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On the Complex Symmetric and Skew-Symmetric Operators with a Simple Spectrum

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Abstract. In this paper we obtain necessary and sufficient conditions for a linear bounded operator in a Hilbert space H to have a three-diagonal complex symmetric matrix with non-zero elements on the first sub-diagonal in an orthonormal basis in H. It is shown that a set of all such operators is a proper subset of a set of all complex symmetric operators with a simple spectrum. Similar necessary and sufficient conditions are obtained for a linear bounded operator in H to have a three-diagonal complex skew-symmetric matrix with non-zero elements on the first sub-diagonal in an orthonormal basis in H.

Key words: complex symmetric operator; complex skew-symmetric operator; cyclic operator; simple spectrum

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1 Introduction

In last years an increasing interest is devoted to the subject of operators related to bilinear forms in a Hilbert space (see [1, 2, 3] and references therein), i.e. to the following forms:

$$[x, y]_J := (x, Jy)_H, \qquad x, y \in H,$$

where J is a conjugation and $(\cdot, \cdot)_H$ is the inner product in a Hilbert space H. The conjugation J is an *antilinear* operator in H such that $J^2 x = x$, $x \in H$, and

$$(Jx, Jy)_H = (y, x)_H, \qquad x, y \in H.$$

Recall that a linear operator A in H is said to be J-symmetric (J-skew-symmetric) if

$$[Ax, y]_J = [x, Ay]_J, \qquad x, y \in D(A), \tag{1}$$

or, respectively,

$$[Ax, y]_J = -[x, Ay]_J, \qquad x, y \in D(A). \tag{2}$$

If a linear bounded operator A in a Hilbert space H is J-symmetric (J-skew-symmetric) for a conjugation J in H, then A is said to be complex symmetric (respectively complex skew-symmetric). The matrices of complex symmetric (skew-symmetric) operators in certain bases of H are complex symmetric (respectively skew-symmetric) semi-infinite matrices. Observe that for a bounded linear operator A conditions (1) and (2) are equivalent to conditions

$$JAJ = A^*, \tag{3}$$

and

$$JAJ = -A^*,$$

respectively.

Recall that a bounded linear operator A in a Hilbert space H is said to have a simple spectrum if there exists a vector $x_0 \in H$ (cyclic vector) such that

$$\overline{\mathrm{Lin}\{A^k x_0,\ k \in \mathbb{Z}_+\}} = H.$$

Observe that these operators are also called *cyclic operators*.

It is well known that a bounded self-adjoint operator with a simple spectrum has a bounded semi-infinite real symmetric three-diagonal (Jacobi) matrix in a certain orthonormal basis (e.g. [4, Theorem 4.2.3]).

The aim of our present investigation is to describe a class $C_+ = C_+(H)$ ($C_- = C_-(H)$) of linear bounded operators in a Hilbert space H, which have three-diagonal complex symmetric (respectively skew-symmetric) matrices with non-zero elements on the first sub-diagonal in some orthonormal bases of H. We obtain necessary and sufficient conditions for a linear bounded operator in a Hilbert space H to belong to the class C_+ (C_-). The class C_+ (C_-) is a subset of the class of all complex symmetric (respectively skew-symmetric) operators in H with a simple spectrum. Moreover, it is shown that $C_+(H)$ is a proper subset.

Notations. As usual, we denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; $\mathrm{Im}\, z = \frac{1}{2i}(z - \overline{z})$, $z \in \mathbb{C}$. Everywhere in this paper, all Hilbert spaces are assumed to be separable. By $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ we denote the scalar product and the norm in a Hilbert space H, respectively. The indices may be omitted in obvious cases. For a set M in H, by $\overline{M}$ we mean the closure of M in the norm $\|\cdot\|_H$. For $\{x_n\}_{n\in\mathbb{Z}_+}$, $x_n \in H$, we write $\mathrm{Lin}\{x_n\}_{n\in\mathbb{Z}_+}$ for the set of linear combinations of elements $\{x_n\}_{n\in\mathbb{Z}_+}$. The identity operator in H is denoted by E_H. For an arbitrary linear operator A in H, the operators A^*, $\overline{A}$, A^{-1} mean its adjoint operator, its closure and its inverse (if they exist). By $D(A)$ and $R(A)$ we mean the domain and the range of the operator A. The norm of a bounded operator A is denoted by $\|A\|$. By $P^H_{H_1} = P_{H_1}$ we mean the operator of orthogonal projection in H on a subspace H_1 in H.

2 The classes $C_\pm(H)$

Let $\mathcal{M} = (m_{k,l})_{k,l=0}^\infty$, $m_{k,l} \in \mathbb{C}$, be a semi-infinite complex matrix. We shall say that $\mathcal{M}$ belongs to the class $\mathfrak{M}_3^+$, if and only if the following conditions hold:

$$m_{k,l} = 0, \qquad k,l \in \mathbb{Z}_+, \qquad |k-l| > 1, \tag{4}$$

$$m_{k,l} = m_{l,k}, \qquad k,l \in \mathbb{Z}_+, \tag{5}$$

$$m_{k,k+1} \neq 0, \qquad k \in \mathbb{Z}_+. \tag{6}$$

We shall say that $\mathcal{M}$ belongs to the class $\mathfrak{M}_3^-$, if and only if the conditions (4), (6) hold and

$$m_{k,l} = -m_{l,k}, \qquad k,l \in \mathbb{Z}_+.$$

Let A be a linear bounded operator in an infinite-dimensional Hilbert space H. We say that A belongs to the class $C_+ = C_+(H)$ ($C_- = C_-(H)$) if and only if there exists an orthonormal basis $\{e_k\}_{k=0}^\infty$ in H such that the matrix

$$\mathcal{M} = ((Ae_l, e_k))_{k,l=0}^\infty, \tag{7}$$

belongs to $\mathfrak{M}_3^+$ (respectively to $\mathfrak{M}_3^-$).

Let $y_0, y_1, \dots, y_n$ be arbitrary vectors in H, $n \in \mathbb{Z}_+$. Set

$$\Gamma(y_0, y_1, \dots, y_n) := \det((y_k, y_l)_H)_{k,l=0}^n.$$

Thus, $\Gamma(y_0, y_1, \dots, y_n)$ is the Gram determinant of vectors $y_0, y_1, \dots, y_n$.

The following theorem provides a description of the class $C_+(H)$.

Theorem 1. *Let A be a linear bounded operator in an infinite-dimensional Hilbert space H. The operator A belongs to the class $C_+(H)$ if and only if the following conditions hold:*

(*i*) *A is a complex symmetric operator with a simple spectrum;*

(*ii*) *there exists a cyclic vector x_0 of A such that the following relations hold:*

$$\Gamma(x_0, x_1, \dots, x_n, x_n^*) = 0, \qquad \forall\, n \in \mathbb{N}, \tag{8}$$

where

$$x_k = A^k x_0, \qquad x_k^* = (A^*)^k x_0, \qquad k \in \mathbb{N};$$

and $Jx_0 = x_0$, for a conjugation J in H such that $JAJ = A^*$.

Proof. *Necessity.* Let H be an infinite-dimensional Hilbert space and $A \in C_+(H)$. Let $\{e_k\}_{k=0}^\infty$ be an orthonormal basis in H such that the matrix $\mathcal{M} = (m_{k,l})_{k,l=0}^\infty$ belongs to $\mathfrak{M}_3^+$, where $m_{k,l} = ((Ae_l, e_k))_{k,l=0}^\infty$. Observe that

$$\begin{aligned} Ae_0 &= m_{0,0}e_0 + m_{1,0}e_1, \\ Ae_k &= m_{k-1,k}e_{k-1} + m_{k,k}e_k + m_{k+1,k}e_{k+1}, \qquad k \in \mathbb{N}. \end{aligned} \tag{9}$$

Suppose that

$$e_r \in \mathrm{Lin}\{A^j e_0,\ 0 \le j \le r\}, \qquad 0 \le r \le n,$$

for some $n \in \mathbb{N}$ (for $n = 0$ it is trivial). By (9) we may write

$$e_{n+1} = \frac{1}{m_{n+1,n}} (Ae_n - m_{n-1,n}e_{n-1} - m_{n,n}e_n) \in \mathrm{Lin}\{A^j e_0,\ 0 \le j \le n+1\}.$$

Here $m_{-1,0} := 0$ and $e_{-1} := 0$. By induction we conclude that

$$e_r \in \mathrm{Lin}\{A^j e_0,\ 0 \le j \le r\}, \qquad r \in \mathbb{Z}_+. \tag{10}$$

Therefore $\overline{\mathrm{Lin}\{A^j e_0,\ j \in \mathbb{Z}_+\}} = H$, i.e. the operator A has a simple spectrum and e_0 is a cyclic vector of A.

Consider the following conjugation:

$$J \sum_{k=0}^\infty x_k e_k = \sum_{k=0}^\infty \overline{x_k} e_k, \qquad x = \sum_{k=0}^\infty x_k e_k \in H.$$

Observe that

$$[Ae_k, e_l]_J = (Ae_k, e_l) = m_{l,k} = m_{k,l} = (Ae_l, e_k) = [Ae_l, e_k]_J, \qquad k, l \in \mathbb{Z}_+.$$

By linearity of the J-form $[\cdot,\cdot]_J$ in the both arguments we get

$$[Ax, y]_J = [Ay, x]_J, \qquad x, y \in H.$$

Thus, the operator A is J-symmetric and relation (3) holds. Notice that $Je_0 = e_0$. It remains to check if relation (8) holds. Set

$$H_r := \mathrm{Lin}\{A^j e_0,\ 0 \leq j \leq r\}, \qquad r \in \mathbb{Z}_+.$$

By (10) we see that $e_0, e_1, \ldots, e_r \in H_r$, and therefore $\{e_j\}_{j=0}^r$ is an orthonormal basis in H_r ($r \in \mathbb{Z}_+$). Since $Je_j = e_j$, $j \in \mathbb{Z}_+$, we have

$$JH_r \subseteq H_r, \qquad r \in \mathbb{Z}_+.$$

Then

$$(A^*)^r e_0 = (JAJ)^r e_0 = JA^r Je_0 = JA^r e_0 \in H_r, \qquad r \in \mathbb{Z}_+.$$

Therefore vectors $e_0, Ae_0, \ldots, A^r e_0, (A^*)^r e_0$, are linearly dependent and their Gram determinant is equal to zero. Thus, relation (8) holds with $x_0 = e_0$.

Sufficiency. Let A be a bounded operator in a Hilbert space H satisfying conditions (i), (ii) in the statement of the theorem. For the cyclic vector x_0 we set

$$H_r := \mathrm{Lin}\{A^j x_0,\ 0 \leq j \leq r\}, \qquad r \in \mathbb{Z}_+.$$

Observe that

$$A^{r+1} x_0 \notin H_r, \qquad r \in \mathbb{Z}_+. \tag{11}$$

In fact, suppose that for some $k \in \mathbb{N}$, we have

$$A^{r+j} x_0 \in H_r, \qquad 1 \leq j \leq k.$$

Then

$$A^{r+k+1} x_0 = AA^{r+k} x_0 = A \sum_{t=0}^{r} \alpha_{r,k;t} A^t x_0 = \sum_{t=0}^{r} \alpha_{r,k;t} A^{t+1} x_0 \in H_r, \qquad \alpha_{r,k;t} \in \mathbb{C}.$$

By induction we obtain

$$A^{r+j} x_0 \in H_r, \qquad j \in \mathbb{Z}_+.$$

Therefore $H = H_r$. We obtain a contradiction since H is infinite-dimensional.

Let us apply the Gram–Schmidt orthogonalization method to the sequence $x_0, Ax_0, A^2 x_0, \ldots$. Namely, we set

$$g_0 = \frac{x_0}{\|x_0\|_H}, \qquad g_{r+1} = \frac{A^{r+1} x_0 - \sum\limits_{j=0}^{r} (A^{r+1} x_0, g_j)_H g_j}{\left\| A^{r+1} x_0 - \sum\limits_{j=0}^{r} (A^{r+1} x_0, g_j)_H g_j \right\|_H}, \qquad r \in \mathbb{Z}_+.$$

By construction we have

$$H_r = \mathrm{Lin}\{g_j,\ 0 \leq j \leq r\}, \qquad r \in \mathbb{Z}_+.$$

Therefore $\{g_j\}_{j=0}^r$ is an orthonormal basis in H_r ($r \in \mathbb{Z}_+$) and $\{g_j\}_{j \in \mathbb{Z}_+}$ is an orthonormal basis in H.

From (8) and (11) we conclude that

$$JA^n x_0 = JA^n Jx_0 = (A^*)^n x_0 \in H_n, \qquad n \in \mathbb{Z}_+.$$

Therefore

$$JH_r \subseteq H_r, \qquad r \in \mathbb{Z}_+. \tag{12}$$

Let

$$Jg_r = \sum_{j=0}^{r} \beta_{r,j} g_j, \qquad \beta_{r,j} \in \mathbb{C}, \qquad r \in \mathbb{Z}_+.$$

Using properties of the conjugation and relation (12) we get

$$\beta_{r,j} = (Jg_r, g_j)_H = (Jg_r, JJg_j)_H = \overline{(g_r, Jg_j)_H} = 0,$$

for $0 \leq j \leq r-1$. Therefore

$$Jg_r = \beta_{r,r} g_r, \qquad \beta_{r,r} \in \mathbb{C}, \qquad r \in \mathbb{Z}_+.$$

Since $\|g_r\|^2 = \|Jg_r\|^2 = |\beta_{r,r}|^2 \|g_r\|^2$, we have

$$\beta_{r,r} = e^{i\varphi_r}, \qquad \varphi_r \in [0, 2\pi), \qquad r \in \mathbb{Z}_+.$$

Set

$$e_r := e^{i\frac{\varphi_r}{2}} g_r, \qquad r \in \mathbb{Z}_+.$$

Then $\{e_j\}_{j=0}^{r}$ is an orthonormal basis in H_r ($r \in \mathbb{Z}_+$) and $\{e_j\}_{j \in \mathbb{Z}_+}$ is an orthonormal basis in H. Observe that

$$Je_r = Je^{i\frac{\varphi_r}{2}} g_r = e^{-i\frac{\varphi_r}{2}} Jg_r = e^{i\frac{\varphi_r}{2}} g_r = e_r, \qquad r \in \mathbb{Z}_+.$$

Define the matrix $\mathcal{M} = (m_{k,l})_{k,l=0}^{\infty}$ by (7). Notice that

$$m_{k,l} = (Ae_l, e_k)_H = [Ae_l, e_k]_J = [e_l, Ae_k]_J = [Ae_k, e_l]_J = (Ae_k, e_l)_H = m_{l,k},$$

where $k, l \in \mathbb{Z}_+$, and therefore $\mathcal{M}$ is complex symmetric.

If $l \geq k+2$ ($k, l \in \mathbb{Z}_+$), then

$$m_{k,l} = (Ae_l, e_k)_H = [Ae_l, e_k]_J = [e_l, Ae_k]_J = (e_l, JAe_k)_H = 0,$$

since $JAe_k \in H_{k+1} \subseteq H_{l-1}$, and $e_l \in H_l \ominus H_{l-1}$. Therefore $\mathcal{M}$ is three-diagonal.

Since $e_r \in H_r$, using the definition of H_r we get

$$Ae_r \subseteq H_{r+1}, \qquad r \in \mathbb{Z}_+.$$

Observe that

$$Ae_r \notin H_r, \qquad r \in \mathbb{Z}_+.$$

In fact, in the opposite case we get

$$Ae_j \in H_r, \qquad 0 \leq j \leq r,$$

and $AH_r \subseteq H_r$. Then $A^k x_0 \in H_r$, $k \in \mathbb{Z}_+$, and $H = H_r$. This is a contradiction since H is an infinite-dimensional space.

Hence, we may write

$$Ae_r = \sum_{j=0}^{r+1} \gamma_{r,j} e_j, \qquad \gamma_{r,j} \in \mathbb{C}, \qquad \gamma_{r,r+1} \neq 0.$$

Observe that

$$m_{r+1,r} = (Ae_r, e_{r+1})_H = \gamma_{r,r+1} \neq 0, \qquad r \in \mathbb{Z}_+.$$

Thus, $\mathcal{M} \in \mathfrak{M}_3^+$ and $A \in C_+(H)$. ■

Remark 1. Condition (ii) of the last theorem may be replaced by the following condition which does not use a conjugation J:

$(ii)^*$ there exists a cyclic vector x_0 of A such that the following relations hold:

$$\Gamma(x_0, x_1, \dots, x_n, x_n^*) = 0, \qquad \forall\, n \in \mathbb{N}, \tag{13}$$

where

$$x_k = A^k x_0, \qquad x_k^* = (A^*)^k x_0, \qquad k \in \mathbb{N},$$

and the following operator:

$$L \sum_{k=0}^{\infty} \alpha_k A^k x_0 := \sum_{k=0}^{\infty} \overline{\alpha_k}\, (A^*)^k x_0, \qquad \alpha_k \in \mathbb{C}, \tag{14}$$

where all but finite number of coefficients α_k are zeros, is a bounded operator in H which extends by continuity to a conjugation in H.

Let us show that conditions (i), (ii) $\Leftrightarrow$ conditions (i), $(ii)^*$.
The necessity is obvious since the conjugation J satisfies relation (14) (with J instead of L).
Sufficiency. Let conditions (i), $(ii)^*$ be satisfied. Notice that

$$LAA^k x_0 = LA^{k+1} x_0 = (A^*)^{k+1} x_0,$$
$$A^* L A^k x_0 = A^* (A^*)^k x_0 = (A^*)^{k+1} x_0, \qquad k \in \mathbb{Z}_+.$$

By continuity we get $LA = A^* L$. Then condition (ii) holds with the conjugation L.

Remark 2. Notice that conditions (13) may be written in terms of the coordinates of x_0 in an arbitrary orthonormal basis $\{u_n\}_{n=0}^{\infty}$ in H:

$$x_0 = \sum_{n=0}^{\infty} x_{0,n} u_n, \qquad A^k x_0 = \sum_{n=0}^{\infty} x_{0,n} A^k u_n, \qquad (A^*)^k x_0 = \sum_{n=0}^{\infty} x_{0,n} (A^*)^k u_n.$$

By substitution these equalities in relation (8) we get some algebraic equations with respect to the coordinates $x_{0,n}$. If cyclic vectors of A are unknown, one can use numerical methods to find approximate solutions of these equations. Then there should be cyclic vectors of A among these solutions.

The following theorem gives an analogous description for the class $C_-(H)$.

Theorem 2. *Let A be a linear bounded operator in an infinite-dimensional Hilbert space H. The operator A belongs to the class $C_-(H)$ if and only if the following conditions hold:*

(i) *A is a complex skew-symmetric operator with a simple spectrum;*

(ii) *there exists a cyclic vector x_0 of A such that the following relations hold:*

$$\Gamma(x_0, x_1, \dots, x_n, x_n^*) = 0, \qquad \forall\, n \in \mathbb{N}, \tag{15}$$

where

$$x_k = A^k x_0, \qquad x_k^* = (A^*)^k x_0, \qquad k \in \mathbb{N}; \tag{16}$$

and $Jx_0 = x_0$, for a conjugation J in H such that $JAJ = -A^*$.

Condition (ii) of this theorem may be replaced by the following condition:

$(ii)^*$ there exists a cyclic vector x_0 of A such that relations (15), (16) hold and the following operator:

$$L\sum_{k=0}^{\infty}\alpha_k A^k x_0 := \sum_{k=0}^{\infty}(-1)^k\overline{\alpha_k}\,(A^*)^k\,x_0, \qquad \alpha_k\in\mathbb{C},$$

where all but finite number of coefficients α_k are zeros, is a bounded operator in H which extends by continuity to a conjugation in H.

The proof of the latter facts is similar and essentially the same as for the case of $C_+(H)$.

The following example shows that condition (ii) (or $(ii)^*$) can not be removed from Theorem 1.

Example 1. Let $\sigma(\theta)$ be a non-decreasing left-continuous bounded function on $[0,2\pi]$ with an infinite number of points of increase and such that

$$\int_0^{2\pi}\ln\sigma'(\theta)d\theta = -\infty. \tag{17}$$

Consider the Hilbert space $L^2([0,2\pi],d\sigma)$ of (classes of equivalence of) complex-valued functions $f(\theta)$ on $[0,2\pi]$ such that

$$\|f\|^2_{L^2([0,2\pi],d\sigma)} := \left(\int_0^{2\pi}|f(\theta)|^2 d\sigma(\theta)\right)^{\frac{1}{2}} < \infty.$$

The condition (17) provides that algebraic polynomials of $e^{i\theta}$ are dense in $L^2([0,2\pi],d\sigma)$ [5, p. 19]. Therefore the operator

$$Uf(\theta) = e^{i\theta}f(\theta), \qquad f\in L^2([0,2\pi],d\sigma),$$

is a cyclic unitary operator in H, with a cyclic vector $f_0(\theta)=1$. Set

$$Jf(\theta) = \overline{f(\theta)}, \qquad f\in L^2([0,2\pi],d\sigma).$$

Then

$$JUJf(\theta) = Je^{i\theta}\overline{f(\theta)} = e^{-i\theta}f(\theta) = U^{-1}f(\theta) = U^*f(\theta).$$

Thus, U is a complex symmetric operator with a simple spectrum and condition (i) of Theorem 1 is satisfied.

However, $U\notin C_+(H)$. In fact, suppose to the contrary that there exists an orthonormal basis $\{e_j\}_{j\in\mathbb{Z}_+}$ such that the corresponding matrix $\mathcal{M}=(m_{k,l})_{k,l=0}^{\infty}$ from (7) belongs to the class $\mathfrak{M}_3^+$. Since U is unitary, we have

$$\mathcal{E} = \mathcal{M}\mathcal{M}^*,$$

with the usual rules of matrix operations, $\mathcal{E}=(\delta_{k,l})_{k,l=0}^{\infty}$. However, the direct calculation shows that the element of the matrix $\mathcal{M}\mathcal{M}^*$ in row 0, column 2 is equal to $m_{0,1}\overline{m_{2,1}}\neq 0$. We obtained a contradiction. Thus, $U\notin C_+(H)$. Consequently, *condition (ii) in Theorem 1 is essential and can not be removed.*

Proposition 1. *Let H be an arbitrary infinite-dimensional Hilbert space. The class $C_+(H)$ is a proper subset of the set of all complex symmetric operators with a simple spectrum in H.*

Proof. Consider an arbitrary infinite-dimensional Hilbert space H. Let V be an arbitrary unitary operator which maps $L^2([0,2\pi],d\sigma)$ (see Example 1) onto H. Then $\widehat{U} := VUV^{-1}$ is a unitary operator in H with a simple spectrum and it has a cyclic vector $\widehat{x}_0 := V1$. Since $JUJ = U^*$, we get

$$JV^{-1}\widehat{U}VJ = V^{-1}\widehat{U}^*V, \qquad VJV^{-1}\widehat{U}VJV^{-1} = \widehat{U}^*.$$

Observe that $\widehat{J} := VJV^{-1}$ is a conjugation in H. Therefore $\widehat{U}$ is a complex symmetric operator in H. Suppose that $\widehat{U} \in C_+(H)$. Let $\mathcal{F} = \{f_k\}_{k=0}^\infty$ be an orthonormal basis in H such that the matrix $M = (m_{k,l})_{k,l=0}^\infty$, $m_{k,l} = (\widehat{U}f_l, f_k)_H$, belongs to $\mathfrak{M}_3^+$. Observe that $\mathcal{G} = \{g_k\}_{k=0}^\infty$, $g_k := V^{-1}f_k$, is an orthonormal basis in $L^2([0,2\pi],d\sigma)$ and

$$(Ug_l, g_k)_{L^2([0,2\pi],d\sigma)} = (V^{-1}\widehat{U}Vg_l, g_k)_{L^2([0,2\pi],d\sigma)} = (\widehat{U}f_l, f_k)_H = m_{k,l}, \qquad k,l \in \mathbb{Z}_+.$$

Therefore $U \in C_+(L^2([0,2\pi],d\sigma))$. This is a contradiction with Example 1. Consequently, we have $\widehat{U} \notin C_+(H)$.

On the other hand, the class $C_+(H)$ is non-empty, since an arbitrary matrix from $\mathcal{M}_3^+$ with bounded elements define an operator B in H which have this matrix in an arbitrary fixed orthonormal basis in H. ■

Remark 3. The classical Jacobi matrices are closely related to orthogonal polynomials [4]. Let us indicate some similar relations for the class $\mathfrak{M}_3^+$. Choose an arbitrary $\mathcal{M} = (m_{k,l})_{k,l=0}^\infty \in \mathfrak{M}_3^+$, where $m_{k,l} \in \mathbb{C}$. Let $\{p_n(\lambda)\}_{n=0}^\infty$, $\deg p_n = n$, $p_0(\lambda) = 1$, be a sequence of polynomials defined recursively by the following relation:

$$m_{n,n-1}p_{n-1}(\lambda) + m_{n,n}p_n(\lambda) + m_{n,n+1}p_{n+1}(\lambda) = \lambda p_n(\lambda), \qquad n = 0,1,2,\ldots, \tag{18}$$

where $m_{0,-1} := 1$, $p_{-1} := 0$. Set $c_n = m_{n,n+1}$, $b_n = m_{n,n}$, $n \in \mathbb{Z}_+$; and $c_{-1} := 1$. By (5), (18) we get

$$c_{n-1}p_{n-1}(\lambda) + b_np_n(\lambda) + c_np_{n+1}(\lambda) = \lambda p_n(\lambda), \qquad n = 0,1,2,\ldots. \tag{19}$$

Let $p_n(\lambda) = \mu_n\lambda^n + \cdots$, $\mu_n \in \mathbb{C}$, $n \in \mathbb{Z}_+$. Comparing coefficients by λ^{n+1} in (19) we get

$$\mu_{n+1} = \frac{1}{c_n}\mu_n, \qquad n \in \mathbb{Z}_+.$$

By induction we see that

$$\mu_n = \left(\prod_{j=0}^{n-1} c_j\right)^{-1}, \qquad n \in \mathbb{N}, \qquad \mu_0 = 1.$$

Set

$$P_n(\lambda) = \prod_{j=0}^{n-1} c_j p_n(\lambda), \qquad n \in \mathbb{N}, \qquad P_0(\lambda) = 1, \qquad P_{-1}(\lambda) = 0.$$

Multiplying the both sides of (19) by $\prod_{j=0}^{n-1} c_j$, $n \geq 1$, we obtain:

$$c_{n-1}^2P_{n-1}(\lambda) + b_nP_n(\lambda) + P_{n+1}(\lambda) = \lambda P_n(\lambda), \qquad n = 0,1,2,\ldots.$$

By Theorem 6.4 in [6] there exists a complex-valued function ϕ of bounded variation on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} P_m(\lambda)P_n(\lambda)d\phi(\lambda) = \left(\prod_{j=0}^{n-1} c_j\right)^2 \delta_{m,n}, \qquad m,n \in \mathbb{Z}_+.$$

Therefore we get

$$\int_{\mathbb{R}} p_m(\lambda)p_n(\lambda)d\phi(\lambda) = \delta_{m,n}, \qquad m,n \in \mathbb{Z}_+.$$

Polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ were used in [7, 8] to state and solve the direct and inverse spectral problems for matrices from $\mathfrak{M}_3^+$. Analogs of some facts of the Weyl discs theory were obtained for the case of matrices from $\mathfrak{M}_3^+$ with additional assumptions [9]: $m_{n,n+1} > 0$, $n \in \mathbb{Z}_+$, and

$$m_{n,n} \in \mathbb{C}: \quad r_0 \leq \operatorname{Im} m_{n,n} \leq r_1,$$

for some $r_0, r_1 \in \mathbb{R}$, $n \in \mathbb{Z}_+$.

On the other hand, the direct and inverse spectral problems for matrices from $\mathfrak{M}_3^-$ were investigated in [10].

Probably, some progress in the spectral theory of complex symmetric and skew-symmetric operators would provide some additional information about corresponding polynomials and vice versa.

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References

[1] Garcia S.R., Putinar M., Complex symmetric operators and applications, *Trans. Amer. Math. Soc.* **358** (2006), 1285–1315.

[2] Garcia S.R., Putinar M., Complex symmetric operators and applications. II, *Trans. Amer. Math. Soc.* **359** (2007), 3913–3931.

[3] Zagorodnyuk S.M., On a J-polar decomposition of a bounded operator and matrices of J-symmetric and J-skew-symmetric operators, *Banach J. Math. Anal.* **4** (2010), no. 2, 11–36.

[4] Akhiezer N.I., Classical moment problem, Fizmatlit, Moscow, 1961 (in Russian).

[5] Geronimus Ya.L., Orthogonal polynomials on the circle and on an interval. Estimates, asymptotic formulas, orthogonal series, Fizmatlit, Moscow, 1958 (in Russian).

[6] Chihara T.S., An introduction to orthogonal polynomials, *Mathematics and its Applications*, Vol. 13, Gordon and Breach Science Publishers, New York – London – Paris, 1978.

[7] Guseinov G.Sh., Determination of an infinite non-self-adjoint Jacobi matrix from its generalized spectral function, *Mat. Zametki* **23** (1978), 237–248 (English transl.: *Math. Notes* **23** (1978), 130–136).

[8] Zagorodnyuk S.M., Direct and inverse spectral problems for $(2N+1)$-diagonal, complex, symmetric, non-Hermitian matrices, *Serdica Math. J.* **30** (2004), 471–482.

[9] Zagorodnyuk S.M., Integral representations for spectral functions of some nonself-adjoint Jacobi matrices, *Methods Funct. Anal. Topology* **15** (2009), 91–100.

[10] Zagorodnyuk S.M., The direct and inverse spectral problems for $(2N+1)$-diagonal complex transposition-antisymmetric matrices, *Methods Funct. Anal. Topology* **14** (2008), 124–131.