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Zero Action on Perfect Crystals for $U_q\big(G_2^{(1)}\big)^{\star}$

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Abstract. The actions of 0-Kashiwara operators on the $U'_q\big(G_2^{(1)}\big)$-crystal B_l in [Yamane S., *J. Algebra* **210** (1998), 440–486] are made explicit by using a similarity technique from that of a $U'_q\big(D_4^{(3)}\big)$-crystal. It is shown that $\{B_l\}_{l\geq 1}$ forms a coherent family of perfect crystals.

Key words: combinatorial representation theory; quantum affine algebra; crystal bases

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1 Introduction

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra. Let I be its index set for simple roots, P the weight lattice, $\alpha_i \in P$ a simple root ($i \in I$), and $h_i \in P^*(= \mathrm{Hom}(P,\mathbb{Z}))$ a simple coroot ($i \in I$). To each $i \in I$ we associate a positive integer m_i and set $\tilde{\alpha}_i = m_i\alpha_i$, $\tilde{h}_i = h_i/m_i$. Suppose $(\langle \tilde{h}_i, \tilde{\alpha}_j\rangle)_{i,j\in I}$ is a generalized Cartan matrix for another symmetrizable Kac–Moody algebra $\tilde{\mathfrak{g}}$. Then the subset $\tilde{P}$ of P consisting of $\lambda \in P$ such that $\langle \tilde{h}_i, \lambda\rangle$ is an integer for any $i \in I$ can be considered as the weight lattice of $\tilde{\mathfrak{g}}$. For a dominant integral weight λ let $B^{\mathfrak{g}}(\lambda)$ be the highest weight crystal with highest weight λ over $U_q(\mathfrak{g})$. Then, in [5] Kashiwara showed the following. (The theorem in [5] is more general.)

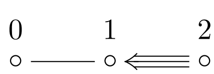
Theorem 1. *Let λ be a dominant integral weight in $\tilde{P}$. Then, there exists a unique injective map $S : B^{\tilde{\mathfrak{g}}}(\lambda) \to B^{\mathfrak{g}}(\lambda)$ such that*

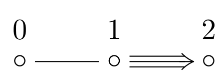
$$\mathrm{wt}\, S(b) = \mathrm{wt}\, b, \qquad S(e_i b) = e_i^{m_i} S(b), \qquad S(f_i b) = f_i^{m_i} S(b).$$

In this paper, we use this theorem to examine the so-called Kirillov–Reshetikhin crystal. Let $\mathfrak{g}$ be the affine algebra of type $D_4^{(3)}$. The generalized Cartan matrix $(\langle h_i, \alpha_j\rangle)_{i,j\in I}$ $(I = \{0,1,2\})$ is given by

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}.$$

Set $(m_0, m_1, m_2) = (3,3,1)$. Then, $\tilde{\mathfrak{g}}$ defined above turns out to be the affine algebra of type $G_2^{(1)}$. Their Dynkin diagrams are depicted as follows

$D_4^{(3)}$:


$G_2^{(1)}$:


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For $G_2^{(1)}$ a family of perfect crystals $\{B_l\}_{l\geq 1}$ was constructed in [7]. However, the crystal elements there were realized in terms of tableaux given in [2], and it was not easy to calculate the action of 0-Kashiwara operators on these tableaux. On the other hand, an explicit action of these operators was given on perfect crystals $\{\hat{B}_l\}_{l\geq 1}$ over $U'_q(D_4^{(3)})$ in [6]. Hence, it is a natural idea to use Theorem 1 to obtain the explicit action of e_0, f_0 on B_l from that on $\hat{B}_{l'}$ with suitable l'. We remark that Kirillov–Reshetikhin crystals are parametrized by a node of the Dynkin diagram except 0 and a positive integer. Both B_l and $\hat{B}_l$ correspond to the pair $(1,l)$.

Our strategy to do this is as follows. We define V_l as an appropriate subset of $\hat{B}_{3l}$ that is closed under the action of $\hat{e}_i^{m_i}$, $\hat{f}_i^{m_i}$ where $\hat{e}_i$, $\hat{f}_i$ stand for the Kashiwara operators on $\hat{B}_{3l}$. Hence, we can regard V_l as a $U'_q(G_2^{(1)})$-crystal. We next show that as a $U_q(G_2^{(1)})_{\{0,1\}}(=U_q(A_2))$-crystal and as a $U_q(G_2^{(1)})_{\{1,2\}}(=U_q(G_2))$-crystal, V_l has the same decomposition as B_l. Then, we can conclude from Theorem 6.1 of [6] that V_l is isomorphic to the $U'_q(G_2^{(1)})$-crystal B_l constructed in [7] (Theorem 2).

The paper is organized as follows. In Section 2 we review the $U'_q(D_4^{(3)})$-crystal $\hat{B}_l$. We then construct a $U'_q(G_2^{(1)})$-crystal V_l in $\hat{B}_{3l}$ with the aid of Theorem 1 and see it coincides with B_l given in [7] in Section 3. Minimal elements of B_l are found and $\{B_l\}_{l\geq 1}$ is shown to form a coherent family of perfect crystals in Section 4. The crystal graphs of B_1 and B_2 are included in Section 5.

2 Review on $U'_q(D_4^{(3)})$-crystal $\hat{B}_l$

In this section we recall the perfect crystal for $U'_q(D_4^{(3)})$ constructed in [6]. Since we also consider $U'_q(G_2^{(1)})$-crystals later, we denote it by $\hat{B}_l$. Kashiwara operators e_i, f_i and ε_i, φ_i on $\hat{B}_l$ are denoted by $\hat{e}_i$, $\hat{f}_i$ and $\hat{\varepsilon}_i$, $\hat{\varphi}_i$. Readers are warned that the coordinates x_i, $\bar{x}_i$ and steps by Kashiwara operators in [6] are divided by 3 here, since it is more convenient for our purpose. As a set

$$\hat{B}_l = \left\{ b=(x_1,x_2,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1)\in(\mathbb{Z}_{\geq 0}/3)^6 \;\middle|\; \begin{array}{l} 3x_3\equiv 3\bar{x}_3 \pmod 2, \\ \sum_{i=1,2}(x_i+\bar{x}_i)+(x_3+\bar{x}_3)/2\leq l/3 \end{array} \right\}.$$

In order to define the actions of Kashiwara operators $\hat{e}_i$ and $\hat{f}_i$ for $i=0,1,2$, we introduce some notations and conditions. Set $(x)_+=\max(x,0)$. For $b=(x_1,x_2,x_3,\bar{x}_3,\bar{x}_2,\bar{x}_1)\in\hat{B}_l$ we set

$$s(b)=x_1+x_2+\frac{x_3+\bar{x}_3}{2}+\bar{x}_2+\bar{x}_1, \tag{2.1}$$

and

$$z_1=\bar{x}_1-x_1, \qquad z_2=\bar{x}_2-\bar{x}_3, \qquad z_3=x_3-x_2, \qquad z_4=(\bar{x}_3-x_3)/2. \tag{2.2}$$

Now we define conditions (E_1)–(E_6) and (F_1)–(F_6) as follows

$$\begin{array}{ll}
(F_1) & z_1+z_2+z_3+3z_4\leq 0, \quad z_1+z_2+3z_4\leq 0, \quad z_1+z_2\leq 0, \quad z_1\leq 0, \\
(F_2) & z_1+z_2+z_3+3z_4\leq 0, \quad z_2+3z_4\leq 0, \quad z_2\leq 0, \quad z_1>0, \\
(F_3) & z_1+z_3+3z_4\leq 0, \quad z_3+3z_4\leq 0, \quad z_4\leq 0, \quad z_2>0, \quad z_1+z_2>0, \\
(F_4) & z_1+z_2+3z_4>0, \quad z_2+3z_4>0, \quad z_4>0, \quad z_3\leq 0, \quad z_1+z_3\leq 0, \\
(F_5) & z_1+z_2+z_3+3z_4>0, \quad z_3+3z_4>0, \quad z_3>0, \quad z_1\leq 0, \\
(F_6) & z_1+z_2+z_3+3z_4>0, \quad z_1+z_3+3z_4>0, \quad z_1+z_3>0, \quad z_1>0.
\end{array} \tag{2.3}$$

The conditions (F_1)–(F_6) are disjoint and they exhaust all cases. (E_i) $(1 \le i \le 6)$ is defined from (F_i) by replacing $>$ (resp. $\le$) with $\ge$ (resp. $<$). We also define

$$A = (0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4). \tag{2.4}$$

Then, for $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in \hat{B}_l$, $\hat{e}_i b$, $\hat{f}_i b$, $\hat{\varepsilon}_i(b)$, $\hat{\varphi}_i(b)$ are given as follows

$$\hat{e}_0 b = \begin{cases} (x_1 - 1/3, \dots) & \text{if } (E_1), \\ (\dots, x_3 - 1/3, \bar{x}_3 - 1/3, \dots, \bar{x}_1 + 1/3) & \text{if } (E_2), \\ (\dots, x_3 - 2/3, \dots, \bar{x}_2 + 1/3, \dots) & \text{if } (E_3), \\ (\dots, x_2 - 1/3, \dots, \bar{x}_3 + 2/3, \dots) & \text{if } (E_4), \\ (x_1 - 1/3, \dots, x_3 + 1/3, \bar{x}_3 + 1/3, \dots) & \text{if } (E_5), \\ (\dots, \bar{x}_1 + 1/3) & \text{if } (E_6), \end{cases}$$

$$\hat{f}_0 b = \begin{cases} (x_1 + 1/3, \dots) & \text{if } (F_1), \\ (\dots, x_3 + 1/3, \bar{x}_3 + 1/3, \dots, \bar{x}_1 - 1/3) & \text{if } (F_2), \\ (\dots, x_3 + 2/3, \dots, \bar{x}_2 - 1/3, \dots) & \text{if } (F_3), \\ (\dots, x_2 + 1/3, \dots, \bar{x}_3 - 2/3, \dots) & \text{if } (F_4), \\ (x_1 + 1/3, \dots, x_3 - 1/3, \bar{x}_3 - 1/3, \dots) & \text{if } (F_5), \\ (\dots, \bar{x}_1 - 1/3) & \text{if } (F_6). \end{cases}$$

$$\hat{e}_1 b = \begin{cases} (\dots, \bar{x}_2 + 1/3, \bar{x}_1 - 1/3) & \text{if } z_2 \ge (-z_3)_+, \\ (\dots, x_3 + 1/3, \bar{x}_3 - 1/3, \dots) & \text{if } z_2 < 0 \le z_3, \\ (x_1 + 1/3, x_2 - 1/3, \dots) & \text{if } (z_2)_+ < (-z_3), \end{cases}$$

$$\hat{f}_1 b = \begin{cases} (x_1 - 1/3, x_2 + 1/3, \dots) & \text{if } (z_2)_+ \le (-z_3), \\ (\dots, x_3 - 1/3, \bar{x}_3 + 1/3, \dots) & \text{if } z_2 \le 0 < z_3, \\ (\dots, \bar{x}_2 - 1/3, \bar{x}_1 + 1/3) & \text{if } z_2 > (-z_3)_+, \end{cases}$$

$$\hat{e}_2 b = \begin{cases} (\dots, \bar{x}_3 + 2/3, \bar{x}_2 - 1/3, \dots) & \text{if } z_4 \ge 0, \\ (\dots, x_2 + 1/3, x_3 - 2/3, \dots) & \text{if } z_4 < 0, \end{cases}$$

$$\hat{f}_2 b = \begin{cases} (\dots, x_2 - 1/3, x_3 + 2/3, \dots) & \text{if } z_4 \le 0, \\ (\dots, \bar{x}_3 - 2/3, \bar{x}_2 + 1/3, \dots) & \text{if } z_4 > 0, \end{cases}$$

$$\begin{aligned} &\hat{\varepsilon}_0(b) = l - 3s(b) + 3\max A - 3(2z_1 + z_2 + z_3 + 3z_4), \\ &\hat{\varphi}_0(b) = l - 3s(b) + 3\max A, \\ &\hat{\varepsilon}_1(b) = 3\bar{x}_1 + 3(\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+, \\ &\hat{\varphi}_1(b) = 3x_1 + 3(x_3 - x_2 + (\bar{x}_2 - \bar{x}_3)_+)_+, \\ &\hat{\varepsilon}_2(b) = 3\bar{x}_2 + \tfrac{3}{2}(x_3 - \bar{x}_3)_+, \qquad \hat{\varphi}_2(b) = 3x_2 + \tfrac{3}{2}(\bar{x}_3 - x_3)_+. \end{aligned} \tag{2.5}$$

If $\hat{e}_i b$ or $\hat{f}_i b$ does not belong to $\hat{B}_l$, namely, if x_j or $\bar{x}_j$ for some j becomes negative or $s(b)$ exceeds $l/3$, we should understand it to be 0. Forgetting the 0-arrows,

$$\hat{B}_l \simeq \bigoplus_{j=0}^{l} B^{G_2^\dagger}(j\Lambda_1),$$

where $B^{G_2^\dagger}(\lambda)$ is the highest weight $U_q(G_2^\dagger)$-crystal of highest weight λ and $G_2^\dagger$ stands for the simple Lie algebra G_2 with the reverse labeling of the indices of the simple roots (α_1 is the short root). Forgetting 2-arrows,

$$\hat{B}_l \simeq \bigoplus_{i=0}^{\lfloor \frac{l}{2} \rfloor} \bigoplus_{\substack{i \le j_0, j_1 \le l-i \\ j_0, j_1 \equiv l-i \pmod 3}} B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1),$$

where $B^{A_2}(\lambda)$ is the highest weight $U_q(A_2)$-crystal (with indices $\{0,1\}$) of highest weight λ.

3 $U'_q(G_2^{(1)})$-crystal

In this section we define a subset V_l of $\hat{B}_{3l}$ and see it is isomorphic to the $U'_q(G_2^{(1)})$-crystal B_l. The set V_l is defined as a subset of $\hat{B}_{3l}$ satisfying the following conditions:

$$x_1, \bar{x}_1, x_2 - x_3, \bar{x}_3 - \bar{x}_2 \in \mathbb{Z}. \tag{3.1}$$

For an element $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1)$ of V_l we define $s(b)$ as in (2.1). From (3.1) we see that $s(b) \in \{0, 1, \ldots, l\}$.

Lemma 1. *For* $0 \le k \le l$

$$\sharp\{b \in V_l \mid s(b) = k\} = \frac{1}{120}(k+1)(k+2)(2k+3)(3k+4)(3k+5).$$

Proof. We first count the number of elements $(x_2, x_3, \bar{x}_3, \bar{x}_2)$ satisfying the conditions of coordinates as an element of V_l and $x_2 + (x_3 + \bar{x}_3)/2 + \bar{x}_2 = m$ ($m = 0, 1, \ldots, k$). According to (a, b, c, d) ($a, d \in \{0, 1/3, 2/3\}$, $b, c \in \{0, 1/3, 2/3, 1, 4/3, 5/3\}$) such that $x_2 \in \mathbb{Z} + a$, $x_3 \in 2\mathbb{Z} + b$, $\bar{x}_3 \in 2\mathbb{Z} + c$, $\bar{x}_2 \in \mathbb{Z} + d$, we divide the cases into the following 18:

(i) $(0, 0, 0, 0)$,	(ii) $(0, 0, 2/3, 2/3)$,	(iii) $(0, 0, 4/3, 1/3)$,
(iv) $(0, 1, 1/3, 1/3)$,	(v) $(0, 1, 1, 0)$,	(vi) $(0, 1, 5/3, 2/3)$,
(vii) $(1/3, 1/3, 1/3, 1/3)$,	(viii) $(1/3, 1/3, 1, 0)$,	(ix) $(1/3, 1/3, 5/3, 2/3)$,
(x) $(1/3, 4/3, 0, 0)$,	(xi) $(1/3, 4/3, 2/3, 2/3)$,	(xii) $(1/3, 4/3, 4/3, 1/3)$,
(xiii) $(2/3, 2/3, 0, 0)$,	(xiv) $(2/3, 2/3, 2/3, 2/3)$,	(xv) $(2/3, 2/3, 4/3, 1/3)$,
(xvi) $(2/3, 5/3, 1/3, 1/3)$,	(xvii) $(2/3, 5/3, 1, 0)$,	(xviii) $(2/3, 5/3, 5/3, 2/3)$.

The number of elements $(x_2, x_3, \bar{x}_3, \bar{x}_2)$ in a case among the above such that $a + (b+c)/2 + d = e$ ($e = 0, 1, 2, 3$) is given by $f(e) = \binom{m-e+3}{3}$. Since there is one case with $e = 0$ (i) and $e = 3$ (xviii) and 8 cases with $e = 1$ and $e = 2$, the number of $(x_2, x_3, \bar{x}_3, \bar{x}_2)$ such that $x_2 + (x_3 + \bar{x}_3)/2 + \bar{x}_2 = m$ is given by

$$f(0) + 8f(1) + 8f(2) + f(3) = \frac{1}{2}(2m+1)(3m^2 + 3m + 2).$$

For each $(x_2, x_3, \bar{x}_3, \bar{x}_2)$ such that $x_2 + (x_3 + \bar{x}_3)/2 + \bar{x}_2 = m$ ($m = 0, 1, \ldots, k$) there are $(k - m + 1)$ cases for $(x_1, \bar{x}_1)$, so the number of $b \in V_l$ such that $s(b) = k$ is given by

$$\sum_{m=0}^{k} \frac{1}{2}(2m+1)(3m^2 + 3m + 2)(k - m + 1).$$

A direct calculation leads to the desired result. ■

We define the action of operators e_i, f_i $(i = 0, 1, 2)$ on V_l as follows.

$$e_0 b = \begin{cases} (x_1 - 1, \dots) & \text{if } (E_1), \\ (\dots, x_3 - 1, \bar{x}_3 - 1, \dots, \bar{x}_1 + 1) & \text{if } (E_2), \\ \left(\dots, x_2 - \frac{2}{3}, x_3 - \frac{2}{3}, \bar{x}_3 + \frac{4}{3}, \bar{x}_2 + \frac{1}{3}, \dots\right) & \text{if } (E_3) \text{ and } z_4 = -\frac{1}{3}, \\ \left(\dots, x_2 - \frac{1}{3}, x_3 - \frac{4}{3}, \bar{x}_3 + \frac{2}{3}, \bar{x}_2 + \frac{2}{3}, \dots\right) & \text{if } (E_3) \text{ and } z_4 = -\frac{2}{3}, \\ (\dots, x_3 - 2, \dots, \bar{x}_2 + 1, \dots) & \text{if } (E_3) \text{ and } z_4 \neq -\frac{1}{3}, -\frac{2}{3}, \\ (\dots, x_2 - 1, \dots, \bar{x}_3 + 2, \dots) & \text{if } (E_4), \\ (x_1 - 1, \dots, x_3 + 1, \bar{x}_3 + 1, \dots) & \text{if } (E_5), \\ (\dots, \bar{x}_1 + 1) & \text{if } (E_6), \end{cases}$$

$$f_0 b = \begin{cases} (x_1 + 1, \dots) & \text{if } (F_1), \\ (\dots, x_3 + 1, \bar{x}_3 + 1, \dots, \bar{x}_1 - 1) & \text{if } (F_2), \\ (\dots, x_3 + 2, \dots, \bar{x}_2 - 1, \dots) & \text{if } (F_3), \\ \left(\dots, x_2 + \frac{1}{3}, x_3 + \frac{4}{3}, \bar{x}_3 - \frac{2}{3}, \bar{x}_2 - \frac{2}{3}, \dots\right) & \text{if } (F_4) \text{ and } z_4 = \frac{1}{3}, \\ \left(\dots, x_2 + \frac{2}{3}, x_3 + \frac{2}{3}, \bar{x}_3 - \frac{4}{3}, \bar{x}_2 - \frac{1}{3}, \dots\right) & \text{if } (F_4) \text{ and } z_4 = \frac{2}{3}, \\ (\dots, x_2 + 1, \dots, \bar{x}_3 - 2, \dots) & \text{if } (F_4) \text{ and } z_4 \neq \frac{1}{3}, \frac{2}{3}, \\ (x_1 + 1, \dots, x_3 - 1, \bar{x}_3 - 1, \dots) & \text{if } (F_5), \\ (\dots, \bar{x}_1 - 1) & \text{if } (F_6), \end{cases}$$

$$e_1 b = \begin{cases} (\dots, \bar{x}_2 + 1, \bar{x}_1 - 1) & \text{if } \bar{x}_2 - \bar{x}_3 \geq (x_2 - x_3)_+, \\ (\dots, x_3 + 1, \bar{x}_3 - 1, \dots) & \text{if } \bar{x}_2 - \bar{x}_3 < 0 \leq x_3 - x_2, \\ (x_1 + 1, x_2 - 1, \dots) & \text{if } (\bar{x}_2 - \bar{x}_3)_+ < x_2 - x_3, \end{cases}$$

$$f_1 b = \begin{cases} (x_1 - 1, x_2 + 1, \dots) & \text{if } (\bar{x}_2 - \bar{x}_3)_+ \leq x_2 - x_3, \\ (\dots, x_3 - 1, \bar{x}_3 + 1, \dots) & \text{if } \bar{x}_2 - \bar{x}_3 \leq 0 < x_3 - x_2, \\ (\dots, \bar{x}_2 - 1, \bar{x}_1 + 1) & \text{if } \bar{x}_2 - \bar{x}_3 > (x_2 - x_3)_+, \end{cases}$$

$$e_2 b = \begin{cases} \left(\dots, \bar{x}_3 + \frac{2}{3}, \bar{x}_2 - \frac{1}{3}, \dots\right) & \text{if } \bar{x}_3 \geq x_3, \\ \left(\dots, x_2 + \frac{1}{3}, x_3 - \frac{2}{3}, \dots\right) & \text{if } \bar{x}_3 < x_3, \end{cases}$$

$$f_2 b = \begin{cases} \left(\dots, x_2 - \frac{1}{3}, x_3 + \frac{2}{3}, \dots\right) & \text{if } \bar{x}_3 \leq x_3, \\ \left(\dots, \bar{x}_3 - \frac{2}{3}, \bar{x}_2 + \frac{1}{3}, \dots\right) & \text{if } \bar{x}_3 > x_3. \end{cases}$$

We now set $(m_0, m_1, m_2) = (3, 3, 1)$.

Proposition 1.

(1) *For any* $b \in V_l$ *we have* $e_i b, f_i b \in V_l \sqcup \{0\}$ *for* $i = 0, 1, 2$.

(2) *The equalities* $e_i = \hat{e}_i^{m_i}$ *and* $f_i = \hat{f}_i^{m_i}$ *hold on* V_l *for* $i = 0, 1, 2$.

Proof. (1) can be checked easily.

For (2) we only treat f_i. To prove the $i = 0$ case consider the following table

	(F_1)	(F_2)	(F_3)	(F_4)	(F_5)	(F_6)
z_1	$-1/3$	$-1/3$	0	0	$-1/3$	$-1/3$
z_2	0	$-1/3$	$-1/3$	$2/3$	$1/3$	0
z_3	0	$1/3$	$2/3$	$-1/3$	$-1/3$	0
z_4	0	0	$-1/3$	$-1/3$	0	0

This table signifies the difference (z_j for $\hat{f}_0 b$) − (z_j for b) when b belongs to the case (F_i). Note that the left hand sides of the inequalities of each (F_i) (2.3) always decrease by 1/3. Since $z_1, z_2, z_3 \in \mathbb{Z}$, $z_4 \in \mathbb{Z}/3$ for $b \in V_l$, we see that if b belongs to (F_i), $\hat{f}_0 b$ and $\hat{f}_0^2 b$ also belong to (F_i) except two cases: (a) $b \in (F_4)$ and $z_4 = 1/3$, and (b) $b \in (F_4)$ and $z_4 = 2/3$. If (a) occurs, we have $\hat{f}_0 b, \hat{f}_0^2 b \in (F_3)$. Hence, we obtain $f_0 = \hat{f}_0^3$ in this case. If (b) occurs, we have $\hat{f}_0 b \in (F_4)$, $\hat{f}_0^2 b \in (F_3)$. Therefore, we obtain $f_0 = \hat{f}_0^3$ in this case as well.

In the $i = 1$ case, if b belongs to one of the 3 cases, $\hat{f}_1 b$ and $\hat{f}_1^2 b$ also belong to the same case. Hence, we obtain $f_1 = \hat{f}_1^3$. For $i = 2$ there is nothing to do. ■

Proposition 1, together with Theorem 1, shows that V_l can be regarded as a $U_q'(G_2^{(1)})$-crystal with operators e_i, f_i ($i = 0, 1, 2$).

Proposition 2. *As a* $U_q(G_2^{(1)})_{\{1,2\}}$*-crystal*

$$V_l \simeq \bigoplus_{k=0}^{l} B^{G_2}(k\Lambda_1),$$

where $B^{G_2}(\lambda)$ *is the highest weight* $U_q(G_2)$*-crystal of highest weight* λ*.*

Proof. For a subset J of $\{0, 1, 2\}$ we say b is J-highest if $e_j b = 0$ for any $j \in J$. Note from (2.5) that $b_k = (k, 0, 0, 0, 0, 0)$ ($0 \le k \le l$) is $\{1, 2\}$-highest of weight $3k\Lambda_1$ in $\hat{B}_{3l}$. By setting $\mathfrak{g} = G_2^\dagger$ (= G_2 with the reverse labeling of indices), $(m_1, m_2) = (3, 1)$, $\tilde{\mathfrak{g}} = G_2$ in Theorem 1, we know that the connected component generated from b_k by $f_1 = \hat{f}_1^3$ and $f_2 = \hat{f}_2$ is isomorphic to $B^{G_2}(k\Lambda_1)$. Hence by Proposition 1 (1) we have

$$\bigoplus_{k=0}^{l} B^{G_2}(k\Lambda_1) \subset V_l. \tag{3.2}$$

Now recall Weyl's formula to calculate the dimension of the highest weight representation. In our case we obtain

$$\sharp B^{G_2}(k\Lambda_1) = \frac{1}{120}(k+1)(k+2)(2k+3)(3k+4)(3k+5).$$

However, this is equal to $\sharp\{b \in V_l \mid s(b) = k\}$ by Lemma 1. Therefore, $\subset$ in (3.2) should be =, and the proof is completed. ■

Proposition 3. *As a* $U_q(G_2^{(1)})_{\{0,1\}}$*-crystal*

$$V_l \simeq \bigoplus_{i=0}^{\lfloor l/2 \rfloor} \bigoplus_{i \le j_0, j_1 \le l-i} B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1),$$

where $B^{A_2}(\lambda)$ *is the highest weight* $U_q(A_2)$*-crystal (with indices* $\{0, 1\}$*) of highest weight* λ*.*

Proof. For integers i, j_0, j_1 such that $0 \le i \le l/2$, $i \le j_0, j_1 \le l - i$, define an element b_{i,j_0,j_1} of V_l by

$$b_{i,j_0,j_1} = \begin{cases} (0, y_1, 3y_0 - 2y_1 + i, y_0 + i, y_0 + j_0, 0) & \text{if } j_0 \le j_1, \\ (0, y_0, y_0 + i, 2y_1 - y_0 + i, 2y_0 - y_1 + j_0, 0) & \text{if } j_0 > j_1. \end{cases}$$

Here we have set $y_a = (l - i - j_a)/3$ for $a = 0, 1$. From (2.5) one notices that b_{i,j_0,j_1} is $\{0, 1\}$-highest of weight $3j_0\Lambda_0 + 3j_1\Lambda_1$ in $\hat{B}_{3l}$. For instance, $\hat{\varepsilon}_0(b_{i,j_0,j_1}) = 0$ and $\hat{\varphi}_0(b_{i,j_0,j_1}) = 3j_0$ since

$s(b_{i,j_0,j_1}) = l$ and $\max A = 2z_1 + z_2 + z_3 + 3z_4 = j_0$. By setting $\mathfrak{g} = \tilde{\mathfrak{g}} = A_2, (m_0, m_1) = (3, 3)$ in Theorem 1, the connected component generated from b_{i,j_0,j_1} by $f_i = \hat{f}_i^3$ $(i = 0, 1)$ is isomorphic to $B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1)$. Hence, by Proposition 1 (1) we have

$$\bigoplus_{i=0}^{\lfloor l/2 \rfloor} \bigoplus_{i \le j_0, j_1 \le l-i} B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1) \subset V_l.$$

However, from Proposition 2 one knows that

$$\sharp V_l = \sum_{k=0}^{l} \sharp B^{G_2}(k\Lambda_1).$$

Moreover, it is already established in [7] that

$$\sum_{k=0}^{l} \sharp B^{G_2}(k\Lambda_1) = \sum_{i=0}^{\lfloor l/2 \rfloor} \sum_{i \le j_0, j_1 \le l-i} \sharp B^{A_2}(j_0\Lambda_0 + j_1\Lambda_1).$$

Therefore, the proof is completed. ■

Theorem 6.1 in [6] shows that if two $U_q'(G_2^{(1)})$-crystals decompose into $\bigoplus_{0 \le k \le l} B^{G_2}(k\Lambda_1)$ as $U_q(G_2)$-crystals, then they are isomorphic to each other. Therefore, we now have

Theorem 2. *V_l agrees with the $U_q'(G_2^{(1)})$-crystal B_l constructed in [7]. The values of ε_i, φ_i with our representation are given by*

$$\begin{aligned} &\varepsilon_0(b) = l - s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4), \qquad \varphi_0(b) = l - s(b) + \max A, \\ &\varepsilon_1(b) = \bar{x}_1 + (\bar{x}_3 - \bar{x}_2 + (x_2 - x_3)_+)_+, \qquad \varphi_1(b) = x_1 + (x_3 - x_2 + (\bar{x}_2 - \bar{x}_3)_+)_+, \\ &\varepsilon_2(b) = 3\bar{x}_2 + \tfrac{3}{2}(x_3 - \bar{x}_3)_+, \qquad \varphi_2(b) = 3x_2 + \tfrac{3}{2}(\bar{x}_3 - x_3)_+. \end{aligned} \tag{3.3}$$

4 Minimal elements and a coherent family

The notion of perfect crystals was introduced in [3] to construct the path realization of a highest weight crystal of a quantum affine algebra. The crystal B_l was shown to be perfect of level l in [7]. In this section we obtain all the minimal elements of B_l in the coordinate representation and also show $\{B_l\}_{l \ge 1}$ forms a coherent family of perfect crystals. For the notations such as P_{cl}, $(P_{cl}^+)_l$, see [3].

4.1 Minimal elements

From (3.3) we have

$$\begin{aligned} \langle c, \varphi(b) \rangle &= \varphi_0(b) + 2\varphi_1(b) + \varphi_2(b) \\ &= l + \max A + 2(z_3 + (z_2)_+)_+ + (3z_4)_+ - (z_1 + z_2 + 2z_3 + 3z_4), \end{aligned}$$

where z_j $(1 \le j \le 4)$ are given in (2.2) and A is given in (2.4). The following lemma was proven in [6], although $\mathbb{Z}$ is replaced with $\mathbb{Z}/3$ here.

Lemma 2. *For $(z_1, z_2, z_3, z_4) \in (\mathbb{Z}/3)^4$ set*

$$\psi(z_1, z_2, z_3, z_4) = \max A + 2(z_3 + (z_2)_+)_+ + (3z_4)_+ - (z_1 + z_2 + 2z_3 + 3z_4).$$

Then we have $\psi(z_1, z_2, z_3, z_4) \ge 0$ and $\psi(z_1, z_2, z_3, z_4) = 0$ if and only if $(z_1, z_2, z_3, z_4) = (0, 0, 0, 0)$.

From this lemma, we have $\langle c, \varphi(b)\rangle - l = \psi(z_1, z_2, z_3, z_4) \geq 0$. Since $\langle c, \varphi(b) - \varepsilon(b)\rangle = 0$, we also have $\langle c, \varepsilon(b)\rangle \geq l$.

Suppose $\langle c, \varepsilon(b)\rangle = l$. It implies $\psi = 0$. Hence from the lemma one can conclude that such element $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1)$ should satisfy $x_1 = \bar{x}_1$, $x_2 = x_3 = \bar{x}_3 = \bar{x}_2$. Therefore the set of minimal elements $(B_l)_{\min}$ in B_l is given by

$$(B_l)_{\min} = \{(\alpha, \beta, \beta, \beta, \beta, \alpha) |\ \alpha \in \mathbb{Z}_{\geq 0}, \beta \in (\mathbb{Z}_{\geq 0})/3, 2\alpha + 3\beta \leq l\}.$$

For $b = (\alpha, \beta, \beta, \beta, \beta, \alpha) \in (B_l)_{\min}$ one calculates

$$\varepsilon(b) = \varphi(b) = (l - 2\alpha - 3\beta)\Lambda_0 + \alpha\Lambda_1 + 3\beta\Lambda_2.$$

4.2 Coherent family of perfect crystals

The notion of a coherent family of perfect crystals was introduced in [1]. Let $\{B_l\}_{l\geq 1}$ be a family of perfect crystals B_l of level l and $(B_l)_{\min}$ be the subset of minimal elements of B_l. Set $J = \{(l, b) \mid l \in \mathbb{Z}_{>0}, b \in (B_l)_{\min}\}$. Let σ denote the isomorphism of $(P_{cl}^+)_l$ defined by $\sigma = \varepsilon \circ \varphi^{-1}$. For $\lambda \in P_{cl}$, T_λ denotes a crystal with a unique element t_λ defined in [4]. For our purpose the following facts are sufficient. For any P_{cl}-weighted crystal B and $\lambda, \mu \in P_{cl}$ consider the crystal

$$T_\lambda \otimes B \otimes T_\mu = \{t_\lambda \otimes b \otimes t_\mu \mid b \in B\}.$$

The definition of T_λ and the tensor product rule of crystals imply

$$\begin{aligned}
&\tilde{e}_i(t_\lambda \otimes b \otimes t_\mu) = t_\lambda \otimes \tilde{e}_i b \otimes t_\mu, \qquad && \tilde{f}_i(t_\lambda \otimes b \otimes t_\mu) = t_\lambda \otimes \tilde{f}_i b \otimes t_\mu,\\
&\varepsilon_i(t_\lambda \otimes b \otimes t_\mu) = \varepsilon_i(b) - \langle h_i, \lambda\rangle, && \varphi_i(t_\lambda \otimes b \otimes t_\mu) = \varphi_i(b) + \langle h_i, \mu\rangle,\\
&wt\,(t_\lambda \otimes b \otimes t_\mu) = \lambda + \mu + wt\, b.
\end{aligned}$$

Definition 1. A crystal B_∞ with an element b_∞ is called a limit of $\{B_l\}_{l\geq 1}$ if it satisfies the following conditions:

- $wt\, b_\infty = 0, \varepsilon(b_\infty) = \varphi(b_\infty) = 0,$
- for any $(l, b) \in J$, there exists an embedding of crystals

  $$f_{(l,b)}: \quad T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \longrightarrow B_\infty$$

  sending $t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)}$ to b_∞,
- $B_\infty = \bigcup_{(l,b)\in J} \operatorname{Im} f_{(l,b)}$.

If a limit exists for the family $\{B_l\}$, we say that $\{B_l\}$ is a coherent family of perfect crystals.

Let us now consider the following set

$$B_\infty = \left\{ b = (\nu_1, \nu_2, \nu_3, \bar{\nu}_3, \bar{\nu}_2, \bar{\nu}_1) \in (\mathbb{Z}/3)^6 \;\middle|\; \begin{array}{l} \nu_1, \bar{\nu}_1, \nu_2 - \nu_3, \bar{\nu}_3 - \bar{\nu}_2 \in \mathbb{Z}, \\ 3\nu_3 \equiv 3\bar{\nu}_3 \pmod 2 \end{array} \right\},$$

and set $b_\infty = (0, 0, 0, 0, 0, 0)$. We introduce the crystal structure on B_∞ as follows. The actions of e_i, f_i $(i = 0, 1, 2)$ are defined by the same rule as in Section 3 with x_i and $\bar{x}_i$ replaced with ν_i and $\bar{\nu}_i$. The only difference lies in the fact that $e_i b$ or $f_i b$ never becomes 0, since we allow a coordinate to be negative and there is no restriction for the sum $s(b) = \sum_{i=1}^{2}(\nu_i + \bar{\nu}_i) + (\nu_3 + \bar{\nu}_3)/2$. For ε_i, φ_i with $i = 1, 2$ we adopt the formulas in Section 3. For ε_0, φ_0 we define

$$\varepsilon_0(b) = -s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4), \qquad \varphi_0(b) = -s(b) + \max A,$$

where A is given in (2.4) and z_1, z_2, z_3, z_4 are given in (2.2) with x_i, $\bar{x}_i$ replaced by ν_i, $\bar{\nu}_i$. Note that $wt\, b_\infty = 0$ and $\varepsilon_i(b_\infty) = \varphi_i(b_\infty) = 0$ for $i = 0, 1, 2$.

Let $b_0 = (\alpha, \beta, \beta, \beta, \beta, \alpha)$ be an element of $(B_l)_{\min}$. Since $\varepsilon(b_0) = \varphi(b_0)$, one can set $\sigma = \mathrm{id}$. Let $\lambda = \varepsilon(b_0)$. For $b = (x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) \in B_l$ we define a map

$$f_{(l,b_0)}: \quad T_\lambda \otimes B_l \otimes T_{-\lambda} \longrightarrow B_\infty$$

by

$$f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda}) = b' = (\nu_1, \nu_2, \nu_3, \bar{\nu}_3, \bar{\nu}_2, \bar{\nu}_1),$$

where

$$\begin{aligned} \nu_1 &= x_1 - \alpha, & \bar{\nu}_1 &= \bar{x}_1 - \alpha, \\ \nu_j &= x_j - \beta, & \bar{\nu}_j &= \bar{x}_j - \beta \quad (j = 2, 3). \end{aligned}$$

Note that $s(b') = s(b) - (2\alpha + 3\beta)$. Then we have

$$\begin{aligned} wt\,(t_\lambda \otimes b \otimes t_{-\lambda}) &= wt\, b = wt\, b', \\ \varphi_0(t_\lambda \otimes b \otimes t_{-\lambda}) &= \varphi_0(b) + \langle h_0, -\lambda\rangle \\ &= \varphi_0(b') + (l - s(b)) + s(b') - (l - 2\alpha - 3\beta) = \varphi_0(b'), \\ \varphi_1(t_\lambda \otimes b \otimes t_{-\lambda}) &= \varphi_1(b) + \langle h_1, -\lambda\rangle = \varphi_1(b') + \alpha - \alpha = \varphi_1(b'), \\ \varphi_2(t_\lambda \otimes b \otimes t_{-\lambda}) &= \varphi_2(b) + \langle h_2, -\lambda\rangle = \varphi_2(b') + 3\beta - 3\beta = \varphi_2(b'). \end{aligned}$$

$\varepsilon_i(t_\lambda \otimes b \otimes t_{-\lambda}) = \varepsilon_i(b')$ $(i = 0, 1, 2)$ also follows from the above calculations.

From the fact that (z_j for b) = (z_j for b') it is straightforward to check that if $b, e_i b \in B_l$ (resp. $b, f_i b \in B_l$), then $f_{(l,b_0)}(e_i(t_\lambda \otimes b \otimes t_{-\lambda})) = e_i f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda})$ (resp. $f_{(l,b_0)}(f_i(t_\lambda \otimes b \otimes t_{-\lambda})) = f_i f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda})$). Hence $f_{(l,b_0)}$ is a crystal embedding. It is easy to see that $f_{(l,b_0)}(t_\lambda \otimes b_0 \otimes t_{-\lambda}) = b_\infty$. We can also check $B_\infty = \bigcup_{(l,b)\in J} \mathrm{Im}\, f_{(l,b)}$. Therefore we have shown that the family of perfect crystals $\{B_l\}_{l\geq 1}$ forms a coherent family.

5 Crystal graphs of B_1 and B_2

In this section we present crystal graphs of the $U'_q(G_2^{(1)})$-crystals B_1 and B_2 in Figs. 1 and 2. In the graphs $b \xrightarrow{i} b'$ stands for $b' = f_i b$. Minimal elements are marked as $*$. Recall that as a $U_q(G_2)$-crystal

$$B_1 \simeq B(0) \oplus B(\Lambda_1), \qquad B_2 \simeq B(0) \oplus B(\Lambda_1) \oplus B(2\Lambda_1).$$

We give the table that relates the numbers in the crystal graphs to our representation of elements according to which $U_q(G_2)$-components they belong to.

$B(0)$: $\boxed{\phi^*} = (0,0,0,0,0,0)$

$B(\Lambda_1)$:

$\boxed{1} = (1,0,0,0,0,0)$ $\boxed{2} = (0,1,0,0,0,0)$ $\boxed{3} = (0,\tfrac{2}{3},\tfrac{2}{3},0,0,0)$ $\boxed{4} = (0,\tfrac{1}{3},\tfrac{4}{3},0,0,0)$
$\boxed{5} = (0,\tfrac{1}{3},\tfrac{1}{3},1,0,0)$ $\boxed{6^*} = (0,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0)$ $\boxed{7} = (0,0,1,1,0,0)$ $\boxed{8} = (0,0,1,\tfrac{1}{3},\tfrac{1}{3},0)$
$\boxed{9} = (0,0,0,\tfrac{4}{3},\tfrac{1}{3},0)$ $\boxed{10} = (0,0,0,\tfrac{2}{3},\tfrac{2}{3},0)$ $\boxed{11} = (0,0,0,0,1,0)$ $\boxed{12} = (0,0,0,0,0,1)$
$\boxed{13} = (0,0,2,0,0,0)$ $\boxed{14} = (0,0,0,2,0,0)$

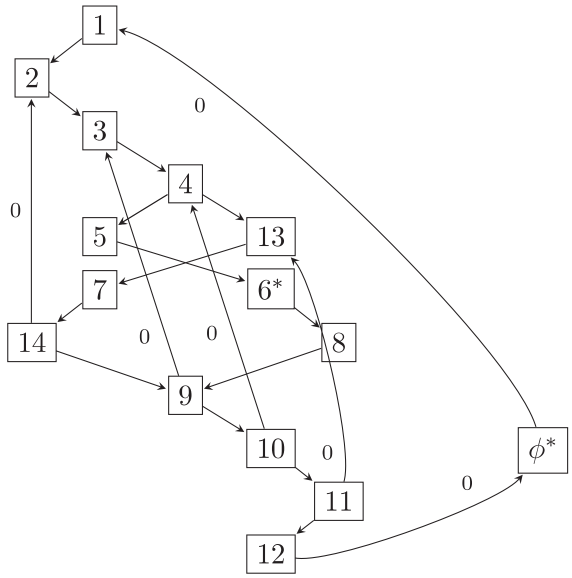

Figure 1. Crystal graph of B_1. $\swarrow$ is f_1 and $\searrow$ is f_2.

$B(2\Lambda_1)$:

$\boxed{15} = (2,0,0,0,0,0)$	$\boxed{16} = (1,1,0,0,0,0)$	$\boxed{17} = \left(1,\frac{2}{3},\frac{2}{3},0,0,0\right)$	$\boxed{18} = \left(1,\frac{1}{3},\frac{4}{3},0,0,0\right)$
$\boxed{19} = \left(1,\frac{1}{3},\frac{1}{3},1,0,0\right)$	$\boxed{20} = \left(1,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right)$	$\boxed{21} = (1,0,1,1,0,0)$	$\boxed{22} = \left(1,0,1,\frac{1}{3},\frac{1}{3},0\right)$
$\boxed{23} = \left(1,0,0,\frac{4}{3},\frac{1}{3},0\right)$	$\boxed{24} = \left(1,0,0,\frac{2}{3},\frac{2}{3},0\right)$	$\boxed{25} = (1,0,0,0,1,0)$	$\boxed{26^*} = (1,0,0,0,0,1)$
$\boxed{27} = (1,0,2,0,0,0)$	$\boxed{28} = (1,0,0,2,0,0)$	$\boxed{29} = (0,2,0,0,0,0)$	$\boxed{30} = \left(0,\frac{5}{3},\frac{2}{3},0,0,0\right)$
$\boxed{31} = \left(0,\frac{4}{3},\frac{4}{3},0,0,0\right)$	$\boxed{32} = \left(0,\frac{4}{3},\frac{1}{3},1,0,0\right)$	$\boxed{33} = \left(0,\frac{4}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right)$	$\boxed{34} = (0,1,1,1,0,0)$
$\boxed{35} = \left(0,1,1,\frac{1}{3},\frac{1}{3},0\right)$	$\boxed{36} = \left(0,1,0,\frac{4}{3},\frac{1}{3},0\right)$	$\boxed{37} = \left(0,1,0,\frac{2}{3},\frac{2}{3},0\right)$	$\boxed{38} = (0,1,0,0,1,0)$
$\boxed{39} = (0,1,0,0,0,1)$	$\boxed{40} = (0,1,2,0,0,0)$	$\boxed{41} = (0,1,0,2,0,0)$	$\boxed{42} = \left(0,\frac{2}{3},\frac{2}{3},0,1,0\right)$
$\boxed{43} = \left(0,\frac{1}{3},\frac{4}{3},0,1,0\right)$	$\boxed{44} = \left(0,\frac{1}{3},\frac{1}{3},1,1,0\right)$	$\boxed{45} = \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{4}{3},0\right)$	$\boxed{46} = (0,0,1,1,1,0)$
$\boxed{47} = \left(0,0,1,\frac{1}{3},\frac{4}{3},0\right)$	$\boxed{48} = \left(0,0,0,\frac{4}{3},\frac{4}{3},0\right)$	$\boxed{49} = \left(0,0,0,\frac{2}{3},\frac{5}{3},0\right)$	$\boxed{50} = (0,0,0,0,2,0)$
$\boxed{51} = (0,0,0,0,1,1)$	$\boxed{52} = (0,0,2,0,1,0)$	$\boxed{53} = (0,0,0,2,1,0)$	$\boxed{54} = \left(0,\frac{2}{3},\frac{2}{3},0,0,1\right)$
$\boxed{55} = \left(0,\frac{1}{3},\frac{4}{3},0,0,1\right)$	$\boxed{56} = \left(0,\frac{1}{3},\frac{1}{3},1,0,1\right)$	$\boxed{57} = \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},1\right)$	$\boxed{58} = (0,0,1,1,0,1)$
$\boxed{59} = \left(0,0,1,\frac{1}{3},\frac{1}{3},1\right)$	$\boxed{60} = \left(0,0,0,\frac{4}{3},\frac{1}{3},1\right)$	$\boxed{61} = \left(0,0,0,\frac{2}{3},\frac{2}{3},1\right)$	$\boxed{62} = (0,0,0,0,0,2)$
$\boxed{63} = (0,0,2,0,0,1)$	$\boxed{64} = (0,0,0,2,0,1)$	$\boxed{65} = \left(0,\frac{2}{3},\frac{8}{3},0,0,0\right)$	$\boxed{66} = \left(0,\frac{1}{3},\frac{10}{3},0,0,0\right)$
$\boxed{67} = \left(0,\frac{1}{3},\frac{7}{3},1,0,0\right)$	$\boxed{68} = \left(0,\frac{1}{3},\frac{7}{3},\frac{1}{3},\frac{1}{3},0\right)$	$\boxed{69} = (0,0,3,1,0,0)$	$\boxed{70} = \left(0,0,3,\frac{1}{3},\frac{1}{3},0\right)$
$\boxed{71} = \left(0,0,2,\frac{4}{3},\frac{1}{3},0\right)$	$\boxed{72} = \left(0,0,2,\frac{2}{3},\frac{2}{3},0\right)$	$\boxed{73} = (0,0,4,0,0,0)$	$\boxed{74} = (0,0,2,2,0,0)$
$\boxed{75} = \left(0,\frac{2}{3},\frac{2}{3},2,0,0\right)$	$\boxed{76} = \left(0,\frac{1}{3},\frac{4}{3},2,0,0\right)$	$\boxed{77} = \left(0,\frac{1}{3},\frac{1}{3},3,0,0\right)$	$\boxed{78} = \left(0,\frac{1}{3},\frac{1}{3},\frac{7}{3},\frac{1}{3},0\right)$
$\boxed{79} = (0,0,1,3,0,0)$	$\boxed{80} = \left(0,0,1,\frac{7}{3},\frac{1}{3},0\right)$	$\boxed{81} = \left(0,0,0,\frac{10}{3},\frac{1}{3},0\right)$	$\boxed{82} = \left(0,0,0,\frac{8}{3},\frac{2}{3},0\right)$
$\boxed{83} = (0,0,0,4,0,0)$	$\boxed{84} = \left(0,\frac{2}{3},\frac{5}{3},1,0,0\right)$	$\boxed{85} = \left(0,\frac{1}{3},\frac{4}{3},\frac{4}{3},\frac{1}{3},0\right)$	$\boxed{86} = \left(0,0,1,\frac{5}{3},\frac{2}{3},0\right)$
$\boxed{87} = \left(0,\frac{2}{3},\frac{5}{3},\frac{1}{3},\frac{1}{3},0\right)$	$\boxed{88} = \left(0,\frac{2}{3},\frac{2}{3},\frac{4}{3},\frac{1}{3},0\right)$	$\boxed{89} = \left(0,\frac{1}{3},\frac{1}{3},\frac{5}{3},\frac{2}{3},0\right)$	$\boxed{90^*} = \left(0,\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},0\right)$
$\boxed{91} = \left(0,\frac{1}{3},\frac{4}{3},\frac{2}{3},\frac{2}{3},0\right)$			

Comparing our crystal graphs with those in [7] we found that some 2-arrows are missing in Fig. 3 of [7].

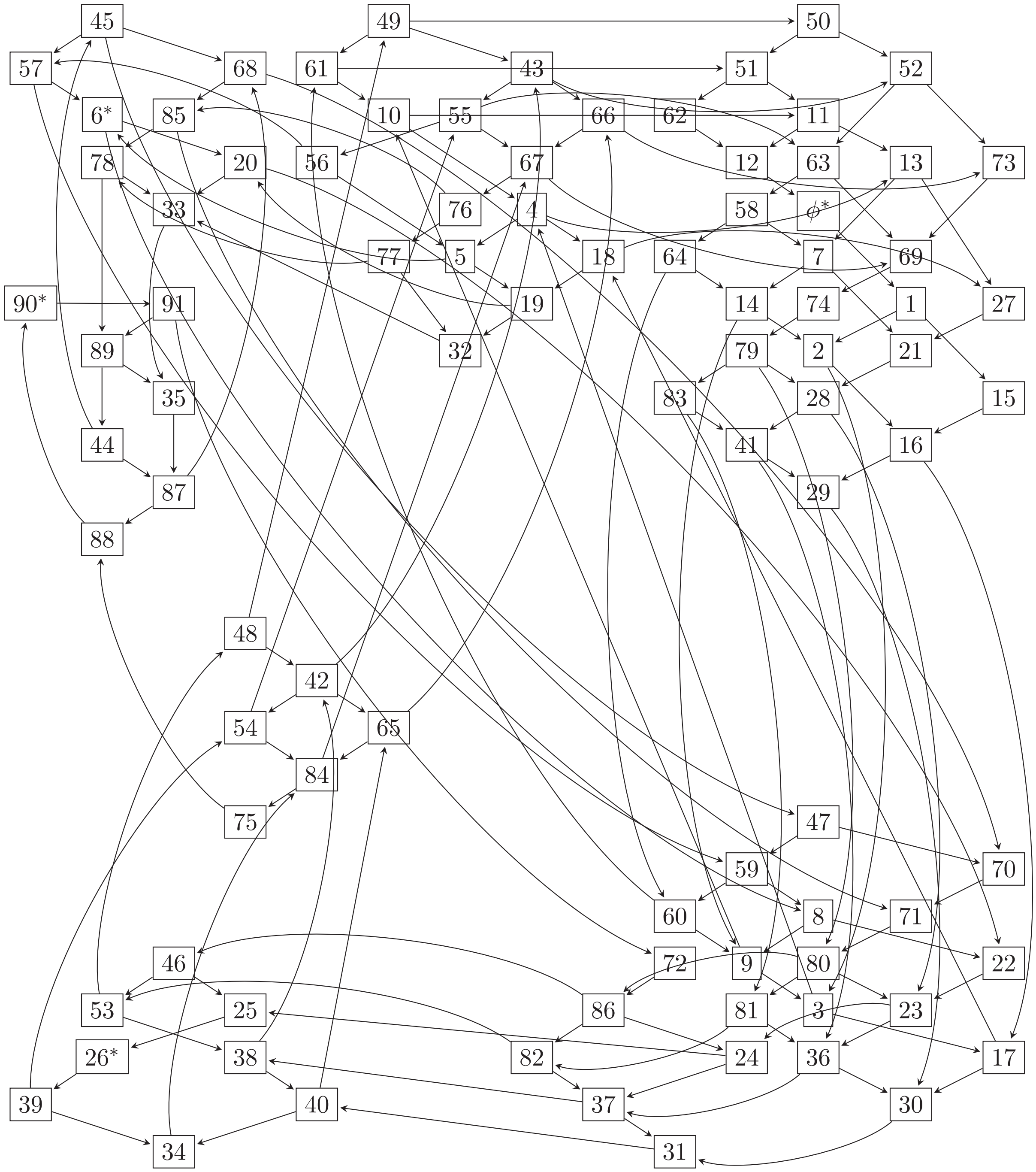

Figure 2. Crystal graph of B_2. $\searrow$ is f_0, $\swarrow$ is f_1 and others are f_2.

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