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Double Affine Hecke Algebras of Rank 1 and the $\mathbb{Z}_3$-Symmetric Askey–Wilson Relations

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Abstract. We consider the double affine Hecke algebra $H = H(k_0, k_1, k_0^\vee, k_1^\vee; q)$ associated with the root system $(C_1^\vee, C_1)$. We display three elements x, y, z in H that satisfy essentially the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. We obtain the relations as follows. We work with an algebra $\hat{H}$ that is more general than H, called the universal double affine Hecke algebra of type $(C_1^\vee, C_1)$. An advantage of $\hat{H}$ over H is that it is parameter free and has a larger automorphism group. We give a surjective algebra homomorphism $\hat{H} \to H$. We define some elements x, y, z in $\hat{H}$ that get mapped to their counterparts in H by this homomorphism. We give an action of Artin's braid group B_3 on $\hat{H}$ that acts nicely on the elements x, y, z; one generator sends $x \mapsto y \mapsto z \mapsto x$ and another generator interchanges x, y. Using the B_3 action we show that the elements x, y, z in $\hat{H}$ satisfy three equations that resemble the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. Applying the homomorphism $\hat{H} \to H$ we find that the elements x, y, z in H satisfy similar relations.

Key words: Askey–Wilson polynomials; Askey–Wilson relations; braid group

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1 Introduction

The double affine Hecke algebra (DAHA) for a reduced root system was defined by Cherednik [2], and the definition was extended to include nonreduced root systems by Sahi [11]. The most general DAHA of rank 1 is associated with the root system $(C_1^\vee, C_1)$ [8]; this algebra involves five nonzero parameters and will be denoted by $H = H(k_0, k_1, k_0^\vee, k_1^\vee; q)$. We mention some recent results on H. In [12] Sahi links certain H-modules to the Askey–Wilson polynomials [1]. This link is given a comprehensive treatment by Noumi and Stokman [7]. In [9] Oblomkov and Stoica describe the finite-dimensional irreducible H-modules under the assumption that q is not a root of unity. In [8] Oblomkov gives a detailed study of the algebraic structure of H, and finds an intimate connection to the geometry of affine cubic surfaces. His point of departure is the case $q = 1$; under that assumption he finds that the spherical subalgebra of H is generated by three elements X_1, X_2, X_3 that mutually commute and satisfy a certain cubic equation [8, Theorem 2.1, Proposition 3.1]. In [4, 5] Koornwinder describes the spherical subalgebra of H under the assumption that q is not a root of unity. His main results [4, Corollary 6.3], [5, Theorem 3.2] are similar in nature to those of Oblomkov, although he formulates these results in a very different way and works with a different presentation of H. In Koornwinder's formulation the spherical subalgebra of H is related to the Askey–Wilson algebra $AW(3)$, which was introduced by Zhedanov in [17]. The original presentation of $AW(3)$ involves three generators and three relations [17, lines (1.1a)–(1.1c)]. Koornwinder works with a slightly different presentation for $AW(3)$ that involves two generators and two relations [4, lines (2.1), (2.2)]. These two relations are sometimes called the Askey–Wilson relations [15]. For the algebra $AW(3)$ a third presentation is known [10, p. 101], [14], [16, Section 4.3] and described as follows. For a sequence of scalars g_x, g_y, g_z, h_x, h_y, h_z the corresponding Askey–Wilson algebra is defined by generators X, Y, Z and relations

$$qXY - q^{-1}YX = g_z Z + h_z, \tag{1}$$
$$qYZ - q^{-1}ZY = g_x X + h_x, \tag{2}$$
$$qZX - q^{-1}XZ = g_y Y + h_y. \tag{3}$$

We will refer to (1)–(3) as the $\mathbb{Z}_3$-*symmetric Askey–Wilson relations*. Upon eliminating Z in (2), (3) using (1) we obtain the Askey–Wilson relations in the variables X, Y. Upon substituting $Z' = g_z Z + h_z$ in (1)–(3) we recover the the original presentation for $AW(3)$ in the variables X, Y, Z'.

In this paper we return to the elements X_1, X_2, X_3 considered by Oblomkov, although for notational convenience we will call them x, y, z. We show that x, y, z satisfy three equations that resemble the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. The resemblance is described as follows. The equations have the form (1)–(3) with h_x, h_y, h_z not scalars but instead rational expressions involving an element t_1 that commutes with each of x, y, z. The element t_1 appears earlier in the work of Koornwinder [4, Definition 6.1]; we will say more about this at the end of Section 2. Our derivation of the three equations is elementary and illuminates a role played by Artin's braid group B_3.

Our proof is summarized as follows. Adapting some ideas of Ion and Sahi [3] we work with an algebra $\hat{H}$ that is more general than H, called the universal double affine Hecke algebra (UDAHA) of type $(C_1^\vee, C_1)$. An advantage of $\hat{H}$ over H is that it is parameter free and has a larger automorphism group. We give a surjective algebra homomorphism $\hat{H} \to H$. We define some elements x, y, z in $\hat{H}$ that get mapped to their counterparts in H by this homomorphism. Adapting [3, Theorem 2.6] we give an action of the braid group B_3 on $\hat{H}$ that acts nicely on the elements x, y, z; one generator sends $x \mapsto y \mapsto z \mapsto x$ and another generator interchanges x, y. Using the B_3 action we show that the elements x, y, z in $\hat{H}$ satisfy three equations that resemble the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. Applying the homomorphism $\hat{H} \to H$ we find that the elements x, y, z in H satisfy similar relations.

2 The double affine Hecke algebra of type $(C_1^\vee, C_1)$

Throughout the paper $\mathbb{F}$ denotes a field. An algebra is meant to be associative and have a 1.

We recall the double affine Hecke algebra of type $(C_1^\vee, C_1)$. For this algebra there are several presentations in the literature; one involves three generators [4, 5, 13] and another involves four generators [6, p. 160], [7, 8, 9]. We will use essentially the presentation of [6, p. 160], with an adjustment designed to make explicit the underlying symmetry.

Definition 2.1. Fix nonzero scalars k_0, k_1, $k_0^\vee$, $k_1^\vee$, q in $\mathbb{F}$. Let $H(k_0, k_1, k_0^\vee, k_1^\vee; q)$ denote the $\mathbb{F}$-algebra defined by generators t_i, $t_i^\vee$ $(i = 0, 1)$ and relations

$$(t_i - k_i)(t_i - k_i^{-1}) = 0, \tag{4}$$
$$(t_i^\vee - k_i^\vee)(t_i^\vee - k_i^{\vee -1}) = 0, \tag{5}$$
$$t_0^\vee t_0 t_1^\vee t_1 = q^{-1}. \tag{6}$$

This algebra is called the *double affine Hecke algebra* (or DAHA) of type $(C_1^\vee, C_1)$.

Note 2.2. In [6, p. 160] Macdonald gives a presentation of H involving four generators. To go from his presentation to ours, multiply each of his generators and the corresponding parameter by $\sqrt{-1}$, and replace his q by q^2.

The following result is well known; see for example [13, Corollary 1].

Lemma 2.3. *Referring to Definition 2.1, for $i \in \{0,1\}$ the elements t_i, $t_i^\vee$ are invertible and*

$$t_i + t_i^{-1} = k_i + k_i^{-1}, \qquad t_i^\vee + t_i^{\vee -1} = k_i^\vee + k_i^{\vee -1}.$$

Proof. Define $r_i = k_i + k_i^{-1} - t_i$ and $r_i^\vee = k_i^\vee + k_i^{\vee -1} - t_i^\vee$. Using (4), (5) we find $t_i r_i = r_i t_i = 1$ and $t_i^\vee r_i^\vee = r_i^\vee t_i^\vee = 1$. The result follows. ∎

We now state our main result. In this result part (ii) follows from [8, Theorem 2.1]; it is included here for the sake of completeness.

Theorem 2.4. *In the algebra $H(k_0, k_1, k_0^\vee, k_1^\vee; q)$ from Definition 2.1, define*

$$x = t_0^\vee t_1 + \big(t_0^\vee t_1\big)^{-1}, \qquad y = t_1^\vee t_1 + \big(t_1^\vee t_1\big)^{-1}, \qquad z = t_0 t_1 + (t_0 t_1)^{-1}.$$

Then the following (i)–(iv) *hold:*

(i) *t_1 commutes with each of x, y, z.*

(ii) *Assume $q^2 = 1$. Then x, y, z mutually commute.*

(iii) *Assume $q^2 \neq 1$ and $q^4 = 1$. Then $\operatorname{Char}(\mathbb{F}) \neq 2$ and*

$$\frac{xy + yx}{2} = \big(k_0^\vee + k_0^{\vee -1}\big)\big(k_1^\vee + k_1^{\vee -1}\big) + \big(k_0 + k_0^{-1}\big)\big(q^{-1} t_1 + q t_1^{-1}\big),$$
$$\frac{yz + zy}{2} = \big(k_1^\vee + k_1^{\vee -1}\big)\big(k_0 + k_0^{-1}\big) + \big(k_0^\vee + k_0^{\vee -1}\big)\big(q^{-1} t_1 + q t_1^{-1}\big),$$
$$\frac{zx + xz}{2} = \big(k_0 + k_0^{-1}\big)\big(k_0^\vee + k_0^{\vee -1}\big) + \big(k_1^\vee + k_1^{\vee -1}\big)\big(q^{-1} t_1 + q t_1^{-1}\big).$$

(iv) *Assume $q^4 \neq 1$. Then*

$$\frac{qxy - q^{-1}yx}{q^2 - q^{-2}} + z = \frac{(k_0^\vee + k_0^{\vee -1})(k_1^\vee + k_1^{\vee -1}) + (k_0 + k_0^{-1})(q^{-1} t_1 + q t_1^{-1})}{q + q^{-1}},$$
$$\frac{qyz - q^{-1}zy}{q^2 - q^{-2}} + x = \frac{(k_1^\vee + k_1^{\vee -1})(k_0 + k_0^{-1}) + (k_0^\vee + k_0^{\vee -1})(q^{-1} t_1 + q t_1^{-1})}{q + q^{-1}},$$
$$\frac{qzx - q^{-1}xz}{q^2 - q^{-2}} + y = \frac{(k_0 + k_0^{-1})(k_0^\vee + k_0^{\vee -1}) + (k_1^\vee + k_1^{\vee -1})(q^{-1} t_1 + q t_1^{-1})}{q + q^{-1}}.$$

The equations in Theorem 2.4(iv) resemble the $\mathbb{Z}_3$-symmetric Askey–Wilson relations, as we discussed in Section 1.

We will prove Theorem 2.4 in Section 5.

We comment on how Theorem 2.4 is related to the work of Koornwinder [4]. Define x, y, z as in Theorem 2.4. Then that theorem describes how x, y, z, t_1 are related. If we translate [4, Definition 6.1, Corollary 6.3] into the presentation of Definition 2.1, then it describes how x, y, t_1 are related, assuming q is not a root of unity and some constraints on k_0, k_1, $k_0^\vee$, $k_1^\vee$. Under these assumptions and modulo the translation the following coincide: (i) the main relations [4, lines (6.2), (6.3)] of [4, Definition 6.1]; (ii) the relations obtained from the last two equations of Theorem 2.4(iv) by eliminating z using the first equation.

3 The universal double affine Hecke algebra of type $(C_1^\vee, C_1)$

In our proof of Theorem 2.4 we will initially work with a homomorphic preimage $\hat{H}$ of $H(k_0, k_1, k_0^\vee, k_1^\vee; q)$ called the universal double affine Hecke algebra of type $(C_1^\vee, C_1)$. Before we get into the details, we would like to acknowledge how $\hat{H}$ is related to the work of Ion and Sahi [3]. Given a general DAHA (not just rank 1) Ion and Sahi construct a group $\tilde{\mathcal{A}}$ called the double affine Artin group [3, Definition 3.4, Theorem 3.10]. The given DAHA is a homomorphic image of the group $\mathbb{F}$-algebra $\mathbb{F}\tilde{\mathcal{A}}$ [3, Definition 1.13]. For the case $(C_1^\vee, C_1)$ of the present paper, their homomorphism has a factorization $\mathbb{F}\tilde{\mathcal{A}} \to \hat{H} \to H(k_0, k_1, k_0^\vee, k_1^\vee; q)$. In this section and the next we will obtain some facts about $\hat{H}$. We could obtain these facts from [3] by applying the homomorphism $\mathbb{F}\tilde{\mathcal{A}} \to \hat{H}$, but for the purpose of clarity we will prove everything from first principles.

We now define $\hat{H}$ and describe some of its basic properties. In Section 4 we will discuss how the group B_3 acts on $\hat{H}$. In Section 5 we will use the B_3 action to prove Theorem 2.4.

Definition 3.1. Let $\hat{H}$ denote the $\mathbb{F}$-algebra defined by generators $t_i^{\pm 1}$, $(t_i^\vee)^{\pm 1}$ $(i = 0, 1)$ and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1, \qquad t_i^\vee t_i^{\vee -1} = t_i^{\vee -1} t_i^\vee = 1, \tag{7}$$

$$t_i + t_i^{-1} \quad \text{is central}, \qquad t_i^\vee + t_i^{\vee -1} \quad \text{is central}, \tag{8}$$

$$t_0^\vee t_0 t_1^\vee t_1 \quad \text{is central}. \tag{9}$$

We call $\hat{H}$ the *universal double affine Hecke algebra* (or UDAHA) of type $(C_1^\vee, C_1)$.

Note 3.2. The double affine Artin group $\tilde{\mathcal{A}}$ of type $(C_1^\vee, C_1)$ is defined by generators $t_i^{\pm 1}$, $(t_i^\vee)^{\pm 1}$ $(i = 0, 1)$ and relations (7), (9) [3, Theorem 3.11].

Definition 3.3. Observe that in $\hat{H}$ the element $t_0^\vee t_0 t_1^\vee t_1$ is invertible; let Q denote the inverse.

Lemma 3.4. *Given nonzero scalars k_0, k_1, $k_0^\vee$, $k_1^\vee$, q in $\mathbb{F}$, there exists a surjective $\mathbb{F}$-algebra homomorphism $\hat{H} \to H(k_0, k_1, k_0^\vee, k_1^\vee; q)$ that sends $Q \mapsto q$ and $t_i \mapsto t_i, t_i^\vee \mapsto t_i^\vee$ for $i \in \{0, 1\}$.*

Proof. Compare the defining relations for $\hat{H}$ and $H(k_0, k_1, k_0^\vee, k_1^\vee; q)$. ■

One advantage of $\hat{H}$ over $H(k_0, k_1, k_0^\vee, k_1^\vee; q)$ is that $\hat{H}$ has more automorphisms. This is illustrated in the next lemma. By an *automorphism* of $\hat{H}$ we mean an $\mathbb{F}$-algebra isomorphism $\hat{H} \to \hat{H}$.

Lemma 3.5. *There exists an automorphism of $\hat{H}$ that sends*

$$t_0^\vee \mapsto t_0, \qquad t_0 \mapsto t_1^\vee, \qquad t_1^\vee \mapsto t_1, \qquad t_1 \mapsto t_0^\vee.$$

This automorphism fixes Q.

Proof. The result follows from Definition 3.1, once we verify that $t_0 t_1^\vee t_1 t_0^\vee = Q^{-1}$. This equation holds since each side is equal to $t_0^{\vee -1} Q^{-1} t_0^\vee$. ■

Lemma 3.6. *In the algebra $\hat{H}$ the element Q^{-1} is equal to each of the following:*

$$t_0^\vee t_0 t_1^\vee t_1, \qquad t_0 t_1^\vee t_1 t_0^\vee, \qquad t_1^\vee t_1 t_0^\vee t_0, \qquad t_1 t_0^\vee t_0 t_1^\vee. \tag{10}$$

Proof. To each side of the equation $t_0^\vee t_0 t_1^\vee t_1 = Q^{-1}$ apply three times the automorphism from Lemma 3.5. ■

Definition 3.7. We define elements x, y, z in $\hat{H}$ as follows.

$$x = t_0^\vee t_1 + \big(t_0^\vee t_1\big)^{-1}, \qquad y = t_1^\vee t_1 + \big(t_1^\vee t_1\big)^{-1}, \qquad z = t_0 t_1 + (t_0 t_1)^{-1}.$$

The following result suggests why x, y, z are of interest.

Lemma 3.8. *Let u, v denote invertible elements in any algebra such that each of $u + u^{-1}$, $v + v^{-1}$ is central. Then*

(i) $uv + (uv)^{-1} = vu + (vu)^{-1}$;

(ii) $uv + (uv)^{-1}$ *commutes with each of* u, v.

Proof. (i) Observe that

$$uv + (uv)^{-1} = uv + vu - \big(v + v^{-1}\big)u - v\big(u + u^{-1}\big) + \big(v + v^{-1}\big)\big(u + u^{-1}\big),$$
$$vu + (vu)^{-1} = uv + vu - u\big(v + v^{-1}\big) - \big(u + u^{-1}\big)v + \big(u + u^{-1}\big)\big(v + v^{-1}\big).$$

In these equations the expressions on the right are equal since $u + u^{-1}$ and $v + v^{-1}$ are central. The result follows.

(ii) We have

$$u^{-1}\big(uv + (uv)^{-1}\big)u = uv + (uv)^{-1}$$

since each side is equal to $vu + (vu)^{-1}$. Therefore $uv + (uv)^{-1}$ commutes with u. One similarly shows that $uv + (uv)^{-1}$ commutes with v. ■

Corollary 3.9. *In the algebra $\hat{H}$ the element t_1 commutes with each of x, y, z.*

Proof. Use Definition 3.7 and Lemma 3.8(ii). ■

4 The braid group B_3

In this section we display an action of the braid group B_3 on the algebra $\hat{H}$ from Definition 3.1. This B_3 action will be used to prove Theorem 2.4.

Definition 4.1. Artin's braid group B_3 is defined by generators b, c and the relation $b^3 = c^2$. For notational convenience define $a = b^3 = c^2$.

The following result is a variation on [3, Theorem 2.6].

Lemma 4.2. *The braid group B_3 acts on $\hat{H}$ as a group of automorphisms such that $a(h) = t_1^{-1} h t_1$ for all $h \in \hat{H}$ and b, c do the following:*

h	$t_0^\vee$	t_0	$t_1^\vee$	t_1
$b(h)$	$t_1^{-1} t_1^\vee t_1$	$t_0^\vee$	t_0	t_1
$c(h)$	$t_1^{-1} t_1^\vee t_1$	$t_0^\vee t_0 t_0^{\vee -1}$	$t_0^\vee$	t_1

Proof. There exists an automorphism A of $\hat{H}$ that sends $h \mapsto t_1^{-1} h t_1$ for all $h \in \hat{H}$. Define

$$T_0^\vee = t_1^{-1} t_1^\vee t_1, \qquad T_0 = t_0^\vee, \qquad T_1^\vee = t_0, \qquad T_1 = t_1. \tag{11}$$

Note that $T_0^\vee$, T_0, $T_1^\vee$, T_1 are invertible and that

$$T_0^\vee + T_0^{\vee -1} = t_1^\vee + t_1^{\vee -1}, \qquad T_0 + T_0^{-1} = t_0^\vee + t_0^{\vee -1},$$
$$T_1^\vee + T_1^{\vee -1} = t_0 + t_0^{-1}, \qquad T_1 + T_1^{-1} = t_1 + t_1^{-1}.$$

In each of these four equations the expression on the right is central so the expression on the left is central. Using (11) and Lemma 3.6,

$$T_0^\vee T_0 T_1^\vee T_1 = t_1^{-1} t_1^\vee t_1 t_0^\vee t_0 t_1 = t_1^{-1} Q^{-1} t_1 = Q^{-1}$$

so $T_0^\vee T_0 T_1^\vee T_1$ is central. By these comments there exists an $\mathbb{F}$-algebra homomorphism $B : \hat{H} \to \hat{H}$ that sends

$$t_0^\vee \mapsto T_0^\vee, \qquad t_0 \mapsto T_0, \qquad t_1^\vee \mapsto T_1^\vee, \qquad t_1 \mapsto T_1.$$

We claim that $B^3 = A$. To prove the claim we show that B^3, A agree at each of $t_0^\vee$, t_0, $t_1^\vee$, t_1. Note that A fixes t_1. Note also that t_1 is fixed by B and hence B^3; therefore B^3 and A agree at t_1. The map B sends

$$t_1^\vee \mapsto t_0 \mapsto t_0^\vee \mapsto t_1^{-1} t_1^\vee t_1 \mapsto t_1^{-1} t_0 t_1 \mapsto t_1^{-1} t_0^\vee t_1.$$

Therefore B^3 sends

$$t_1^\vee \mapsto t_1^{-1} t_1^\vee t_1, \qquad t_0 \mapsto t_1^{-1} t_0 t_1, \qquad t_0^\vee \mapsto t_1^{-1} t_0^\vee t_1,$$

so B^3, A agree at each of $t_1^\vee$, t_0, $t_0^\vee$. We have shown $B^3 = A$. By this and since A is invertible, we see that B is invertible and hence an automorphism of $\hat{H}$. Define

$$S_0^\vee = t_1^{-1} t_1^\vee t_1, \qquad S_0 = t_0^\vee t_0 t_0^{\vee -1}, \qquad S_1^\vee = t_0^\vee, \qquad S_1 = t_1. \tag{12}$$

Note that $S_0^\vee$, S_0, $S_1^\vee$, S_1 are invertible and

$$S_0^\vee + S_0^{\vee -1} = t_1^\vee + t_1^{\vee -1}, \qquad S_0 + S_0^{-1} = t_0 + t_0^{-1},$$
$$S_1^\vee + S_1^{\vee -1} = t_0^\vee + t_0^{\vee -1}, \qquad S_1 + S_1^{-1} = t_1 + t_1^{-1}.$$

In each of these four equations the expression on the right is central so the expression on the left is central. Using (12) and Lemma 3.6,

$$S_0^\vee S_0 S_1^\vee S_1 = t_1^{-1} t_1^\vee t_1 t_0^\vee t_0 t_1 = t_1^{-1} Q^{-1} t_1 = Q^{-1}$$

so $S_0^\vee S_0 S_1^\vee S_1$ is central. By these comments there exists an $\mathbb{F}$-algebra homomorphism $C : \hat{H} \to \hat{H}$ that sends

$$t_0^\vee \mapsto S_0^\vee, \qquad t_0 \mapsto S_0, \qquad t_1^\vee \mapsto S_1^\vee, \qquad t_1 \mapsto S_1.$$

We claim that $C^2 = A$. To prove the claim we show that C^2, A agree at each of $t_0^\vee$, t_0, $t_1^\vee$, t_1. Both C^2 and A fix t_1. The map C sends $t_0^\vee \mapsto t_1^{-1} t_1^\vee t_1 \mapsto t_1^{-1} t_0^\vee t_1$ so C^2, A agree at $t_0^\vee$. The map C sends $t_1^\vee \mapsto t_0^\vee \mapsto t_1^{-1} t_1^\vee t_1$ so C^2, A agree at $t_1^\vee$. The map C sends

$$t_0 \mapsto t_0^\vee t_0 t_0^{\vee -1} \mapsto t_1^{-1} t_1^\vee t_1 t_0^\vee t_0 t_0^{\vee -1} t_1^{-1} t_1^{\vee -1} t_1.$$

In the above line the expression on the right equals $t_1^{-1} t_0 t_1$. To see this, note that $t_1^\vee t_1 t_0^\vee t_0 = t_0 t_1^\vee t_1 t_0^\vee$ since each side equals Q^{-1} by Lemma 3.6. We have shown that C^2, A agree at t_0. By the above comments C^2, A agree at each of $t_0^\vee$, t_0, $t_1^\vee$, t_1 so $C^2 = A$. Therefore C is invertible and hence an automorphism of $\hat{H}$. We have shown that the desired B_3 action exists. ■

The next result is immediate from Lemma 4.2 and its proof.

Lemma 4.3. *The B_3 action from Lemma 4.2 does the following to the central elements (8), (9). The generator a fixes every central element. The generators b, c fix Q and satisfy the table below.*

h	$t_0^\vee + t_0^{\vee -1}$	$t_0 + t_0^{-1}$	$t_1^\vee + t_1^{\vee -1}$	$t_1 + t_1^{-1}$
$b(h)$	$t_1^\vee + t_1^{\vee -1}$	$t_0^\vee + t_0^{\vee -1}$	$t_0 + t_0^{-1}$	$t_1 + t_1^{-1}$
$c(h)$	$t_1^\vee + t_1^{\vee -1}$	$t_0 + t_0^{-1}$	$t_0^\vee + t_0^{\vee -1}$	$t_1 + t_1^{-1}$

5 The proof of Theorem 2.4

Recall the elements x, y, z of $\hat{H}$ from Definition 3.7. In this section we describe how the group B_3 acts on these elements. Using this information we show that x, y, z satisfy three equations that resemble the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. Using these equations we obtain Theorem 2.4.

Theorem 5.1. *The B_3 action from Lemma 4.2 does the following to the elements x, y, z from Definition 3.7. The generator a fixes each of x, y, z. The generator b sends $x \mapsto y \mapsto z \mapsto x$. The generator c swaps x, y and sends $z \mapsto z'$ where*

$$\begin{aligned} Qz + Q^{-1}z' + xy &= Q^{-1}z + Qz' + yx \\ &= \left(t_0^\vee + t_0^{\vee-1}\right)\left(t_1^\vee + t_1^{\vee-1}\right) + \left(t_0 + t_0^{-1}\right)\left(Q^{-1}t_1 + Qt_1^{-1}\right). \end{aligned}$$

Proof. The generator a fixes each of x, y, z by Corollary 3.9 and since $a(h) = t_1^{-1}ht_1$ for all $h \in \hat{H}$. The generator b sends $x \mapsto y \mapsto z \mapsto x$ by Definition 3.7, Corollary 3.9, and Lemma 4.2. Similarly the generator c swaps x, y. Define $z' = c(z)$. We show that z' satisfies the equations in the theorem statement. We first show that

$$Q^{-1}t_0 + Qc(t_0) + yt_0^\vee = (t_1^\vee t_1)^{-1}\left(t_0^\vee + t_0^{\vee-1}\right) + Q^{-1}\left(t_0 + t_0^{-1}\right). \tag{13}$$

By Lemma 4.2, $c(t_0) = t_0^\vee t_0 t_0^{\vee-1}$. By this and Definition 3.3,

$$Qc(t_0) = \left(t_1^\vee t_1\right)^{-1} t_0^{\vee-1}. \tag{14}$$

By Lemma 3.6,

$$t_1^\vee t_1 t_0^\vee = Q^{-1}t_0^{-1}. \tag{15}$$

Using (14), (15) and $y = t_1^\vee t_1 + (t_1^\vee t_1)^{-1}$ we obtain (13). Next we show that

$$Q^{-1}t_0^{-1} + Qc\left(t_0^{-1}\right) + yt_0^{\vee-1} = t_1^\vee t_1\left(t_0^\vee + t_0^{\vee-1}\right) + Q\left(t_0 + t_0^{-1}\right). \tag{16}$$

By Lemma 4.3,

$$c(t_0) + c\left(t_0^{-1}\right) = t_0 + t_0^{-1}.$$

Combining this with (13) we obtain (16) after a brief calculation. In (13) we multiply each term on the right by t_1 and use $c(t_1) = t_1$ to get

$$Q^{-1}t_0t_1 + Qc(t_0t_1) + yt_0^\vee t_1 = \left(t_1^\vee t_1\right)^{-1} t_1\left(t_0^\vee + t_0^{\vee-1}\right) + Q^{-1}t_1\left(t_0 + t_0^{-1}\right). \tag{17}$$

In (16) we multiply each term on the left by t_1^{-1} and use $c(t_1^{-1}) = t_1^{-1}$ together with the fact that y commutes with t_1 to get

$$Q^{-1}(t_0t_1)^{-1} + Qc\left((t_0t_1)^{-1}\right) + y\left(t_0^\vee t_1\right)^{-1} = t_1^{-1}t_1^\vee t_1\left(t_0^\vee + t_0^{\vee-1}\right) + Qt_1^{-1}\left(t_0 + t_0^{-1}\right). \tag{18}$$

We have

$$\left(t_1^\vee t_1\right)^{-1} t_1 + t_1^{-1}t_1^\vee t_1 = t_1^\vee + t_1^{\vee-1} \tag{19}$$

since both sides equal $t_1^{-1}(t_1^\vee + t_1^{\vee-1})t_1$. We now add (17), (18) and simplify the result using (19) to obtain

$$Q^{-1}z + Qz' + yx = \left(t_0^\vee + t_0^{\vee-1}\right)\left(t_1^\vee + t_1^{\vee-1}\right) + \left(t_0 + t_0^{-1}\right)\left(Q^{-1}t_1 + Qt_1^{-1}\right). \tag{20}$$

We now apply c to each side of (20) and evaluate the result. To aid in this evaluation we recall that c swaps x, y; also c swaps z, z' since $c^2 = a$ and $a(z) = z$. By these comments and Lemma 4.3 we obtain

$$Qz + Q^{-1}z' + xy = \left(t_0^\vee + t_0^{\vee-1}\right)\left(t_1^\vee + t_1^{\vee-1}\right) + \left(t_0 + t_0^{-1}\right)\left(Q^{-1}t_1 + Qt_1^{-1}\right).$$ ∎

Theorem 5.2. *In the algebra $\hat{H}$ the elements x, y, z are related as follows:*

$$
\begin{aligned}
&Qxy - Q^{-1}yx + \big(Q^2 - Q^{-2}\big)z \\
&\qquad = \big(Q - Q^{-1}\big)\big(\big(t_0^\vee + t_0^{\vee -1}\big)\big(t_1^\vee + t_1^{\vee -1}\big) + \big(t_0 + t_0^{-1}\big)\big(Q^{-1}t_1 + Qt_1^{-1}\big)\big), \\
&Qyz - Q^{-1}zy + \big(Q^2 - Q^{-2}\big)x \\
&\qquad = \big(Q - Q^{-1}\big)\big(\big(t_1^\vee + t_1^{\vee -1}\big)\big(t_0 + t_0^{-1}\big) + \big(t_0^\vee + t_0^{\vee -1}\big)\big(Q^{-1}t_1 + Qt_1^{-1}\big)\big), \\
&Qzx - Q^{-1}xz + \big(Q^2 - Q^{-2}\big)y \\
&\qquad = \big(Q - Q^{-1}\big)\big(\big(t_0 + t_0^{-1}\big)\big(t_0^\vee + t_0^{\vee -1}\big) + \big(t_1^\vee + t_1^{\vee -1}\big)\big(Q^{-1}t_1 + Qt_1^{-1}\big)\big).
\end{aligned}
$$

Proof. To get the first equation, eliminate z' from the equations of Theorem 5.1. To get the other two equations use the B_3 action from Lemma 4.2. Specifically, apply b twice to the first equation and use the data in Lemma 4.3, together with the fact that b cyclically permutes x, y, z. ■

Proof of Theorem 2.4. Apply the homomorphism $\hat{H} \to H(k_0, k_1, k_0^\vee, k_1^\vee)$ from Lemma 3.4. Part (i) follows via Corollary 3.9, and parts (ii)–(iv) follow from Theorem 5.2 together with Lemma 2.3. ■

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