_i}{\alpha_i} x_i^{m_i}(1 - e^{\alpha_i m_i t}) \right.$$
$$\left. - m_i x_i^{m_i} \int_0^t \left[\psi_{3,i}(s,x)\right]^{-1-m_i} Z_{3,i}(\psi_s^3(x)) e^{\alpha_i m_i s} ds\right)^{\frac{-1}{m_i}}$$

and the implicit form of the Y_3-flow reads as

$$\psi_t^3(x) = x e^{\alpha t}\left(1 + \frac{\beta}{\alpha} x^m (1 - e^{\alpha m t}) - m x^m \int_0^t \left[\psi_s^3(x)\right]^{-1-m} Z_3(\psi_s^3(x)) e^{\alpha m s} ds\right)^{-\frac{1}{m}}. \tag{29}$$

3.7 Estimation of the Y_3-flow

By the same arguments as in the previous, we get the following estimates of the Y_3-flow.

Lemma 10. *If the following assumptions are true*

i) all the coefficients α_i are negative, $-a \leq \alpha_i \leq -b < 0$;

ii) all the coefficients β_i are non positive, $-a' \leq \beta_i \leq -b'$;

iii) the exponents m_i are even natural numbers with $0 < m_0 \leq m_i \leq m_0'$;

iv)

$$\|Z_{3i}(x)\| \leq c_0' |x_i|^{2+m_i} \quad \text{if } x \in B(0,1),$$
$$\|Z_{3i}(x)\| \leq c_0'' |x_i|^{1+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B(0,1)$$

with $c_0 = \max\{c_0', c_0''\}$, $b_0 = b' - c_0 > 0$, $a_0 = a' + c_0$.

Then

1) there exist constants $m > 0$ and $m' > 0$ such that the Y_3-flow has the estimates, $\forall\, t \geq 0$

$$\|x\| e^{-at}\left(1 + \frac{a_0}{a}\|x\|^m (1 - e^{-amt})\right)^{-\frac{1}{m}}$$
$$\leq \|\psi_t^3(x)\| \leq \|x\| e^{-bt}\left(1 + \frac{b_0}{b}\|x\|^{m'}(1 - e^{-bm't})\right)^{-\frac{1}{m'}};$$

2) for any $t > 0$ there are positive constants c_1 and c_2 such that

$$c_1 \|x\| e^{-at} \leq \|\psi_t^3(x)\| \leq c_2 \|x\| e^{-bt}; \tag{30}$$

3) the vector field Y_3 is semi-complete.

By similar calculations as in previous lemmas, we get the following estimates to the first derivative of the Y_3-flow.

Lemma 11. *Suppose that*

i) all the coefficients α_i are negative, $-a \leq \alpha_i \leq -b < 0$;

ii) all the coefficients $\beta_i$are non positive, $-a' \leq \beta_i \leq -b'$;

iii) the exponents m_i are even natural numbers such that $0 < m_0 \leq m_i \leq m_0'$;

iv)

$$\|D^l Z_{3i}(x)\| \leq c_l' |x_i|^{2-l+m_i} \quad \text{if } x \in B(0,1),$$
$$\|D^l Z_{3i}(x)\| \leq c_l'' |x_i|^{1-l+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B(0,1)$$

with $l = 0, 1$;

v)

$$a_0 = a' + c_0, \qquad b_0 = b' - c_0 > 0$$

and

$$a_1 = a'(1+m_0) + c_1, \qquad b_1 = b'(1+m_0) - c_1 > 0$$

with $c_l = \max\{c_l', c_l''\}$.

Then there exist constants $m > 0$ *and* $m' > 0$ *such that for any* $t \geq 0$

$$e^{-at}\left(1 + \frac{b_0}{b}\|x\|^m\left(1 - e^{-bmt}\right)\right)^{-\frac{a_1}{b_0 m}} \leq \left\|D\psi_t^3(x)\right\| \leq e^{-bt}\left(1 + \frac{a_0}{a}\|x\|^{m'}\left(1 - e^{-am't}\right)\right)^{-\frac{b_1}{a_0 m'}}$$

and for any $t \geq 0$, *there is a constant* $M_1 > 0$ *such that*

$$\left\|D\psi_t^3(x)\right\| \leq M_1 e^{-bt}. \tag{31}$$

4 Global stability of prolongations of flows

With notations of the previous sections, we will give global stability of some flows.

4.1 Global stability of the Y_1-flow

Lemma 12. *Let the vector fields*

$$Y_1 = \sum_{i=1}^{n}\left(\alpha_i x_i + Z_{1i}(x)\right)\frac{\partial}{\partial x_i}$$

with the following assumptions

i) all the coefficients are negative, $-a \leq \alpha_i \leq -b < 0$;

ii)

$$\|Z_1(x)\| \leq c_0' \|x\|^{1+m} \quad \forall\, x \in B(0,1) \text{ and } \forall m \geq 1,$$
$$\|Z_1(x)\| \leq c_0'' \|x\| \quad \forall\, x \in \mathbb{R}^n \setminus B(0,1);$$

iii) $b_0 = b - c_0 > 0$, *where* $c_0 = \max\{c_0', c_0''\}$.

Then the origin 0 *is a globally asymptotically stable equilibrium to the* Y_1*-flow* ψ_t^1 *on* $\mathbb{R}^n$.

Proof. Let $\psi_t^1 = \exp(tY_1)$ be the Y_1-flow, then by the assumptions and the estimates given by Lemma 2 we get that

$$\left\|\psi_t^1(x)\right\| \leq \|x\| e^{-b_0 t} \quad \forall\, t \geq 0 \text{ and } \forall x \in \mathbb{R}^n$$

and by Proposition 1, the origin 0 is $G.A.S.$ for ψ_t^1 on $\mathbb{R}^n$. ∎

Example 3. We consider the vector field

$$X_3 = \sum_{i=1}^{n}\left(\alpha_i x_i + \beta_i x_i^{1+m_i}\right)\frac{\partial}{\partial x_i}$$

of Example 1 with $a \leq \alpha_i \leq b < 0$, $a' \leq \beta_i \leq b' \leq 0$. The X_3-flow $\phi_t^3 = \exp(tX_3)$ is then given by

$$\phi_t^3(x) = xe^{\alpha t}\left(1 + \frac{\beta}{\alpha}x^m\left(1 - e^{\alpha m t}\right)\right)^{\frac{-1}{m}}.$$

Let $\rho > 0$ be arbitrary and fixed real number. By the estimates (13), we have for any $x \in B(0,\rho)$ and any $t \geq t_0 \geq 0$

$$\left\|\phi_t^3(x)\right\| \leq \|x\| e^{-bt}.$$

By Proposition1 the origin 0 is a $G.A.S.$ for the flow ϕ_t^3 on $\mathbb{R}^n$.

4.2 Global stability of the first prolongation of the Y_1-flow

Lemma 13. *With the same assumptions as in Lemma 12 and the following conditions*

$$\|DZ_1(x)\| \leq c_1' \|x\|^m \quad \forall\, x \in B(0,1) \text{ and } \forall\, m \geq 1,$$
$$\|DZ_1(x)\| \leq c_1'' \quad \forall\, x \in \mathbb{R}^n \setminus B(0,1)$$

with $b_1 = b - c_1 > 0$ *and* $c_1 = \max\{c_1', c_1''\}$.

Then the origin 0 *is a globally asymptotically stable for the first prolongation of the* Y_1*-flow* ψ_t^1 *on* $\mathbb{R}^n$.

Proof. By the estimates (18) and the hypothesis we deduce that

$$\|D\psi_t^1(x)\nu\| \leq \|\nu\| e^{-b_1 t} \quad \forall\, t > 0, \quad \forall\, \nu \in \mathbb{R}^n$$

and by Proposition 1,we obtain that the origin 0 is a $G.A.S.$ equilibrium on $\mathbb{R}^n$ for $\eta_1^1(t,x,v) = D\psi_t^1(x)\nu$. ■

4.3 Global stability of the k^{th} prolongation of the Y_1-flow

Suppose that

i) all the coefficients are negative, $-a \leq \alpha_i \leq -b < 0$;

ii) for any $l = 1, \ldots, k-1$

$$\|D^l Z_1(x)\| \leq c_l' \|x\|^{1-l+m} \quad \text{for any } x \in B(0,1) \text{ and for any integer } m \geq l-1,$$
$$\|D^l Z_1(x)\| \leq c_l'' \quad \forall\, x \in \mathbb{R}^n \setminus B(0,1),$$
$$a_0 = a + c_0, \qquad b_0 = b - c_0 > 0,$$
$$a_1 = a + c_1, \qquad b_1 = b - c_1 > 0$$

with $c_l = \max\{c_l', c_l''\}$, $b_l = c_l \;\forall\, l \geq 2$.

Put $\eta_1^l(t,x,\nu,\ldots,\nu) = D^k\psi_t^1(x)\nu^k$, where $\nu \in \mathbb{R}^n$. Since by Lemmas 12 and 13 the origin 0 is an $G.A.S.$ equilibrium for η_1^l, with $l = 0,1$, on $\mathbb{R}^n$, we suppose that this property remains true for $l = 0,1,\ldots,k-1$ with $k \geq 2$ i.e. for any $\rho > 0$ and any $x \in B(0,\rho)$ there exist constants $M_l > 0$ such that for any $t \geq t_0 > 0$

$$\|D^l\psi_t^1(x)\| \leq M_l e^{-b_1 t}.$$

We will show that the origin 0 is a $G.A.S.$ equilibrium for η_1^k on $\mathbb{R}^n$. $\eta_1^k(t,x,\nu,\ldots,\nu) = D^k\psi_t^1(x)\nu^k$ is solution of the dynamic system

$$\frac{d}{dt}\eta_1^k = D_y Y_1 \cdot \eta_1^k + G_1^k(t,x,\nu), \qquad \eta_1^k(0,x,\nu,\ldots,\nu) = \nu$$

with $y = \psi_t^1(x)$ and

$$G_1^k(t,x,\nu) = \sum_{l=2}^{k} D_y^l Y_1(y) \sum_{\substack{i_1+\dots+i_l=k \\ i_j>0}} \left(\prod_{j=1}^{l} D^{i_j}\psi_t^1(x)\nu^{i_j} \right)$$
$$= \sum_{l=2}^{k-1} D_y^l Z_1(y) \sum_{\substack{i_1+\dots+i_l=k \\ i_j>0}} \left(\prod_{j=1}^{l} D^{i_j}\psi_t^1(x)\nu^{i_j} \right) + D_y^k Z_1(y) \left(D\psi_t^1(x)\nu \right)^k .$$

Consequently we get

$$\eta_1^k(t,x,\nu,\dots,\nu) = D\psi_t^1(x)\nu + \int_0^t D\Psi_{t-s}^1(\psi_s^1(x)) G_1^k(s,x,\nu)ds.$$

The integral is well defined at $s=0$, since

$$\lim_{s\to 0^+} D\psi_{t-s}^1(\psi_s^1(x)) = D\psi_t^1(x)$$

and there exist constants $A_l > 0$ such that

$$\lim_{s\to 0^+} G_1^k(s,x,\nu) = \sum_{l=2}^{k} A_l D_y^l Z_1(y)\nu^k.$$

We will show that it converges uniformly with respect to x as $t + \infty$. Put

$$I_k = \int_0^t \|D\psi_{t-s}^1(\psi_s^1(x))\| \|G_1^k(s,x,\nu)\| ds.$$

Since $\|D^l Z_1(x)\| \le c_l \ \forall\ l \ge 1,\ \forall\ x \in \mathbb{R}^n$, there are constants $b_l > 0$ such that $\forall\ y \in \mathbb{R}^n$, $\|D_y^l Y_1(y)\| \le b_l$ and by the assumption of recurrence there exist constants $M_l > 0$ such that

$$\|D^l \psi_t^1(x)\| \le M_l e^{-b_1 t} \quad \forall\ t \ge 0.$$

We deduce that there is a constant $C_k > 0$ such that

$$I_k \le \sum_{l=2}^{k} b_l M_l \int_0^t e^{-b_1(t-s+sl)} ds \le C_k e^{-b_1 t}.$$

So for any $x \in \mathbb{R}^n$ one has

$$\lim_{t\to+\infty} I_k \le \sum_{l=2}^{k} M_l b_l \|\nu\|^l \int_0^{+\infty} e^{-b_1 s} ds = \frac{1}{b_1}\sum_{l=2}^{k} M_l b_l \|\nu\|^l$$

and the integral I_k is uniformly convergent with respect to $x \in \mathbb{R}^n$ as $t \to +\infty$. Consequently

$$\lim_{t\to+\infty} \|\eta_1^k\| = \lim_{t\to+\infty} \|D\psi_t^1(x)\nu\| + \int_0^{+\infty} \lim_{t\to+\infty} \|D\psi_{t-s}^1(\psi_s^1(x))\| \|G_1^k(s,x,\nu)\| ds = 0$$

and there is a constant $M_k' > 0$ such that

$$\|\eta_1^k\| \le \|D\psi_t^1(x)\nu\| + \int_0^{+\infty} \|D\psi_{t-s}^1(\psi_s^1(x))\| \|G_1^k(s,x,\nu)\| ds \le M_k' \|\nu\|^k e^{-b_1 t}.$$

This show by Proposition 1 that the origin 0 is a $G.A.S.$ equilibrium to η_1^k on $\mathbb{R}^n$. We formulate our proving as follows

Proposition 3. *Let $k \ge 0$ be any integer. The origin 0 is a $G.A.S.$ equilibrium of order k for the Y_1-flow and there is a constant $M_k > 0$ such that $\forall\ t > 0$*

$$\|D^k \psi_t^1(x)\| \le M_k e^{-b_1 t}, \qquad \|D^k \psi_{-t}^1(x)\| \le M_k e^{a_1 t}. \tag{32}$$

5 Global stability of a flow generated by nonlinear perturbed vector fields

First we will start with monomial vector fields.

5.1 Global stability of the X_2-flow

Let

$$X_2 = \sum_{i=1}^{n} \beta_i x_i^{1+m_i} \frac{\partial}{\partial x_i}$$

with

(i) all the coefficients $\beta_i \leq 0$ such that $-a' \leq \beta_i \leq -b'$;

(ii) all the exponents m_i are even natural integers with $0 < m_0 \leq m_i \leq m_0'$.

Let $\phi_t^2 = \exp(tX_2)$ be the X_2-flow. By the estimations (19) we obtain

$$\left\|\phi_t^2(x)\right\| \leq \|x\| \left(1 + a' m_0' t \|x\|^{m_0'}\right)^{\frac{-1}{m_0'}}.$$

Let $\rho > 0$ be arbitrary fixed, for any $x \in B(0, \rho)$ and any $t \geq t_0 > 0$ there is a constant $M_0 > 0$ such that

$$\|\phi_t^2(x)\| \leq M_0 \|x\| t^{-\frac{1}{m_0'}}.$$

By Proposition 1, the origin is a globally asymptotically stable equilibrium to the flow ϕ_t^2 on $\mathbb{R}^n$.

Let $l = 1, 2, \ldots$ any positive integer. By Proposition 2, we have: for any fixed $\rho > 0$, and all $x \in B(0, \rho)$ and $t \geq t_0 > 0$, there exist constants $M_l > 0$ and $M_l' > 0$ such that

$$\|D^l \phi_t^2(x)\| \leq M_l \, t^{-1-\frac{1}{m_0'}} \qquad \text{and} \qquad \|D^l \phi_0^2(x)\| \leq M_l'.$$

So the origin 0 is a $G.A.S.$ equilibrium for $D^l \phi_t^2(x)$ on $\mathbb{R}^n$.

Resuming our proving, we get

Proposition 4. *Let $k \geq 0$ be any integer. Under the above conditions (i) and (ii), the origin 0 is a G.A.S. of order k for the X_2-flow on $\mathbb{R}^n$.*

5.2 Global stability of high order of the Y_2-flow

Let

$$Y_2 = \sum_{i=1}^{n} \left(\beta_i x_i^{1+m_i} + Z_{2,i}(x)\right) \frac{\partial}{\partial x_i}$$

be a smooth vector field on $\mathbb{R}^n$ such that

i) all the coefficients $\beta_i \leq 0$ are non negative with $-a' \leq \beta_i \leq -b'$;

ii) m_i are even natural numbers with $0 < m_0 \leq m_i \leq m_0'$;

iii) for $k = 0, \ldots, 1 + m_i$

$$\begin{aligned} &\|D^k Z_{2i}(x)\| \leq c_k' \, |x_i|^{2-k+m_i} \quad \text{if } x \in B(0,1)\,; \\ &\|D^k Z_{2i}(x)\| \leq c_k'' \, |x_i|^{1-k+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B(0,1)\,; \end{aligned}$$

iv) for any $k \geq 2 + m_i$

$$\|D^k Z_{2i}(x)\| \leq c_k;$$

v)

$$a_0 = a' + c_0, \qquad a_1 = a'(1 + m_0) + c_1,$$
$$b_0 = b' - c_0 > 0, \qquad b_1 = b'(1 + m_0) - c_1 > a_0 m_0'$$

with $c_k = \max\{c_k', c_k''\}$.

Remark 1. If $x \in B(0,1)$ then $\|D^k Z_{2i}(x)\| \leq c_k' |x_i|^{2-k+m_i} \leq c_k' |x_i|^{1-k+m_i}$. Putting $c_l = \max\{c_l', c_l''\}$, we deduce that for any $x \in \mathbb{R}^n$ have $\|D^k Z_{2i}(x)\| \leq c_k |x_i|^{1-k+m_i}$.

5.2.1 Global stability of the Y_2-flow on $\mathbb{R}^n$

Let $\psi_t^2 = \exp(tY_2)$ be the Y_2-flow and let $\rho > 0$ be arbitrary and fixed, so by the estimates (25) for all $x \in B(0,\rho)$ and all $t \geq t_0 > 0$ there is a constant $M_0 > 0$ such that

$$\|\psi_t^2(x)\| \leq M_0 \|x\| t^{-\frac{1}{m_0'}}.$$

So by Proposition 1, the origin 0 is a $G.A.S.$ equilibrium for the Y_2-flow ψ_t^2 on $\mathbb{R}^n$.

5.2.2 Global stability of prolongation of the Y_2-flow on $\mathbb{R}^n$

We proceed by recurrence. Since it is already true for $k = 0$, we suppose that for any $l = 1, \ldots, k-1$, with $k \geq 2$, the origin 0 is a $G.A.S.$ to $D^l \psi_t^2(x)$ on $\mathbb{R}^n$ that is to say for any fixed $\rho > 0$, all $x \in B(0,\rho)$ and all $t \geq t_0 > 0$ there are constants $M_l > 0$ such that

$$\|D^l \psi_t^2(x)\| \leq M_l t^{-\frac{b_1}{a_0 m_0'}} \qquad \text{and} \qquad \|D^l \psi_0^2(x)\| \leq M_l'.$$

We will show that 0 is a $G.A.S.$ for $D^k \psi_t^2(x)$ on $\mathbb{R}^n$.
Put $\eta_2^k(t, x, \nu, \ldots, \nu) = D^k \psi_t^2(x) \nu^k \ \forall \ \nu \in \mathbb{R}^n$ which is solution of the dynamic system

$$\frac{d}{dt}\eta_2^k = D_y Y_2 \cdot \eta_2^k + G_2^k(t, x, \nu), \qquad \eta_2^k(0, x, \nu, \ldots, \nu) = \nu$$

with $y = \psi_t^2(x)$ and

$$G_2^k(t, x, \nu) = \sum_{l=2}^{k} D_y^l Y_2(y) \sum_{\substack{i_1 + \cdots + i_l = k \\ i_j > 0}} \left(\prod_{j=1}^{l} D^{i_j} \psi_t^2(x) \nu^{i_j} \right).$$

By the method of the resolvent, we deduce

$$\eta_2^k(t, x, \nu, \ldots, \nu) = D\psi_t^2(x)\nu + \int_0^t D\psi_{t-s}^2(\psi_s^2(x)) G_2^k(s, x, \nu) ds.$$

Clearly the integral

$$I_k^1 = \int_0^1 \|D\psi_{t-s}^2(\psi_s^2(x))\| \|G_2^k(s, x, \nu)\| ds$$

is well defined at $s = 0$ and $s = t$, since

$$\lim_{s\to 0^+} D\psi^2_{t-s}(\psi^2_s(x)) = D\psi^2_t(x).$$

By the recurrent assumption $D^l\psi^2_0(x)$ are bounded and there exist constants $A_l > 0$ such that

$$\lim_{s\to 0^+} \|G^k_2(s,x,\nu)\| \leq \sum_{l=2}^{k} A_l \|D^l_x Y_2(x)\nu^l\|.$$

In the same way

$$\lim_{s\to t^-} D\psi^2_{t-s}(\psi^2_s(x)) = \text{identity}.$$

Now, we have to show that

$$I^2_k = \int_1^t \|D\psi^2_{t-s}(\psi^2_s(x))\|\|G^k_2(s,x,\nu)\|ds$$

converges uniformly on any compact set $K \subset \mathbb{R}^n$ as $t \to 0$.

Let $x \in K$, by the relations (26) and (28) we get for all $t \geq 0$

$$\|x\|\left(1 + a_0 m_0 t \|x\|^{m_0}\right)^{\frac{-1}{m_0}} \leq \|\psi^2_t(x)\| \leq \|x\|\left(1 + b_0 m'_0 t \|x\|^{m'_0}\right)^{\frac{-1}{m'_0}},$$
$$\left(1 + b_0 m_0 t \|x\|^{m_0}\right)^{-\frac{a_1}{b_0 m_0}} \leq \|D\psi^2_t(x)\| \leq \left(1 + a_0 m'_0 t \|x\|^{m'_0}\right)^{-\frac{b_1}{a_0 m'_0}}.$$

So $\|y\| = \|\psi^2_t(x)\| \leq \|x\|$ and $\left\|D\psi^2_{t-s}(\psi^2_s(x))\right\|$ is bounded. Since for any $x \in \mathbb{R}^n$ and any $l = 1, \dots, 1+m_i$, $\|D^l Z_{2i}(x)\| \leq c_l |x_i|^{1-l+m_i}$ then $D^l_y Y_2(y)$ are bounded. Now by the assumption of recurrence there exist constants $M_l > 0$ such that for any $t > 0$

$$\|D^l\psi^2_t(x)\| \leq M_l t^{-\frac{b_1}{a_0 m'_0}}$$

with $a_0 m'_0 < b_1$ i.e. $\frac{b_1}{a_0 m'_0} > 1$, and we deduce the existence of constants $C_l > 0$ such that

$$\lim_{t\to+\infty} I^2_k \leq \sum_{l=2}^{k} C_l \int_1^{+\infty} s^{-\frac{lb_1}{a_0 m'_0}} ds \leq \sum_{l=2}^{k} C_l \left(\frac{lb_1}{a_0 m'_0} - 1\right)^{-1}.$$

The integral I^2_k converges uniformly on any compact $K \subset \mathbb{R}^n$ as $t \to +\infty$.

Now since the integral is well defined at $s = 0$, then

$$\lim_{t\to 0} \|\eta^k_2(t,x,\nu,\dots,\nu)\| \leq \lim_{t\to 0} \|D\psi^2_t(x)\nu\| = \|\nu\|$$

hence there is a constant $M'_k > 0$ such that

$$\|D^k\psi^2_0(x)\| \leq M'_k.$$

In the same way as above the integral $\int_0^t \|D\psi^2_{t-s}(\psi^2_s(x))\|\|G^k_2(s,x,\nu)\|ds$ is well defined and putting $\tau = \frac{s}{t}$ we obtain

$$\eta^k_2(t,x,\nu,\dots,\nu) = D\psi^2_t(x)\nu + t\int_0^1 D\psi^2_{t(1-\tau)}(\psi^2_{t\tau}(x))G^k_2(t\tau,x,\nu)d\tau.$$

Since $b_1 = b'(1+m_0) - c_1 > a_0 m_0'$, by the estimates (26) and (28), we deduce the existence of a constant $M_k > 0$ such that

$$\|\eta_2^k(t,x,\nu,\dots,\nu)\| \leq \|D\psi_t^2(x)\nu\| + t\int_0^1 \|G_2^k(t\tau,x,\nu)\|d\tau$$
$$\leq \|D\psi_t^2(x)\nu\| + t\sum_{l=2}^{k}\int_0^1 \frac{(t\tau)^{-\frac{lb_1}{a_0 m_0'}}\|x\|^{1+m_0'-l}}{\left(1+b_0 m_0' t\tau\|x\|^{m_0'}\right)^{\frac{1+m_0'-l}{m_0'}}}d\tau \leq M_k t^{-\frac{b_1}{a_0 m_0'}}.$$

Which shows that the origin 0 is a $G.A.S.$ equilibrium for η_2^k on $\mathbb{R}^n$. We formulate this fact as

Proposition 5. *Let $k \geq 0$ be any integer. Under the above conditions (i), (ii), (iii), (iv) and (v), the origin 0 is a G.A.S. of order k on $\mathbb{R}^n$ for the Y_2-flow and there is a constant $M_k > 0$ such that for any $t \geq t_0 > 0$*

$$\|D^k\psi_t^2(x)\| \leq M_k t^{-\frac{b_1}{a_0 m_0'}}. \tag{33}$$

5.3 Global stability of prolongations of the Y_3-flow

Let

$$Y_3 = \sum_{i=1}^{n}\left(\alpha_i x_i + \beta_i x_i^{1+m_i} + Z_{3i}(x)\right)\frac{\partial}{\partial x_i}$$

with

i) all the coefficient α_i are negative with $-a \leq \alpha_i \leq -b$;

ii) all the coefficients $\beta_i \leq 0$ and $-a' \leq \beta_i \leq -b'$;

iii) the exponents m_i are even natural numbers with $0 < m_0 \leq m_i \leq m_0'$;

iv) For any $k = 0,\dots,1+m_i$

$$\|D^k Z_{3i}(x)\| \leq c_k' |x_i|^{2-k+m_i} \quad \text{if } x \in B(0,1),$$
$$\|D^k Z_{3i}(x)\| \leq c_k'' |x_i|^{1-k+m_i} \quad \text{if } x \in \mathbb{R}^n \setminus B(0,1);$$

v) for any $k \geq 2+m_i$

$$\|D^k Z_{3i}(x)\| \leq c_k;$$

vi)

$$a_0 = a' + c_0, \qquad a_1 = a'(1+m_0) + c_1,$$
$$b_0 = b' - c_0 > 0, \qquad b_1 = b'(1+m_0) - c_1 > 0$$

with $c_k = \max\{c_k', c_k''\}$.

Remark 2. If $x \in B(0,1)$ then $\|D^k Z_{3i}(x)\| \leq c_k' |x_i|^{2-k+m_i} \leq c_k' |x_i|^{1-k+m_i}$.

Let $c_l = \max\{c_l', c_l''\}$, for any $x \in \mathbb{R}^n$ one has $\|D^k Z_{3i}(x)\| \leq c_k |x_i|^{1-k+m_i}$.

5.3.1 Global stability of the Y_3-flow $\mathbb{R}^n$

Denote by $\psi_t^3 = \exp(tY_3)$, by the estimates (30), we have

$$\|\psi_t^3(x)\| \le C\|x\|e^{-bt} \quad \forall\, t > 0 \text{ and } \forall\, x \in \mathbb{R}^n,$$

where $C > 0$ is a constant. So by Proposition 1, 0 is a $G.A.S.$ on $\mathbb{R}^n$. We proceed by recurrence; since the property is true in case $k = 0$, we assume that the property remains true for any $l = 1, \dots, k-1$, with k fixed i.e. 0 is a global $G.A.S.$ of $\eta_3^l(t, x, \nu, \dots \nu) = \|D^l \psi_t^3(x)\nu^k\|$ on $\mathbb{R}^n$ and there exist constants $M_l > 0$ such that for any $t > 0$

$$\|D^l \psi_t^3(x)\| \le M_l e^{-bt}.$$

We will show that 0 is a $G.A.S.$ equilibrium to η_3^k on $\mathbb{R}^n$.

$\eta_3^k(t, x, \nu, \dots, \nu)$ is a solution to the dynamic system

$$\frac{d}{dt}\eta_3^k = D_y \eta_3^k + G_3^k(t, x, \nu)$$

with $y = \psi_t^3(x)$ and

$$G_3^k(t, x, \nu) = \sum_{l=2}^{k} D_y^l Y_3(y) \sum_{\substack{i_1+\dots+i_l=k \\ i_j>0}} \left(\prod_{j=1}^{l} D^{i_j} \psi_t^3(x) \nu^{i_j} \right).$$

By the method of the resolvent, we get

$$\eta_3^k(t, x, \nu, \dots, \nu) = D\psi_t^3(x)\nu + \int_0^t D\psi_{t-s}^3(\psi_s^3(x)) G_3^k(s, x, \nu)\, ds$$

and by the same argument as for the Y^1-flow, we deduce that for any integer $k \ge 0$ there exist a constant M_k such that $\forall\, t \ge 0$

$$\|D^k \psi_t^3(x)\| \le M_k \|x\| e^{-bt}.$$

By Proposition 1, we have

Proposition 6. *Under the above conditions* (i)*,* (ii)*,* (iii)*,* (iv)*,* (v) *and* (vi)*, the origin* 0 *is a* $G.A.S.$ *equilibrium of order* k *on* $\mathbb{R}^n$ *to the* Y_3*-flow .*

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